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Affine Completeness of Generalised Dihedral Groups

Jürgen Ecker

Abstract. In this paper we study affine completeness of generalised dihedral groups. We give a formula for the number of unary compatible functions on these groups, and we characterise for every $k \in \mathbb{N}$ the k-affine complete generalised dihedral groups. We find that the direct product of a 1-affine complete group with itself need not be 1-affine complete. Finally, we give an example of a nonabelian solvable affine complete group. For nilpotent groups we find a strong necessary condition for 2-affine completeness.

1 Introduction

Let $G = (G, \cdot)$ be a finite group and let N be a normal subgroup of G. For $g \in G$ and $S \subseteq G$, let $[g]$ and $[S]$ denote the normal subgroups of G generated by g and S, respectively. As in [9], let $\gamma_i(G)$ denote the i-th term of the lower central series of G.

Let $k \in \mathbb{N}$, and let $\varphi\colon G^k \to G$ be a k-ary function on G. We say φ is *compatible with* N iff for all $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in G^k$ such that $x_i y_i^{-1} \in N$ for every $1 \le i \le k$, we have $\varphi(x)\varphi(y)^{-1} \in N$. We call φ *compatible* iff it is compatible with every normal subgroup of G. Equivalently [8, Lemma 3a], φ is compatible, if for all $x = (x_1, \dots, x_k), y = (y_1, \dots, y_k) \in G^k$, $\varphi(x)\varphi(y)^{-1} \in [\{x_1 y_1^{-1}, \dots, x_k y_k^{-1}\}]$. Let $\mathrm{Comp}_k(G)$ denote the set of all k-ary compatible functions on the group G. Then $(\mathrm{Comp}_k(G), \cdot)$ is a group, where $\cdot$ is the point-wise multiplication of functions. For every $i \in \{1, \dots, k\}$ and for every $g \in G$, the projection $\pi_i\colon G^k \to G$, $(x_1, \dots, x_k) \mapsto x_i$, and the constant function $\bar{g}\colon G^k \to G$, $(x_1, \dots, x_k) \mapsto g$ are k-ary compatible functions on G. Let $\mathrm{Pol}_k(G)$ be the subgroup of $\mathrm{Comp}_k(G)$ generated by $\{\pi_i \mid 1 \le i \le k\} \cup \{\bar{g} \mid g \in G\}$. We call the elements of $\mathrm{Pol}_k(G)$ the *k-ary polynomial functions on G*. For the definition of a k-ary polynomial function on an arbitrary algebra A we refer to [5]. We call G *k-affine complete* if $\mathrm{Pol}_k(G) = \mathrm{Comp}_k(G)$, and *affine complete* if G is k-affine complete for every $k \in \mathbb{N}$. By [8, Lemma 1], if G is k-affine complete for some $k \in \mathbb{N}$, then G is l-affine complete for each $l \le k$.

Let C_n denote the cyclic group of order n. For a finite abelian group A, the *generalised dihedral group of A*, $\mathrm{Dih}(A)$, is the semi-direct product of A with C_2, where the non-identity element of C_2 takes each $a \in A$ to a^{-1}. We write all group operations as multiplications.

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The objective of this paper is to determine the k-affine complete generalised dihedral groups for every $k \in \mathbb{N}$. In Section 2 we treat the case of unary functions and prove the following theorem.

Theorem 1.1 *Let A be a finite abelian group. Then* $\mathrm{Dih}(A)$ *is* 1*-affine complete if and only if A is* 1*-affine complete.*

To this end, we first study unary compatible functions on a group that map all elements of the group into a certain normal subgroup (Proposition 2.4). In Section 4 we study compatible and polynomial functions of higher arity and prove the following result.

Theorem 1.2 *Let A be a finite abelian group. Let P be a* 2*-group and let Q be a group of odd order such that $A = PQ$. Then for $G = \mathrm{Dih}(A)$ the following conditions are equivalent:*

(i) *G is affine complete.*
(ii) *G is* 2*-affine complete.*
(iii) (a) $\exp P = 2$ *and*
  (b) *Q is affine complete.*

To achieve this we generalise the results for unary functions in [2] and, in Proposition 4.1, give a strong necessary condition for a nilpotent group to be 2-affine complete.

Theorems 1.1 and 1.2 reduce the problem of deciding affine completeness to the case of a finite abelian group. For every $k \in \mathbb{N}$, the finite k-affine complete abelian groups have been characterised in [6, 8]. For easier reference we gather some results in the following lemma.

Lemma 1.3 ([6, 8]) *Let p be a prime and let*

$$G = \mathrm{C}_{p^{\alpha_1}} \times \mathrm{C}_{p^{\alpha_2}} \times \cdots \times \mathrm{C}_{p^{\alpha_r}},$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$ and $r \in \mathbb{N}$. Then G is 1*-affine complete iff one of the following conditions holds.*

(i) *$r > 1$ and $\alpha_1 = \alpha_2$,*
(ii) *$r > 1$, $p = 2$, and $\alpha_1 = \alpha_2 + 1$,*
(iii) *$r = 1$, $p = 2$, and $\alpha_1 = 1$.*

Furthermore, G is not 2*-affine complete in cases* (ii) *and* (iii) *and affine complete in case* (i)*. The direct product of two finite groups G_1 and G_2 of coprime order is k-affine complete iff G_1 and G_2 are k-affine complete.*

2 The 1-Affine Complete Generalised Dihedral Groups

We start with a few simple observations concerning the structure of generalised dihedral groups. For every abelian group A and every integer $d \geq 0$, the groups $\mathrm{Dih}(A) \times (\mathrm{C}_2)^d$ and $\mathrm{Dih}(A \times (\mathrm{C}_2)^d)$ are isomorphic. Let $d \geq 0$ be such that A is the direct product of d cyclic groups of even order and some or no groups of odd order. Then the derived subgroup of $G = \mathrm{Dih}(A)$ is $\gamma_2(G) = \{a^2 \mid a \in A\}$ (by [9, 5.1.5.]), hence the abelian quotient is $G/\gamma_2(G) \cong (\mathrm{C}_2)^{d+1}$. For all $g \in G \setminus A$ and $a \in A$, we have $[g, a] = a^2$, thus $[g] \supseteq \gamma_2(G)$.

The following lemma gives the number of unary polynomial functions on generalised dihedral groups.

Lemma 2.1 ([7]) *Let A be a finite abelian group. Let $d \geq 0$ be such that A is a direct product of d cyclic groups of even order and some or no groups of odd order. Then*

$$|\,\mathrm{Pol}_1(\mathrm{Dih}(A))| = \begin{cases} 4 \cdot |\,\mathrm{Pol}_1(A)|^2 & \text{if } d = 0, \\ \dfrac{1}{2^d} \cdot |\,\mathrm{Pol}_1(A)|^2 & \text{if } d > 0. \end{cases}$$

Lemma 2.2 *Let G be a finite group and $k \in \mathbb{N}$. Let N be a normal subgroup of G, such that G/N is k-affine complete. Then the mapping*

$$\rho\colon\ \mathrm{Comp}_k(G) \to \mathrm{Comp}_k(G/N), \quad \varphi \mapsto \varphi^N,$$

where $\varphi^N(gN) := \varphi(g)N$, for every $g \in G^k$, is a group epimorphism.

Proof Clearly, ρ is a group homomorphism

$$(\mathrm{Comp}_k(G), \cdot\,) \to (\mathrm{Comp}_k(G/N), \cdot\,).$$

We show that ρ is an epimorphism. To this end we fix an arbitrary $\phi \in \mathrm{Comp}_k(G/N)$. Since G/N is k-affine complete, ϕ is induced by a polynomial p over G/N. Lifting the coefficients of p from G/N to G, we obtain a polynomial q on G. Let $\bar{q}$ be the polynomial function on G induced by q. Then $\rho(\bar{q}) = \bar{q}^N = \phi$. Thus ρ is surjective. ∎

Corollary 2.3 *Let G be a finite group and $k \in \mathbb{N}$. Let N be a normal subgroup of G, such that G/N is k-affine complete. Then*

$$|\,\mathrm{Comp}_k(G)| = |\,\mathrm{Pol}_k(G/N)| \cdot |\{\psi \in \mathrm{Comp}_k(G) \mid \psi(G^k) \subseteq N\}|.$$

Proof Let ρ be the epimorphism constructed in Lemma 2.2. The kernel of ρ is $\{\psi \in \mathrm{Comp}_k(G) \mid \psi(G^k) \subseteq N\}$. The result follows from the isomorphism theorem. ∎

Now we study the part $\{\psi \in \mathrm{Comp}_k(G) \mid \psi(G^k) \subseteq N\}$ in Corollary 2.3 for $k = 1$.

Proposition 2.4 *Let G be a finite group, and let $T \trianglelefteq A \trianglelefteq G$, such that*

(i) *every normal subgroup of A is normal in G, and*
(ii) *for every $I \trianglelefteq G$ with $I \not\leq A$ we have $I \geq T$.*

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Let C be a transversal for the cosets of A in G. Define the mapping

$$\Psi\colon \{\varphi \in \mathrm{Comp}_1(A) \mid \varphi(A) \subseteq T\}^{|C|} \to \{\varphi \in \mathrm{Comp}_1(G) \mid \varphi(G) \subseteq T\}$$
$$(\varphi_c)_{c\in C} \mapsto \varphi,$$

where $\varphi\colon G \to G$ is defined by $\varphi(ac) := \varphi_c(a)$ (for all $a \in A$ and $c \in C$). Then Ψ is bijective.

Proof Let $(\varphi_c)_{c\in C}$ be a family of unary compatible functions on A such that for every $c \in C$, $\varphi_c(A) \subseteq T$. Let $\varphi := \Psi((\varphi_c)_{c\in C})$. Clearly, φ maps G into T. We show that φ is a compatible function on G. Let $g, h \in G$.

If $gh^{-1} \in A$, then there exist unique $a, b \in A$ and $c \in C$, such that $g = ac$ and $h = bc$. Then $\varphi(g)\varphi(h)^{-1} = \varphi_c(a)\varphi_c(b)^{-1} \in [ab^{-1}] = [ac(bc)^{-1}] = [gh^{-1}]$. We note that by condition (i), it makes no difference if we generate the normal subgroups in A or in G. If $gh^{-1} \notin A$, then by condition (ii), $[gh^{-1}] \geq T$. Since $\varphi(G) \subseteq T$, we have $\varphi(g)\varphi(h)^{-1} \in [gh^{-1}]$. Thus φ is compatible.

Clearly, Ψ is injective. It remains to show that Ψ is surjective. Let φ be a compatible function on G which maps G into T. For each $c \in C$ we define a function $\varphi_c\colon A \to T$ by $\varphi_c(a) := \varphi(ac)$. Then $\Psi((\varphi_c)_{c\in C}) = \varphi$, and for every $c \in C$, $\varphi_c(A) \subseteq T$. It remains to show that each φ_c is compatible. To this end, we fix $c \in C$ and $a, b \in A$, and we compute $\varphi_c(a)\varphi_c(b)^{-1} = \varphi(ac)\varphi(bc)^{-1} \in [ac(bc)^{-1}] = [ab^{-1}]$. Thus Ψ is bijective. ■

Corollary 2.5 *Let $G = \mathrm{Dih}(A)$, where A is a finite abelian group. Let $S := \{a^2 \mid a \in A\}$. Then*

$$\bigl|\{\varphi \in \mathrm{Comp}_1(G) \mid \varphi(G) \subseteq \gamma_2(G)\}\bigr| = \bigl|\{\varphi \in \mathrm{Comp}_1(A) \mid \varphi(A) \subseteq S\}\bigr|^2.$$

Corollary 2.6 *If A is the direct product of $d \geq 0$ cyclic groups of even order and some or no cyclic groups of odd order, then*

$$|\mathrm{Comp}_1(\mathrm{Dih}(A))| = 2^{d+2} \cdot \Bigl(\frac{|\mathrm{Comp}_1(A)|}{|\mathrm{Pol}_1(A/S)|}\Bigr)^2,$$

where $S := \{a^2 \mid a \in A\}$.

Proof We know that S is normal in A. By Lemma 1.3, the quotient $A/S \cong (\mathrm{C}_2)^d$ is 1-affine complete. Let $\mathcal{A} := \{\varphi \in \mathrm{Comp}_1(A) \mid \varphi(A) \subseteq S\}$. By Corollary 2.3,

$$|\mathrm{Comp}_1(A)| = |\mathrm{Pol}_1(A/S)| \cdot |\mathcal{A}|. \tag{1}$$

Let $G = \mathrm{Dih}(A)$. Then $H := G/\gamma_2(G)$ is elementary abelian of order 2^{d+1}, so by Lemma 1.3, it is 1-affine complete. Since H is abelian,

$$\mathrm{Pol}_1(H) = \{p\colon H \to H,\ x \mapsto h \cdot x^k \mid h \in H,\ 0 \leq k < \exp(H)\}.$$

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By Lemma 2.2 and Corollary 2.5,

$$\begin{aligned}|\operatorname{Comp}_1(G)| &= |\operatorname{Pol}_1(H)| \cdot \left|\{\varphi \in \operatorname{Comp}_1(G) \mid \varphi(G) \subseteq \gamma_2(G)\}\right| \\ &= 2^{d+2} \cdot |\mathcal{A}|^2.\end{aligned}$$

Using (1), we can substitute for $|\mathcal{A}|$ and obtain the desired result. ∎

We are now in the position to prove Theorem 1.1 by simply combining Lemma 2.1 and Corollary 2.6.

Proof of Theorem 1.1 Let $d \geq 0$ be such that A is a direct product of d cyclic groups of even order and some or no groups of odd order. Let $S := \{a^2 \mid a \in A\}$. If $d = 0$, then $S = A$ and by Corollary 2.6,

$$|\operatorname{Comp}_1(\operatorname{Dih}(A))| = 4 \cdot |\operatorname{Comp}_1(A)|^2.$$

If $d > 0$, then $|\operatorname{Pol}_1(A/S)| = |\operatorname{Pol}_1((C_2)^d)| = 2^{d+1}$ and by Corollary 2.6,

$$|\operatorname{Comp}_1(\operatorname{Dih}(A))| = 2^{d+2} \cdot \left(\frac{|\operatorname{Comp}_1(A)|}{2^{d+1}}\right)^2 = \frac{1}{2^d} \cdot |\operatorname{Comp}_1(A)|^2.$$

In both cases, Lemma 2.1 allows us to deduce $|\operatorname{Pol}_1(\operatorname{Dih}(A))| = |\operatorname{Comp}_1(\operatorname{Dih}(A))|$ if and only if $|\operatorname{Pol}_1(A)| = |\operatorname{Comp}_1(A)|$. ∎

3 Direct Products of 1-Affine Complete Groups

The direct product of a finite abelian group with itself is affine complete. In this section we give an example of a 2-group D, such that D is 1-affine complete, but $D \times D$ is not. We only need one more easy lemma, which describes the compatible functions mapping into a minimal normal subgroup.

Lemma 3.1 *Let G be a finite group and let A be a minimal normal subgroup of G. Let A^* be the sum of all normal subgroups of G having trivial intersection with A. Let φ be a function on G with $\varphi(G) \subseteq A$. Then φ is compatible if and only if it is constant on the cosets of A^*. As a consequence, $\left|\{\varphi \in \operatorname{Comp}_1(G) \mid \varphi(G) \subseteq A\}\right| = |A|^{[G:A^*]}$.*

Proof Let φ be a compatible function with $\varphi(G) \subseteq A$. Let us consider a normal subgroup N of G, with $A \cap N = \{1\}$. For $g, h \in G$ and $gh^{-1} \in N$, we have $\varphi(g)\varphi(h)^{-1} \in N \cap A = \{1\}$, forcing $\varphi(g) = \varphi(h)$. So φ is constant on the cosets of every $N \trianglelefteq G$ with $N \cap A = \{1\}$. Let N_1 and N_2 be two such normal subgroups. Consider $S := N_1N_2$. If $gh^{-1} \in S$, then there exist $n_1 \in N_1$ and $n_2 \in N_2$, such that $g = n_1n_2h$. Hence $\varphi(h) = \varphi(n_2h) = \varphi(n_1n_2h) = \varphi(g)$. Thus φ is constant on the cosets of A^*. Conversely, let φ be a function mapping G into A and let φ be constant on the cosets of A^*. Then φ is compatible on G: let $I \trianglelefteq G$. If $I \geq A$, then φ is compatible with I, because $\varphi(G) \subseteq A \subseteq I$. If $I \ngeq A$, then $I \cap A = \{1\}$, whence $I \leq A^*$. So, φ is compatible with I, because φ is constant on the cosets of I. Finally, there are precisely $|A|^{[G:A^*]}$ functions mapping G into A, which are constant on the cosets of A^*. ∎

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Example 3.2 The group $D = \mathrm{C}_2 \times \mathrm{Dih}(\mathrm{C}_4) \cong \mathrm{Dih}(\mathrm{C}_2 \times \mathrm{C}_4)$ is 1-affine complete, by Theorem 1.1 and Lemma 1.3. We show that the group $G = D \times D$ is not 1-affine complete. The group $A := \{1\} \times \gamma_2(D)$ is a minimal normal subgroup of G, and the quotient G/A is isomorphic to $D \times (\mathrm{C}_2)^3 \cong \mathrm{Dih}((\mathrm{C}_2)^4 \times \mathrm{C}_4)$, which is 1-affine complete by Theorem 1.1 and Lemma 1.3. Every normal subgroup not containing A is contained in $D \times (\mathrm{C}_2 \times \gamma_2(\mathrm{Dih}(\mathrm{C}_4)))$, which is a normal subgroup of index 4 in G. Hence the sum of all normal subgroups not containing A has at least index 4 in G. By Lemmas 2.1, 2.2, and 3.1, $|\operatorname{Comp}_1(G)| \geq |\operatorname{Pol}_1(\mathrm{Dih}((\mathrm{C}_2)^4 \times \mathrm{C}_4))| \cdot 2^4 = 2^{-5} \cdot |\operatorname{Pol}_1((\mathrm{C}_2)^4 \times \mathrm{C}_4)|^2 \cdot 2^4 = 2^{15}$. Since G is nilpotent of class 2 and $\exp G = 4$, the number of polynomial functions can be computed with the help of [3, Thm. 1 and Prop. 1]. We get $|\operatorname{Pol}_1(G)| = |G| \cdot \lambda(G) \cdot [G\colon \mathrm{Z}(G)] = 2^8 \cdot 2^2 \cdot 2^4 = 2^{14}$. Thus G is not 1-affine complete.

4 The Affine Complete Generalised Dihedral Groups

In this last section we are going to prove Theorem 1.2. The following proposition gives a strong necessary condition for a nilpotent group to be 2-affine complete. It is inspired by [1, Lemma 2.3], but is not a consequence of this result. Furthermore, we explicitly construct a 2-ary compatible function which is not polynomial.

Proposition 4.1 *Let G be a nilpotent group. Suppose that there exist normal subgroups S and T of G such that*

(i) *for every $I \trianglelefteq G$, either $I \leq S$ or $I \geq T$,*
(ii) *$T \not\leq \gamma_3(G)$,*
(iii) *$S < G$.*

Let $t \in T \setminus \gamma_3(G)$. Then the 2-ary function ϕ on G defined by

$$\phi(x, y) = \begin{cases} t & \text{if } x \notin S \text{ and } y \notin S, \\ 1 & \text{otherwise} \end{cases}$$

is a compatible but not a polynomial function on G. Thus G is not 2-affine complete.

Proof The function ϕ is compatible (*cf.* [1, Lemma 2.2]). Suppose that ϕ is polynomial. Employing Hall's collection process described in [4, Chap. 5], we find unary polynomial functions p_1 and p_2 on G, an integer k, and a 2-ary polynomial function ρ on G, such that for all $x, y \in G$, $\phi(x, y) = p_1(x) \cdot p_2(y) \cdot [x, y]^k \cdot \rho(x, y)$, $\rho(x, y) \in \gamma_3(G)$, and $\rho(x, 1) = \rho(1, y) = 1$. Since, by definition, $\phi(x, 1) = \phi(1, y) = 1$ for all $x, y \in G$, both p_1 and p_2 must map G to $\{1\}$. Now let $g \in G \setminus S$. Then $t = \phi(g, g) = [g, g]^k \cdot \rho(g, g) = \rho(g, g) \in \gamma_3(G)$, a contradiction. Thus ϕ is not polynomial. ∎

In the case where the 2-Sylow subgroup of a generalised dihedral group is (elementary) abelian, we can obtain positive results. The following lemma generalises [2, Lemma 2.2 and Cor. 2.3]. In the proof we make use of [2, Lemma 2.1] which we state here, because it will be used in the proof of Lemma 4.4.

Notation 4.2 Let G be a group and $X, Y \trianglelefteq G$. Then we write $X \prec Y$ if and only if

$$X < Y \quad \text{and} \quad \nexists Z \trianglelefteq G : X < Z < Y.$$

Lemma 4.3 ([2, Lemma 2.1]) *Let G be a finite group, which is a semi-direct product $G = AB$ (A normal). Let $Z := B \cap C_G(A)$. Assume that the following conditions are satisfied:*

- *For each $b \in B \setminus Z$, the mapping $\varphi_b \colon A \to A$, $a \mapsto b^{-1}ab$ is fixed point free.*
- *For all $X, Y \trianglelefteq B$ with $X \prec Y \leq Z$, we have $C_B(Y/X) > Z$.*

Then the function $e \colon G \to G$ defined by

$$\text{for all } a \in A,\, b \in B: \quad e(ab) = \begin{cases} a & \text{if } b \in Z, \\ 1 & \text{otherwise,} \end{cases}$$

is a polynomial function on G.

Lemma 4.4 *Let G be a group satisfying the assumptions of Lemma* 4.3*. Let $k \in \mathbb{N}$ and $Z := B \cap C_G(A)$. Let $n := [B : Z]$ and let $r_1, \dots, r_n$ be a transversal for the cosets of Z in B. Then the following holds.*

(i) *The mapping Ψ defined by*

$$\Psi \colon \{p \in \mathrm{Pol}_k(G) \mid p(G^k) \subseteq A\} \to \left\{ p|_{A^k} \;\middle|\; p \in \mathrm{Pol}_k(G), p(G^k) \subseteq A \right\}^{n^k}$$
$$p \mapsto (p_j)_{j \in \{1,\dots,n\}^k},$$

where for $j = (j_1, \dots, j_k) \in \{1, \dots, n\}^k$, p_j is defined by

$$p_j(a_1, \dots, a_k) := p(a_1 r_{j_1}, \dots, a_k r_{j_k}),$$

is bijective.

(ii) *Let f be a k-ary function on G such that $f(G^k) \subseteq A$. Then f is polynomial if and only if for every $j = (j_1, \dots, j_k) \in \{1, \dots, n\}^k$ there exists a k-ary polynomial function p_j on G such that for all $a_1, \dots, a_k \in A$ and $z_1, \dots, z_k \in Z$, $f(a_1 r_{j_1} z_1, \dots, a_k r_{j_k} z_k) = p_j(a_1, \dots, a_k)$.*

Proof (i) Let $p \in \mathrm{Pol}_k(G)$ such that $p(G^k) \subseteq A$. Then each p_j defined as above maps A^k into A.

We prove that Ψ is injective. Let p and q be two k-ary polynomial functions, such that $p(G^k) \subseteq A$, $q(G^k) \subseteq A$, and $\Psi(p) = \Psi(q)$. We fix $a_1, \dots, a_k \in A$ and $b_1, \dots, b_k \in B$. Let $h = (h_1, \dots, h_k) \in \{1, \dots, n\}^k$ be such that for every $i \in \{1, \dots, k\}$, $r_{h_i}^{-1} b_i \in Z$. Then $p(a_1 b_1, \dots, a_k b_k) = p(a_1 r_{h_1} r_{h_1}^{-1} b_1, \dots, a_k r_{h_k} r_{h_k}^{-1} b_k) = p(a_1 r_{h_1}, \dots, a_k r_{h_k}) = p_h(a_1, \dots, a_k)$, since $p(G^k) \subseteq A$ and $A \cap Z = \{1\}$. In the same way we obtain $q(a_1 b_1, \dots, a_k b_k) = q_h(a_1, \dots, a_k)$. Since $\Psi(p) = \Psi(q)$, p_h and q_h are equal, thus $p = q$.

It remains to show that Ψ is surjective. In light of the given hypotheses, Lemma 4.3 ensures that there exists a unary polynomial function e on G such that for all $a \in A$ and for all $b \in B$,

$$e(ab) = \begin{cases} a & \text{if } b \in Z, \\ 1 & \text{otherwise.} \end{cases}$$

For $x \in G$ we define $\beta(x) := \left(\prod_{i=1}^{n} e(xr_i^{-1})\right)^{-1} \cdot x$. First, we show that for all $a \in A$ and $b \in B$, we have $\beta(ab) = b$. Let $a \in A$, $b \in B$, and let $x = ab$. Then

$$(*) \qquad e(xr_i^{-1}) = e(abr_i^{-1}) = \begin{cases} a & \text{if } b \in r_i Z, \\ 1 & \text{otherwise.} \end{cases}$$

So $\beta(x) = a^{-1}ab = b$.

For $i \in \{1, \ldots, n\}$, $a \in A$, and $x \in G$ we define $e_i(a, x) := e(a\beta(x)r_i^{-1})$. Then for all $i \in \{1, \ldots, n\}$, $a \in A$ and $x \in G$, we have

$$(**) \qquad e_i(a, x) = e(a\beta(x)r_i^{-1}) = \begin{cases} a & \text{if } \beta(x) \in r_i Z, \\ 1 & \text{otherwise.} \end{cases}$$

For every $j = (j_1, \ldots, j_k) \in \{1, \ldots, n\}^k$ we fix a k-ary polynomial function p_j from $\{p|_{A^k} \mid p \in \mathrm{Pol}_k(G),\ p(G^k) \subseteq A\}$. For $1 \le i \le k$ we define

$$\begin{aligned} s_{j,0}(x_1, \ldots, x_k) &:= p_j(e(x_1 r_{j_1}^{-1}), \ldots, e(x_k r_{j_k}^{-1})), \\ s_{j,i}(x_1, \ldots, x_k) &:= e_{j_i}(s_{j,i-1}(x_1, \ldots, x_k), x_i), \\ q_j(x_1, \ldots, x_k) &:= s_{j,k}(x_1, \ldots, x_k). \end{aligned}$$

Let us fix an arbitrary k-tuple $j = (j_1, \ldots, j_k) \in \{1, \ldots, n\}^k$, $a_1, \ldots, a_k \in A$, and $b_1, \ldots, b_k \in B$, and let $x_i = a_i b_i$, for every i with $1 \le i \le k$. Now we are going to show that $q_j(x_1, \ldots, x_k) = p_j(a_1, \ldots, a_k)$, if for every $i \in \{1, \ldots, k\}$, $b_i \in r_{j_i}Z$, and $q_j(x_1, \ldots, x_k) = 1$ otherwise. First, we observe that if for every $i \in \{1, \ldots, n\}$, $b_i \in r_{j_i}Z$, then by $(*)$, $s_{j,0}(x_1, \ldots, x_k) = p_j(a_1, \ldots, a_k)$. Now we fix an i with $1 \le i \le k$. By our definition, $s_{j,i}(x_1, \ldots, x_k) = e_{j_i}(s_{j,i-1}(x_1, \ldots, x_k), x_i)$. By $(**)$, this is equal to $s_{j,i-1}(x_1, \ldots, x_k)$, if $\beta(x_i) \in r_{j_i}Z$, and equal to 1 otherwise.

We claim that a preimage of $(p_j)_{j \in \{1,\ldots,n\}^k}$ under Ψ is

$$q(x_1, \ldots, x_k) := \prod_{\substack{j \in \{1,\ldots,n\}^k \\ j=(j_1,\ldots,j_k)}} q_j(x_1, \ldots, x_k).$$

Clearly, q is a k-ary polynomial function on G. For fixed $x_1, \ldots, x_k \in G$ at most one factor $q_j(x_1, \ldots, x_k)$ is not equal to 1, whence the order of multiplication of the q_j is not relevant. Thus we have constructed a polynomial function q on G such that $\Psi(q) = (p_j)_{j \in \{1,\ldots,n\}^k}$.

(ii) Let f be a k-ary polynomial function on G such that $f(G^k) \subseteq A$. Let $j = (j_1, \dots, j_k) \in \{1, \dots, n\}^k$. We define a function $p_j \in \mathrm{Pol}_k(G)$ by $p_j(g_1, \dots, g_k) := f(g_1 r_{j_1}, \dots, g_k r_{j_k})$. Then for arbitrary $a_1, \dots, a_k \in A$ and $z_1, \dots, z_k \in Z$,

$$p_j(a_1, \dots, a_k) = f(a_1 r_{j_1}, \dots, a_k r_{j_k}) = f(a_1 r_{j_1} z_1, \dots, a_k r_{j_k} z_k),$$

since $A \cap Z = \{1\}$.

Conversely, let f be a k-ary function on G and let $(p_j)_{j \in \{1,\dots,n\}^k}$ be a family of polynomial functions on G, such that for every $j = (j_1, \dots, j_k) \in \{1, \dots, n\}^k$, for all $a_1, \dots, a_k \in A$ and for all $z_1, \dots, z_k \in Z$, $f(a_1 r_{j_1} z_1, \dots, a_k r_{j_k} z_k) = p_j(a_1, \dots, a_k)$. Then each p_j is an element of the set $\{p|_{A^k} \mid p \in \mathrm{Pol}_k(G),\ p(G^k) \subseteq A\}$. Thus by (i), there exists a k-ary polynomial function q on G, such that $q(G^k) \subseteq A$ and for all $j \in \{1, \dots, n\}^k$, $a_1, \dots, a_k \in A$, and $z_1, \dots, z_k \in Z$, we have

$$p_j(a_1, \dots, a_k) = q(a_1 r_{j_1}, \dots, a_k r_{j_k}) = q(a_1 r_{j_1} z_1, \dots, a_k r_{j_k} z_k).$$

Thus $f = q$, so f is a polynomial function on G. ■

Proposition 4.5 *Let G be a finite group, which is a semi-direct product $G = AB$, and let $k \in \mathbb{N}$. Let $Z := B \cap C_G(A)$. Assume that the following conditions are satisfied:*

- *For each $b \in B \setminus Z$, there exists an $e_b \in \mathbb{Z}$, such that the mapping $\varphi_b \colon A \to A$, $a \mapsto b^{-1}ab$ is fixed point free and $\varphi_b(a) = a^{e_b}$.*
- *For all $X, Y \trianglelefteq B$ with $X \prec Y \leq Z$, we have $C_B(Y/X) > Z$.*
- *A and B are k-affine complete.*

Then G is k-affine complete.

Proof Let c be a k-ary compatible function on G. Then the function $c^A \colon (G/A)^k \to G/A$, $(g_1 A, \dots, g_k A) \mapsto c(g_1, \dots, g_k)A$, is a compatible function on G/A. Since $G/A \cong B$ and B is k-affine complete, there exists a k-ary polynomial function q_1 on G/A, such that $c^A = q_1$. Lifting the coefficients of q_1 from G/A to G, we obtain a polynomial function q_2 on G such that $q_2^A = q_1$. Now define a k-ary function c_1 on G by $c_1(g_1, \dots, g_k) := c(g_1, \dots, g_k)(q_2(g_1, \dots, g_k))^{-1}$. Also, c_1 is a compatible function on G. We show that c_1 is a polynomial function on G. First note that $c_1(G^k) \subseteq A$, since $q_2^A = c^A$. Let $n := [B : Z]$ and let $r_1, \dots, r_n$ be a transversal for the cosets of Z in B. We fix an arbitrary $j = (j_1, \dots, j_k) \in \{1, \dots, n\}^k$ and define the k-ary function c_2 on G by $c_2(g_1, \dots, g_k) := c_1(g_1 r_{j_1}, \dots, g_k r_{j_k})$. Then $c_2(G^k) \subseteq A$, and c_2 is a compatible function on G. Since every normal subgroup of A is normal in G by our assumptions, $c_2|_{A^k}$ is a k-ary compatible function on A. Since A is k-affine complete, there exists a k-ary polynomial function p_j on G such that $p_j|_{A^k} = c_2|_{A^k}$. We fix $a_1, \dots, a_k \in A$ and $z_1, \dots, z_k \in Z$. Since Z is normal in G, there exist $z_1', \dots, z_k' \in Z$ such that $c_1(a_1 r_{j_1} z_1, \dots, a_k r_{j_k} z_k) = c_1(a_1 z_1' r_{j_1}, \dots, a_k z_k' r_{j_k}) = c_2(a_1 z_1', \dots, a_k z_k')$. Since $c_2(G^k) \subseteq A$ and $A \cap Z = \{1\}$,

$$c_2(a_1 z_1', \dots, a_k z_k') = c_2(a_1, \dots, a_k) = p_j(a_1, \dots, a_k).$$

By Lemma 4.4(ii), c_1 is a polynomial function on G. By our definition, $c(g_1, \dots, g_k) = c_1(g_1, \dots, g_k) q_2(g_1, \dots, g_k)$ for all $g_1, \dots, g_k \in G$, hence c is a polynomial function on G. ■

For the proof of Theorem 1.2, we need one more lemma, which, in particular, tells us that for any $k \in \mathbb{N}$, the 2-Sylow subgroup of a k-affine complete generalised dihedral group is k-affine complete.

Lemma 4.6 *Let $A = PQ$ be the direct product of the groups P and Q, and let G be a semi-direct product of A with the group C, such that both P and Q are normal in G. Let $k \in \mathbb{N}$ and $S := PC$. Assume that for all $s \in S$ and $q \in Q$, we have $s \in [sq]$. If S is not k-affine complete, then G is not k-affine complete.*

Proof Let ϕ be a k-ary compatible function on S, which is not polynomial. We extend ϕ to a k-ary compatible function on G, which is not polynomial on G. To this end we define $e\colon G \to G$, $pqc \mapsto pc$. We show that e is a unary compatible function on G. Let $p_1, p_2 \in P$, $q_1, q_2 \in Q$ and $c_1, c_2 \in C$. Then $s := e(p_1q_1c_1)e(p_2q_2c_2)^{-1} = p_1c_1c_2^{-1}p_2^{-1} \in S$. Since $[P, Q] = \{1\}$, we have $p_1q_1c_1(p_2q_2c_2)^{-1} = q_1sq_2^{-1}$. There exists $q' \in Q$, such that $q_1sq_2^{-1} = sq'$. Since by our assumptions, $s \in [sq']$, we obtain $e(p_1q_1c_1)e(p_2q_2c_2)^{-1} \in [p_1q_1c_1(p_2q_2c_2)^{-1}]$, thus e is compatible. Now define $\psi(g_1, \dots, g_k) := \phi(e(g_1), \dots, e(g_k))$. Then ψ is a k-ary compatible function on G and $\psi(G^k) \subseteq S$. Suppose that ψ is a polynomial function on G. Then there exist $r \in \mathbb{N}$ and $q_0 \in Q$, $s_0 \in S$, and for every i with $1 \le i \le r$ and every j with $1 \le j \le k$ there exist $q_{i,j} \in Q$, $s_{i,j} \in S$, and $e_{i,j} \in \mathbb{Z}$ such that for all $x_1, \dots, x_k \in G$,

$$\psi(x_1, \dots, x_k) = q_0s_0\Big(\prod_{i=1}^{r}\prod_{j=1}^{k} x_j^{e_{i,j}} q_{i,j}s_{i,j}\Big).$$

Now we prove that for all $(t_1, \dots, t_k) \in S^k$, we have

$$\psi(t_1, \dots, t_k) = s_0\Big(\prod_{i=1}^{r}\prod_{j=1}^{k} t_j^{e_{i,j}} s_{i,j}\Big).$$

To this end we fix a k-tuple $(t_1, \dots, t_k) \in S^k$. The subgroup Q is normal in G. So for all $g \in G$ and $q \in Q$ there exists an element $q' \in Q$ such that $gq = q'g$. So we can find a $q \in Q$, such that $q_0s_0\big(\prod_{i=1}^{r}\prod_{j=1}^{k} t_j^{e_{i,j}}q_{i,j}s_{i,j}\big) = q \cdot s_0\big(\prod_{i=1}^{r}\prod_{j=1}^{k} t_j^{e_{i,j}}s_{i,j}\big)$. Since $\psi(t_1, \dots, t_k) \in S$ and $Q \cap S = \{1\}$, we must have $q = 1$.

So, $\psi|_{S^k}$ is induced by a polynomial over S. But $\psi|_{S^k} = \phi$, and ϕ is not a polynomial function on S, a contradiction. ∎

It remains to arrange the results obtained in this section.

Proof of Theorem 1.2 Remember that $\mathrm{Dih}(A)$ is the semi-direct product of A with a group C of order 2.

(i) $\Rightarrow$ (ii): by definition.

(ii) $\Rightarrow$ (iii)(b): if Q is not affine complete, then A is not 1-affine complete by Lemma 1.3, and by Theorem 1.1, $\mathrm{Dih}(A)$ is not 1-affine complete. Hence $\mathrm{Dih}(A)$ is not 2-affine complete.

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(ii) $\Rightarrow$ (iii)(a): if $|A| = 1$, then $|G| = 2$, hence G is not 2-affine complete. Next, we consider the case $A = P$. If $\exp A > 2$, then G is a non-abelian nilpotent group. For every normal subgroup N of G we either have $N \leq A$ or $N \geq \gamma_2(G)$. So by Proposition 4.1, G is not 2-affine complete. It remains to check the case $P < A$. If $\exp(P) \neq 2$, then by Lemma 1.3 and Lemma 4.6, G is not 2-affine complete.

(iii) $\Rightarrow$ (i): If $A = P$, then G is an elementary abelian 2-group, which by Lemma 1.3 is affine complete. It remains to check the case $P < A$. Then PC is a subgroup of G isomorphic to the elementary abelian group $\mathrm{Dih}(P)$ and G is a semi-direct product of Q with PC. Since G satisfies the conditions of Proposition 4.5 for every $k \in \mathbb{N}$, G is affine complete. ■

Example 4.7 The group $\mathrm{Dih}(\mathrm{C}_2 \times (\mathrm{C}_3)^2)$ is affine complete and solvable, but not nilpotent.

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Institut für Algebra
Johannes Kepler Universität Linz
4040 Linz
Austria

and

Computer- und Mediensicherheit
FH OÖ Campus Hagenberg
4232 Hagenberg
Austria
e-mail: juergen.ecker@fh-hagenberg.at