Canad. Math. Bull. Vol. **17** (1), 1974

ON A PROBLEM OF PURDY RELATED TO SPERNER SYSTEMS

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Purdy asked whether the following conjecture is true:

Conjecture. Let E be a set of $2n$ elements. If $S=\{S_1, S_2, \ldots, S_t\}$ is a Sperner system of E, i.e. $S_i \not\subset S_j$ for $i\neq j$, $i, j=1, 2, \ldots, t$; and if

$$S_i \cup S_j \neq E \qquad i, j = 1, 2, \ldots, t \tag{1}$$

then

$$t \leq \binom{2n}{n-1}.$$

The proof of the conjecture will be obtained using the following theorem of Katona (Acta Math. **15** (1964), 329–337):

THEOREM. *Let $A=\{A_1, A_2, \ldots, A_s\}$ be a set of subsets of M, $|M|=m$, such that $|A_i|=l$, $|A_i \cap A_j|\geq k$ for $i, j=1, 2, \ldots, s$.* Denote by A^g the set of sets B such that for some μ $B \subset A_\mu$ and $|B|=g$. Then, if $1\leq g\leq l$, $1\leq k\leq l$, $g+k\geq l$ *the following holds:*

$$|A^g| \geq s\binom{2l-k}{g}\Big/\binom{2l-k}{l} \tag{2}$$

Proof of the conjecture. First, by an argument [1] involving the decomposition L of the lattice of all subsets of E into mutually disjoint symmetrical chains, the members of S having more than $n-1$ elements may be replaced by the same number of n-sets, This may be done without contracting (1). Namely, if $B_1, B_2, \ldots, B_q$ are the considered members of S, then they are situated on different chains of L. Replace B_i by the member C_i of the chain, containing B_i, which is an n-set. Then, since $B_i \supseteq C_i$, the condition $B_i \cup B_j\neq E$ involves $C_i \cup C_j\neq E$, while if $F\in S$ and $|F|<n$ then $|F \cup C_i|<2n$ and hence $F \cup C_i\neq E$.

The complement of every C_i is also an n-set and therefore $q\leq\frac{1}{2}\binom{2n}{n}$. Since $\frac{1}{2}\binom{2n}{n}\leq\binom{2n}{n-1}$, the conjecture is proved in the case $S=\{B_1, B_2, \ldots, B_q\}$.

Let $D=\{D_1, D_2, \ldots, D_p\}$ be the set of members of S each containing fewer than n elements. Then, since $\binom{2n}{n-1}$ is the number of chains in L containing subsets

https://doi.org/10.4153/CMB-1974-029-1 Published online by Cambridge University Press

of the considered size and since no two members of S can occur in the same chain, $p \leq \binom{2n}{n-1}$. This proves the case $S=\{D_1, D_2, \ldots, D_p\}$.

For the general case we shall prove that

$$p \leq \binom{2n}{n-1} - q, \tag{3}$$

where the right side is nonnegative by the former inequality on q.

Inequality (3) follows from the fact that if F is a $(n-1)$-set and for some μ $F \subset C_\mu$, then the chain of L containing F, contains no members of D and there are at least q such sets F.

The last statement follows from the theorem, for $g=n-1$, namely, the sets $C_1, C_2, \ldots, C_q$ satisfy its assumptions with $k=1$, $l=n$, $s=q$. Therefore the number of different sets F is by (2), at least q.

This completes the proof, since in all cases

$$t = q+p \leq q+\binom{2n}{n-1} - q = \binom{2n}{n-1}.$$

REMARK. By the same method the assertion of the conjecture can be proved under condition $S_i \cap S_j \neq \varnothing$ instead of (1). This is a result related to Theorem 1 in [2].

ACKNOWLEDGEMENT. Thanks are due to Professor P. Erdös for calling my attention to Purdy's problem and to the Remark.

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https://doi.org/10.4153/CMB-1974-029-1 Published online by Cambridge University Press