LEIBNIZ'S MEREOLOGY: A LOGICAL RECONSTRUCTION

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Abstract. The aim of this paper is to give a full exposition of Leibniz's mereological system. My starting point will be his papers on Real Addition, and the distinction between the containment and the part-whole relation. In the first part (§2), I expound the Real Addition calculus; in the second part (§3), I introduce the mereological calculus by restricting the containment relation via the notion of homogeneity which results in the parthood relation (this corresponds to an extension of the Real Addition calculus via what I call the Homogeneity axiom). I analyze in detail such a notion, and argue that it implies a gunk conception of (proper) part. Finally, in the third part (§4), I scrutinize some of the applications of the containment-parthood distinction showing that a number of famous Leibnizian doctrines depend on it.

§1. Introduction. The main claim of this paper is that at the heart of Leibniz's philosophy lies the distinction between two closed but different relations: containment (sometimes called the *ingredienthood* or the *inesse* relation) and parthood. The first aim of this paper is to study these two relations from a logical point of view, looking at the commonalities and differences of the two *calculi* to which they give rise. The second aim is to explain the importance of not confusing the two relations. It will be shown that the distinction between containment and parthood sheds light on a number of notoriously difficult Leibnizian claims.

Put differently, this paper consists in the study of Leibniz's mereology. The main difference from contemporary mereological theories[1] lies in the fact that Leibniz defines the part-whole relation as a special case of the containment relation. As a consequence, a proper study of Leibniz's mereology must start with the consideration of this more general relation.

Leibniz studied the logic of the containment relation in different writings. In his early logical writings (dated around 1679) he took the containment relation as primitive and developed a logical calculus on this basis.[2] Later on, for instance in the *Generales Inquisitiones*, he took identity (and composition) as primitive, and defined containment on the basis of them.[3] However, our main focus will be a number of essays dated around 1686 and 1687 concerning a calculus known as the Real Addition (or the Plus-Minus) calculus, where containment is introduced on the basis of identity and Real Addition. One might wonder why our attention is being limited to these essays, given that the containment calculus is present in many other writings. The reason is simple: while

Received: July 27, 2024.
2020 *Mathematics Subject Classification*: Primary 01A45, Secondary 03A05.
Key words and phrases: Leibniz mereology, real addition, theory of quantity, monadology, infinity.

1 For a general introduction to contemporary mereology see [17, 23, 51, 54].
2 See [43] for a deep study of Leibniz's logic as based on the containment relation.
3 'A contains B' is defined via identity and composition as 'A is identical to AB', where AB is the composition of A and B.

doi:10.1017/S1755020325000012

the early logical writings and those connected with the *Generales Inquisitiones* strictly concern logic, in the precise sense of the relationships between terms and propositions,[4] in the writings on Real Addition the domain of application of the calculus is not only the domain of terms and concepts, but is made of things in general.[5] Real Addition is an operation that applies to any kind of thing, no matter its nature. For this reason, [44] interprets the calculus developed in these essays as a mereology.

Despite agreeing with [44] that the containment calculus is closer to what we call mereology rather than logic, I won't use the term 'mereology' to refer to it. Since the containment relation is not the part-whole relation, I have decided to reserve the term 'mereology' for the theory governing the latter relation, not the former. This is only a terminological difference with [44]. I shall thus speak of Real Addition calculus to mean the calculus whose main relation is containment, while I shall use 'mereology' to denote the calculus of which the part-whole is the main relation.[6]

The Real Addition calculus has received quite a lot of attention in the literature[7], especially from Wolfgang Lenzen. Therefore my aim here will not be that of providing a full reconstruction of it, for which I refer the reader to the secondary literature. Rather, I shall reconstruct the calculus having in mind contemporary mereology, showing which mereological principles are valid in it. This is pivotal to the main aim of this paper, which is the study of Leibniz's mereology. This theory will be introduced as an extension—via a suitable axiom—of the Real Addition calculus. It must be noted that while Leibniz studied the Real Addition calculus in depth, he never bothered to develop the mereological calculus. What I intend to do here is to collect the different hints he gave us on the parthood relation to try to understand which logical properties his notion of parthood enjoys.

The paper is divided into three parts. In the first part (§2), I shall introduce Leibniz's calculus of Real Addition—the system RA—and study it from the perspective of contemporary mereology. I shall argue that the calculus is very close to Classical Extensional Mereology (CEM, sometimes called General Extensional Mereology, GEM), with a notable difference: while in CEM composition is unrestricted, the Real Addition operation is a binary operation. However, I shall also present an extension of

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[4] One of the aims of the logical essays is to develop a logic of terms where every kind of proposition is expressed in categorical form. Here the containment relation is thought of as a relation between terms, and the concepts they express.

[5] This is stressed by [44, p. 50] (in [19]) who notes that "usually, Leibniz employs the Latin neuter, singular or plural, mainly of pronouns to denote the 'things' that play the role of a 'part' or 'whole'". However, interpreters have usually translated such neutral pronoun with the word 'term'.

[6] One of the first occurrences of the idea that parts are homogeneous to the whole is the text *De Magnitudine* of 1676: "A whole is that to whose nature several homogeneous things can be understood to belong, which are called parts" (A VI 3, 483, my translation). At this time, however, this is only one definition among many, as a number of texts testify (see e.g., A VI 4, 197–200), in which Leibniz does not mention homogeneity in the definition of parthood. Only starting from the papers on Real Addition does Leibniz set his mind to the definition of parthood via *inesse* and homogeneity. In some cases, he points out that he is not always rigorous in characterizing parthood: "A whole is that of which many things constituting it come together, which are said parts. In a more rigorous way, we take a whole to be that which parts are homogeneous". ([29, p. 476] - circa 1700-1702). One aim of this paper is to make clear the importance of being rigorous about this distinction.

[7] See [39, 44, 48, 52].

the calculus, labeled RA^{∞}, where Real Addition is treated as an infinitary operation. I shall argue that such extension is implicitly used by Leibniz in different applications of the containment relation. The second part (§3) is dedicated to Leibniz's own notion of the part-whole relation as a restriction of the containment relation through the notion of homogeneity, and discusses the latter at length. This restriction will be formally captured by extending the system RA with an axiom in order to obtain the system M. We then obtain Leibniz's mereology—the system LM—by adding a further axiom that relates the notion of quantity to that of part. I shall scrutinize the formal feature of the notion of part showing that it is a gunk conception of parthood. Finally, in the third and more philosophical part (§4), I deal with the applications of the containment-parthood distinction. Here we shall see how fruitful this distinction is to understand a number of typical Leibnizian philosophical theses.

§2. Leibniz's real addition calculus - RA.

2.1. Identity in Leibniz's calculus. As a basic logical system we shall assume First-Order Logic (FOL) with identity (=). Identity is regulated by the two following standard axioms:

a) $\forall x(x = x)$

b) $\forall x \forall y(x = y \rightarrow (\alpha \rightarrow \alpha'))$ where α' is like α except that some free occurrences of x have been replaced by y.

The relation of identity plays a pivotal role in Leibniz's logical calculi. Definition 1 of both *Non inelegans specimen demonstrandi in abstractis* ([30]: A VI 4a, 845-855) and *Specimen calculus coincidentium et inexistentium* (A VI 4a, 830-845: from now on I shall call them respectively NI and CC) characterize the identity relation by means of a substitution rule: "Identical or coincident are those things of which one can be substituted everywhere for the other preserving truth [*salva veritate*]". Let us call this rule SSV (substitution salva veritate). By means of SSV, Leibniz derives symmetry and transitivity of identity. Suppose A=B. Then by SSV we can make a double substitution: we can replace A with B, and B with A, obtaining B=A (identity is symmetric).[8] If we suppose that A=B and B=C, the former allows us to replace in the latter B with A to obtain A=C (identity is transitive). Reflexivity is not proved in these papers by Leibniz, but in the *Generales Inquisitiones*, paragraph 10, 'A=A' is said to be a proposition true in itself (*propositio per se vera*). Moreover, he even suggests that if for every term A there is a formula such as A=B, then the same formula allows us to substitute B with A in itself to obtain A=A. That for each A there is a formula such as A=B is indeed guaranteed by the Idempotence axiom (see next paragraph).[9]

If we now compare Leibniz's approach to identity with the standard first-order theory of identity, we immediately notice that they are logically equivalent.[10] Point a) above amounts to the reflexivity of the identity relation; point b) expresses the

8 As [42, §2] claims, this proof shows that Leibniz allows for 'reflexive substitutions', i.e., substitutions in the very same identity-sentence that allows the substitution.

9 On reflexivity see [42, p. 698].

10 Logical equivalence does not imply metaphysical equivalence. [46, chap. 4] stresses the fact that while the contemporary version of the principle of indiscernibility works with abstract predicates, for Leibniz abstract predicates are only a fiction, since "the indiscernibility of substances requires to consider the individual accidents that form the complete concept of it."

https://doi.org/10.1017/S1755020325000012 Published online by Cambridge University Press

indiscernibility of identical things: if a is identical to b, then everything true of a is also true of b, and so they can be substituted for each other *salva veritate*.[11] It is the fact that a and b are identical that allows us the substitution while preserving truth.[12]

2.2. Real addition and the containment relation. Other than identity, the primitive notion that Leibniz uses is real addition, for which he employs the symbol '$\oplus$' (to keep the notion distinct from the arithmetical addition).[13] The notion of real addition is similar to the notion of mereological sum or fusion of contemporary mereology: the idea is that we can add or fuse different things and so obtain aggregates of those objects. There are two explicit axioms that regulate how real addition works, plus associativity, which is never made explicit, but often used:

1) $\forall x(x \oplus x = x)$
2) $\forall x \forall y(x \oplus y = y \oplus x)$
3) $\forall x \forall y \forall z((x \oplus y) \oplus z = x \oplus (y \oplus z))$.

Axiom 1 states the idempotence of real addition (which is of course a property not shared by arithmetical addition); axiom 2 expresses commutativity; axiom 3 associativity. It is important to notice that such properties are—what we might call—'structure-blind': idempotence means that the repetition of a certain element does not matter; commutativity means that the order does not matter; associativity means that the order in which we might group the elements does not matter: all these aspects—repetition, order, and grouping—are structural aspects of composite things.[14]

Postulate 1 of NI and Postulate 2 of CC deal with Real Addition:

> Postulate 1: Any plurality of things whatever [*Plura quaecunque*] can be added to constitute a single thing; as for example, if we have A and B, we can write $A + B$, and call it L.
> Postulate 2: Any plurality of things, as A and B, can be added to compose a single thing, $A \oplus B$ or L.

[11] Leibniz explicitly restricts SSV to non-opaque contexts. See [26] for a concise discussion of such a topic.

[12] I would like to stress that we are here dealing with the principle called 'indiscernibility of identity', and not with the 'identity of indiscernibles'. This latter principle was endorsed by Leibniz only with regard to concrete objects, and not in relation to abstract mathematical entities. On this point see [46, chap. 4], the paragraph entitled 'Identité logique et identité métaphysique'.

[13] In NI Leibniz uses the simple sign '+'; '$\oplus$' is used in CC. In the main text, I shall always use the latter for uniformity. However, in the quotations from NI, I shall leave the symbol +. [37] explains in detail how Leibniz developed the Real Addition Calculus by gradually transferring the rules of arithmetic to the realm of 'entities'. For example, arithmetical equality between numbers becomes identity between objects; addition of magnitudes becomes real addition of objects, etc. I cannot enter into a full discussion of this topic here for matter of space, so I refer the reader to Lenzen's clear and insightful paper.

[14] The idea that real addition and, more generally, mereological notions are structure-blind can be found in the following passage: "In this place, however, we have nothing to do with the theory of the variations which consist simply in changes of order [i.e., the theory of permutations], and AB is for us the same as BA. And also we here take no account of repetition—that is AA is for us the same as A. Thus wherever these laws just mentioned can be used, the present calculus can be applied." (A VI 4a, 834, English translation from [40, p. 386]). Notice that here Leibniz simply juxtaposes the terms to denote real addition.

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Leibniz is here endorsing a principle of finite composition:

Finite Composition Principle (FCP) $\forall x \forall y \exists z (x \oplus y = z)$.

This idea is specified in the following terms:

> I reply that our general construction depends upon the second postulate, in which is contained the assumption that any thing and any other thing can be put together as components. Thus God, soul, body, point, and heat compose an aggregate of these five things. And in this fashion also quadrilateral and trilateral can be put together as components. [...] But if anyone wished to apply this general calculus of compositions of whatever sort to a special mode of composition; for example if one wished to unite "trilateral" and "circle" and "quadrilateral" not only to compose an aggregate but so that each of these concepts shall belong to the same subject, then it is necessary to observe whether they are compatible. Thus immovable straight lines at a distance from one another can be added to compose an aggregate but not to compose a continuum. (CC, comment between Propositions 22 and 23. Translation from [40], slightly modified).

The passage is important because it stresses that we can sum up together whatever thing we want. From an ontological point of view, however, the aggregate so obtained is not something more than its constituents (i.e., it does not require any further ontological commitment). Leibniz's conception of aggregates is clearly nominalist: what FCP allows us to do is to introduce a singular term denoting an aggregate, and to use this term in sentences of our theories. However, this is not sufficient to prove that the referent of that term actually exists. As a matter of fact, what really exists for Leibniz are only simple substances, which are not aggregates.[15] In other words, the 'definition' of aggregates that the Real Addition Calculus delivers is only nominal, not real. Talking of wholes and parts (but the same clearly applies to aggregates), Leibniz writes that "[...] the parts, considered together with their position, differ only in name from the whole, and the name 'whole' is used in reasonings [*in rationes ponatur*], as a shorthand term for the parts themselves" ([27]: GM VII 274).[16]

It is by means of real addition (and identity) that Leibniz defines the containment relation (in what follows Cxy must be read as x contains y, or y is contained in x). Definition 3 of CC says:

> A is in L, or L contains A, is the same as to say that L can be made to coincide with a plurality of things, taken together, of which A is one [*pro pluribus inter quae est A*] (A VI 4a 832, translation slightly modified from [40, p. 386])

In general, Leibniz is here providing the following definition for the containment relation:

15 There is a sense in which also bodies - as aggregates of substances - exist, but their being results from that of substances in them. On this see §4.1 below.

16 For the ontological and metaphysical interpretation of such notions see [13].

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(**Definition of Containment**) $Cxy \equiv \exists z(y \oplus z = x)$

Before illustrating the properties of such a relation, it is worth taking a look at two theorems (respectively Propositions 13 and 14 of CC) that provide us with a specification of this definition.[17]

Proposition 13: If $x \oplus y = x$, then Cxy.

Proof Suppose $x \oplus y = x$; but $C((x \oplus y), y)$—namely $x \oplus y$ contains y—by definition of the C-relation. By SSV licensed by the supposition, we have Cxy. □

Proposition 14: If Cxy, then $x \oplus y = x$.

Proof Suppose Cxy; then by definition of the C-relation, $\exists t(y \oplus t = x)$. Call this t, p. So we have $y \oplus p = x$. If we now add to both sides of the equation y[18], we have $y \oplus p \oplus y = x \oplus y$. By Commutativity and Idempotence on the left-side of the identity sign, we obtain $y \oplus p = x \oplus y$. By SSV, and symmetry of identity, we conclude $x \oplus y = x$. □

The two theorems imply that $Cxy \equiv (x \oplus y = x)$. This provides us with a more specific definition of the C-relation and allows Leibniz to substitute any occurrence of Cxy with $x \oplus y = x$ and vice versa.

We can now turn to the properties of the C-relation. These are easily derivable from the properties of real addition:

Reflexivity: Cxx

Proof by axiom 1 (idempotence) $x \oplus x = x$. So we have that $\exists z(x \oplus z = x)$. By definition of C-relation, Cxx. □

Transitivity: If Cxy and Cyz, then Cxz

Proof Suppose Cxy and Cyz. From the former, we have $\exists w(y \oplus w = x)$. Call such a w, p. We thus have $(y \oplus p = x)$. From the latter, $\exists w(z \oplus w = y)$. Call such a w, s. So we have $(z \oplus s = y)$. By SSV, we can replace y in the first sentence with $z \oplus s$, so we obtain $z \oplus s \oplus p = x$. This means that $\exists w(z \oplus w = x)$ is true, and so Cxz. □

Antisymmetry: If Cxy and Cyx, then $x = y$

Proof suppose Cxy and Cyx. From the former, $\exists w(y \oplus w = x)$, so $(y \oplus p = x)$. By SSV we can substitute x in Cyx with $y \oplus p$ to obtain $C(y, (y \oplus p))$. But this means that Cyp. By Proposition 14, $y \oplus p = y$. By SSV the left-side of the equation can be replaced with x, and we obtain $x = y$. □

To sum up, the containment relation is reflexive, transitive, and anti-symmetric; it is thus a partial order (as stressed by both [43] and [44]). Notice that these are exactly the features that the notion of (improper) part has in classical extensional mereology.

Before proceeding there are still two definitions that are worth mentioning here. These are the definitions of *communicating things* and *non-communicating things*, which

[17] In various proofs below we follow Leibniz's use of the SSV, instead of applying the axioms concerning identity presented at the beginning of §2 as our formal setting would require. In any case, the two procedures are equivalent, as stressed above. We decided to proceed in this way for clarity and simplicity.

[18] That we can add the same elements to both sides of an equation is proved in Proposition 9 of CC.

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are Leibniz's way of defining the relations of overlapping and disjointness[19]:

Overlapping $x \circ y \equiv \exists w(Cxw \wedge Cyw)$

Disjointness $D(x, y) \equiv \neg(x \circ y)$

Two things overlap when they have something in common; they are disjoint if there is nothing in common between them.[20]

2.3. The subtraction operation. In the essays *Specimen Calculi Coincidentium* (A VI 4a, 816-822)[21] and in NI, Leibniz also develops a subtraction operation, clearly presented as the inverse of the operation of Real Addition. In *Specimen Calculi Coincidentium*, Leibniz writes

> We have used the sign '+' to designate a unique aggregate [*unum collectivum*] obtained from many things, in which there exist many things [*in quo plura sint*], and which coincides with them when they are taken together [*et quod ipsis simul sumtis coincidat*]. Now, the sign '–' will be used to designate that something is subtracted from something else, such that it is the inverse [*contrarium*] of the sign '+'. Thus if $A + B = C$, then we have $A = C - B$, and we call A the remainder [*Residuum*]. But it is necessary that A and B do not have anything in common. Otherwise we could have $A + A = A$, and from that $A - A = A$. But $A - A = Nihil$ from which it would follow that $A = Nihil$. (A VI 4a, 819; the translation is mine)

There are three conditions that must be met in order for $E - B = A$ to be a well defined operation. First, almost trivially, the result of the operation (i.e., the object A) must be unique; second, $A \oplus B = E$ must be the case, which means that E contains B; third, A (i.e., $E - B$) and B must be disjoint (if they were not disjoint, we would not have subtracted the whole of B from E). The principle that emerges is that something is in A (the result of the subtraction $E - B$) if and only if it is in D and it is disjoint from B:

$$[A = E - B] \equiv C(E, B) \rightarrow \forall w((C(A, w) \leftrightarrow (C(E, w) \wedge D(w, B)))$$

The consequent of the conditional captures the uniqueness of A; moreover it implies the disjointness of A and B as well. Finally, the antecedent formalizes the requirement that $A \oplus B = E$ is the case. We can generalize this as follows:

Subtraction 1: $\forall x \forall y \exists z[(x - y = z) \equiv C(x, y) \rightarrow \forall w(C(z, w) \leftrightarrow (C(x, w) \wedge D(w, y)))]$

[19] Definition 4 of NI says: "If something M is contained in A and at the same time in B, it is said to be common to them, and A and B are said to be communicating. If they do not have anything in common [...], they are said to be non-communicating". (A VI 4a 847, my translation).

[20] Lenzen [39] restricts these definitions to the cases where the common element is non-empty. This is enough to stop the derivation of a number of problems that we are going to face later on concerning the presence of the term '*Nihil*' in the calculus. We shall deal with this issue in due course.

[21] This essay must not be confused with *Specimen calculus coincidentium et inexistentium*, which is the one we are referring to as CC.

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Subtraction 1 works under the condition that $x - y$ is defined only if Cxy is the case. This corresponds to what [39] calls the conservative approach.[22] However, as Lenzen points out, in other passages Leibniz considers a more liberal approach, where subtraction is defined for arbitrary objects.[23] It may be useful to give a definition of subtraction by cases:

Subtraction 2:

$$\forall x \forall y \exists z[(x - y = z) \equiv \begin{cases} \forall w(C(z,w) \leftrightarrow (C(x,w) \wedge D(w,y)))], & \neg Cyx \\ z = Nihil], & Cyx \end{cases}$$

Notice that the second case covers the possibility that something is subtracted from itself: an expression as '$x - x$' is well-defined. Leibniz captures this idea by introducing in the calculus the term '*Nihil*' via the following axiom:

Axiom Nihil: $\forall x(x - x = Nihil)$

Axiom Nihil is only a special case of Subtraction 2, which covers also the case in which x is completely in y and y is subtracted from x.

2.3.1. The problem of the term 'Nihil'. The admission of the term *'Nihil'* is very problematic, because it naturally leads to contradiction. Suppose that the term refers to an empty object, let us say $\emptyset$. Then in the Real Addition calculus, we have that $x \oplus Nihil = x$. By definition of C-relation, this is equivalent to $\forall x C(x, Nihil)$, i.e., $\emptyset$ is contained in every object. This means that $\emptyset$ overlaps every object: $\forall x(x \circ Nihil)$. In other words, given any two arbitrary objects they always have something in common. But this implies that there are no disjoint objects, contradicting Subtraction 2 (indeed, disjoint objects are required if we want a subtraction operation).

There are different ways in which this contradiction can be avoided. A first proposal made by Lenzen in [39] consists in restricting the definitions of overlapping and disjointness in order to exclude the term *'Nihil'*. For example, we may define two objects A and B to be disjoint if there is nothing in common between them apart from *Nihil*. This amounts to substituting the two previous definitions with the following:

Overlapping $x \circ^{-} y \equiv \exists w(Cxw \wedge Cyw) \wedge \neg(w = Nihil)$

Disjointness $D(x,y) \equiv \neg(x \circ^{-} y)$

In this way, we may claim that there are disjoint objects even if *Nihil* is contained in everything. This is a perfectly reasonable move, but there is no sign in Leibniz's work that he ever envisioned a similar strategy. A second possibility, suggested in [11], is to interpret the term '*Nihil*' as an empty term which, despite not having a reference, can be the subject of true propositions. This seems to capture well talk of *Nihil* or nothingness in a number of different contexts within Leibniz's production (see [11, §§7 and 8]). Formally this requires a Positive Free Logic that restricts the quantifier's rule to terms that denote. In this way, from the claim that *Nihil* is contained in everything, we cannot derive that there is something ($\exists x$) that is contained in everything, which is enough to block the derivation of the contradiction. One may object that the appeal to a Free

[22] This approach has been investigated at length by [20].
[23] For example, theorem IX, point 1 of NI applies subtraction between disjoint objects.

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Logic is very anachronistic, but the objection would be misplaced. Free logic would only be used as a contemporary tool to exactly describe how empty terms relate to other terms and quantifiers. From an historical point of view, what matters is only the possibility that Leibniz may have conceived of '*Nihil*' as a non-denoting term. Finally, there is a third way out suggested by Leibniz himself. In A VI 4a 817, points 15 and 16 Leibniz claims that if $A \oplus B = A$, then B is contained in A only if B is something, i.e., $\neg(B = Nihil)$. The contradiction here is avoided because *Nihil* is contained in no object whatever. While the implicit claim that *Nihil* is not something may suggest that the term must be taken as non-denoting, the idea expressed by this passage amounts to restricting the definition of containment: $Cxy \equiv \exists z(y \oplus z = x) \wedge \neg(y = Nihil)$. This implies that *Nihil* is contained neither in other objects nor in itself. This view is probably influenced by the the Scholastic motto '*nihili nulla esse attributa*' that Leibniz often repeats (e.g., in A VI 4 570). For our purposes here, we will adopt this last proposal. The reason is that not only does it allow us to say that there is no x contained in everything (as happens with the first two proposals), but it also implies that *Nihil* is never contained in something else. This will be important when moving to the mereological calculus in §3.2.[24]

2.3.2. Is subtraction the inverse of real addition? When Leibniz introduces the subtraction operation (in the passage quoted above), he is guided by the idea that, like in the case of arithmetic, Real Subtraction is the inverse of Real Addition. But he soon realizes that the axiom $(A \oplus B) - B = A$, which holds without any restriction in arithmetic, must be restricted to disjoint entities in the case of Real Addition:

> Now we use the sign '–' to designate something which must be detracted from something else, namely the inverse of what happens with the sign '+'. And in this way if $A + B = C$, it will be that $A = C - B$, and A is called the remainder. But it is necessary that A and B have nothing in common. Indeed for example $A + A = A$, so it

24 Subtraction 2 does not allow for privative terms, i.e., terms that denote something less than nothing. In some passages, Leibniz suggests that an expression such as $Nihil - B$, where B denotes a positive entity, would denote something less than nothing (A VI 4 851: "Hinc detractiones possunt facere nihilum seu non-Ens simplex, imo minus nihilo"). This is not only awkward for a calculus whose main applications are things in general, but if one does not impose any restriction on *Nihil* as those described in this paragraph, one would face a deep problem: if we consider an expression such as $(Nihil - B) + B$, then this would be equal to $-B + B$, i.e., to *Nihil*. So we would have $(Nihil - B) + B = Nihil$. By definition of the containment relation this implies that B is contained in *Nihil*: $C(Nihil, B)$. Since *Nihil* is contained in everything, and so in B, by antisymmetry, we can conclude that $B = Nihil$. B was arbitrary, so everything is *Nihil*. How do the three proposals deal with this difficulty? In [39], Lenzen suggests that the best way to avoid this consequence is to impose that $Nihil - B = Nihil$. The same result can be obtained within a Positive Free Logic (second proposal) where *Nihil* is an empty term. From the fact that *Nihil* is contained in B, it does not follow that *there exists* something in common between *Nihil* and B. So the two terms are disjoint, which means that when we subtract B from *Nihil*, we get *Nihil*. The difficulty is blocked but privative terms are not accepted. Finally, the third proposal rejects the problematic idea that *Nihil* is contained in any other objects. Not only is the difficulty blocked, but this proposal seems compatible with privative terms. However, we won't develop this topic of privative terms further since this would require a long digression from our main interests. On this delicate issue, I refer the reader to [38].

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will be that $A = A - A$. But we know that $A - A = Nihil$, so we will have $A = Nihil$ against the hypothesis. (A VI 4a, 819)[25]

2.4. Extensionality and the RA system. One important consequence of the axioms stated so far is that the calculus is extensional, in the sense that we are in a position to prove the Extensionality Principle according to which two things are the same if and only if they contain the same 'ingredients':

EXT $\forall z(Cxz \leftrightarrow Cyz) \leftrightarrow x = y$

Proof From right to left is trivial (assume $x = y$, then substitute y with x in the left-hand side proposition). From left to right: assume $\forall z(Cxz \leftrightarrow Cyz)$. This means 1. $\forall z(Cxz \rightarrow Cyz)$ and 2. $\forall z(Cyz \rightarrow Cxz)$. By Reflexivity of containment we have both Cxx and Cyy. From the former and 1 (with z instantiated with x) we obtain by *modus ponens* Cyx. From the latter and 2 (with z instantiated with y) we obtain by *modus ponens* Cxy. By Antisymmetry of containment we conclude $x = y$. □

To sum up what we have seen until now, Leibniz's Real Addition calculus is based on the real addition operation (regimented by FCP and axioms 1, 2, and 3), from which we can define the C-relation and prove that C is reflexive, transitive, and antisymmetric. Moreover, Subtraction 2 holds. From this base EXT follows. I refer the reader to [39] for a full list of the theorems derivable from this calculus.[26] As a consequence we have the following axioms (based on FOL plus identity, with $\oplus$ as the only primitive term).
1) $\forall x(x \oplus x = x)$
2) $\forall x \forall y(x \oplus y = y \oplus x)$
3) $\forall x \forall y \forall z((x \oplus y) \oplus z = x \oplus (y \oplus z))$.
Finite Composition Principle: $\forall x \forall y \exists z(x \oplus y = z)$.
Subtraction 2:

$$\forall x \forall y \exists z[(x - y = z) \equiv \begin{cases} \forall w(C(z,w) \leftrightarrow (C(x,w) \wedge D(w,y)))], & \neg Cyx \\ z = Nihil], & Cyx \end{cases}$$

Clearly, containment is defined by excluding the term *Nihil* from its range of application:

$$Cxy \equiv \exists z(y \oplus z = x) \wedge \neg(y = Nihil)$$

The RA calculus is thus weaker than Classical Extensional Mereology (CEM) since the composition operation (real addition) does not apply to infinitely many things, contrary to what happens in CEM. However, we will argue that the application of the notion of aggregate and the containment relation to metaphysics require an infinitary composition operation. In the next paragraph we explain how to extend the RA calculus in order to accommodate an infinitary composition operation. We then start defending the idea that this is a legitimate interpretation to take. This defense has two stages: a negative one, where we debunk possible objections against this view, and a positive one (which is postponed to the third part of this paper) where we show that

[25] On this point see the discussion in [39, p. 102].
[26] Notice that the two famous decomposition principles known as Weak and Strong Supplementation, respectively $\forall x \forall y(Cxy \rightarrow \exists z(Cxz \wedge Dzy))$ and $\forall x \forall y(\neg Cyx \rightarrow \exists z(Cxz \wedge Dzy))$, are valid as well since both are weaker than Subtraction 2 and can easily be derived from it.

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there are plenty of applications of the distinction between containment and parthood that require an infinitary real addition operation.

2.5. Making real addition an infinitary operation. We are now going to extend RA into an infinitary calculus that we shall label RA^{∞}. Its axioms are exactly those of RA once FCP has been replaced with an unrestricted sum operation that can be applied to any collection or plurality of objects, no matter how many they are (finitely or infinitely many). We shall write $\Sigma_{\phi}z$ to denote that z is the sum (or aggregate) of everything satisfying the condition ϕ.

Definition: $\Sigma_{\phi}z \equiv \forall x(\phi(x) \rightarrow Czx) \wedge \forall y(\forall x(\phi(x) \rightarrow Cyx) \rightarrow Cyz)$

The first conjunct guarantees that everything satisfying ϕ is contained in the sum (so the sum is an upper bound for the containment relation); the second conjunct guarantees that the sum is a minimal upper bound: if everything satisfying ϕ is contained in y, then their sum z is also contained in y. Equipped with this definition, we can state a principle of unrestricted composition (UCP), or unrestricted sum:

UCP: $\exists x\phi(x) \rightarrow \exists z(\Sigma_{\phi}z)$

The uniqueness of the sum operation follows straightforwardly from the antisymmetry of containment (see [17, p. 29]).[27]

This formalization makes real addition an infinitary operation: when there are infinitely many things satisfying ϕ, UCP delivers us an infinite aggregate.

We will see the importance of this infinitary interpretation of real addition in §4, dealing with the applications in metaphysics of these notions. For the time being I limit myself to a 'negative' defense of the possibility of interpreting real addition as an infinitary operation by replying to three objections.

Objection 1: The infinitary interpretation of real addition presupposes the notion of actual infinite. We can sum together only things that are available; if there are infinitely many things satisfying ϕ, then this infinite must be actual, and not merely potential. But Leibniz, along with all (or most) philosophers and mathematicians before Cantor, only admitted the notion of potential infinite, namely the idea of procedures and operations that can be iterated indefinitely.

Reply 1: It is simply false that the only sense of infinity that Leibniz admitted is the potential. Nowadays all Leibnizian scholars agree that Leibniz conceived of matter as being *actually* divided into infinitely many parts. This is an actual infinite since each part of matter is fully determined. More generally Leibniz holds that what actually exists is completely determined, and since there are infinitely many substances, the created world constitutes an actual infinite.[28] In addition, recently some scholars have advanced the idea that Leibniz accepted the actual infinity in mathematics as well.[29] Some texts are very explicit on this point:

27 Notice that the containment relation that appears in the definition of the infinitary real addition cannot be the same containment relation that we have defined via (finitary) real addition. To avoid any circularity, we take here containment as primitive and we assume that it is a partial order.

28 More on Leibniz's conception of the created world in §4.3.

29 This has been defended by [3, 5, 45].

https://doi.org/10.1017/S1755020325000012 Published online by Cambridge University Press

> There is an actual infinite in the mode of a distributive whole, not in that of a collective whole. Thus something can be enunciated concerning all numbers [de omnibus numeris], but not collectively. Thus it can be said that to every even number there is a corresponding odd number, and vice versa; but it cannot accurately be said that the multiplicity of even numbers is therefore equal to that of odd ones. ([28]: GP II 315)

In this text, Leibniz considers the integers as constituting an actual infinite.[30] However, his idea is that they do not form a whole or a collection. In other words, they do not constitute a categorematic infinite, i.e., an infinite object. For this reason, generality over actually infinitely many things must be understood distributively, not collectively. For my present purpose, it is not important to take a stand on whether Leibniz admitted the actual infinite in mathematics or not.[31] What really matters is that Leibniz, at least in some contexts, advocates a notion of actual infinite clearly not reducible to the potential one.

Objection 2: Leibniz famously rejected infinite numbers (see [10]). This rejection amounts to the rejection of infinite wholes and magnitudes. These cannot exist because their existences contradict the part-whole axiom according to which the whole is always bigger than its parts. There is thus no categorematic infinite for Leibniz; but UCP delivers us exactly categorematic infinites, i.e., aggregates with infinitely many elements. Given Leibniz's rejection of infinite wholes, he could not have accepted an infinitary composition operation.

Reply 2: Let's see why this argument fails. The problem is that it exploits the part-whole relation and the part-whole principle, which are not yet available in the RA and the RA^∞ calculi. Parthood is defined via containment, and so it requires real addition. But it also requires the notion of homogeneity (see next section), which is nowhere to be found in these calculi. Within RA (or RA^∞) we cannot appeal to properties of the parthood relation, since parthood cannot even be defined. Therefore, we cannot appeal to the fact that an infinite aggregate is not a whole to establish the properties of real addition. Moreover, exactly because infinite aggregates of RA^∞ obtained via UCP are not (in general) wholes with parts, they have no magnitude (what has magnitude must be a whole with parts—more on this in §3). Therefore, the rejection of infinite magnitudes is compatible with admitting infinite aggregates.

One may think that allowing only a distributive form of generality with regard to infinitely many things (as Leibniz says in the quotation above) is incompatible with UCP, since the latter, delivering us infinite aggregates, allows for a form of collective generality. But notice that it is Leibniz himself who often speaks collectively of infinitely many things (for example, when he talks of bodies as infinite aggregates). However, this does not have to be interpreted as an incoherence on Leibniz's part. Rather, the idea mentioned above that aggregates "differ only in name" from their constituents (and are thus only shorthand expressions for them) implies that we can talk collectively of infinitely many things as long as we know that this is only a fiction. As long as we stick

[30] For a discussion of this text see [5, 45]. In the latter, Rabouin makes the important point that the category of potential infinite is not Leibnizian and that Leibniz prefers to speak of indefinite or syncategorematic infinite. This clearly raises doubts about attributing to Leibniz the acceptance of the potential infinite.

[31] [7], [1], and [53] argue against this view.

to this nominalistic interpretation of aggregates, there is no danger in fictionally talking collectively of what, more rigorously, allows only a distributive form of generality.

Objection 3: Leibniz is very clear that not even God can perform an infinite operation. The complete concept of Julius Caesar contains all the infinitely many attributes of Caesar, including the fact that he crossed the Rubicon. Still this fact is contingent and, as is well-known, Leibniz explained its contingency with the claim that neither us nor God can derive it from the complete concept. A supposed derivation of it would contain infinitely many steps, but neither us nor God can perform an operation of this kind. God knows that Caeser crossed the Rubicon because he immediately sees all of its attributes. This view sheds doubt on the possibility of attributing to Leibniz any infinitary operation. Similar considerations can be raised starting from the infinitesimal calculus, where infinite sums (in convergent series) are interpreted by Leibniz as abbreviations for the fact that the partial finite sums of the series converge to a certain value.

Reply 3: This objection is based on a misunderstanding. The claim that Leibniz may have admitted an infinitary real addition operation does not mean that somebody (us or God) can literally operate with infinitely many terms. For example, it does not imply that someone can derive a contingent truth from a complete concept or that an infinite sum in the calculus is something more than its corresponding partial finite sums. The notion of sum at play here is simply different from the notion of sum in arithmetics or in the calculus. An infinitary real addition operation only allows us to consider infinite aggregates with some of their ingredients, and the only operations that we are allowed to perform are those permitted by the calculus (e.g., if A is in B—with B an infinite aggregate—we may consider the complement of A in B, i.e., $B - A$). Given the nominalistic reading of aggregates that I am proposing, this simply means that we are allowed to introduce a single term to refer to infinitely many things at once, as Leibniz often did.

§3. Leibniz's mereology: the system LM. In this section we introduce and study Leibniz's mereological system—the system LM—by extending the system RA with two axioms. Leibniz never presented a formalization of such a system, while limiting himself to defining parthood via containment.[32] What I am trying to do in this section is to understand the logical properties of the parthood relation once it is introduced in the way Leibniz introduced it. In §4 we shall see how useful the distinction containment-parthood is in a number of different contexts.

[32] The reason for this is that he probably considered the mereological calculus as a particular interpretation of the Real Addition calculus. Indeed, he writes: "As the speciosa generalis is merely the representation and treatment of combinations by signs, and as various laws of combination can be discovered, the result of this is that various methods of computation arise. Here, however, no account is taken of the variation which consists in a change of order alone, and AB is the same for us as BA. Next, no account is taken here of repetition; i.e., AA is the same for us as A. Consequently, whenever these laws are observed, the present calculus can be applied" (A VI 4a, 834; English trans. from [24]). The idea is that the same calculus can be interpreted in different ways. Mereology is thus only a particular interpretation of it which obtains as soon as we limit our attention to homogeneous things. Since it is only an interpretation of a more general calculus, it is natural that Leibniz took the laws of real addition to hold true of mereology as well.

3.1. Homogeneity. Leibniz defines the part-whole relation as a restriction of the containment relation via the relation of homogeneity:

> [...] the part must be homogeneous to the whole; and therefore if A and B are two homogeneous things and A contains B, A will be the whole and B the part. (GM VII 274)

If we use Pxy as denoting that x is part of y and Hxy to formalize the idea that x is homogeneous to y, then we have:

Definition of Parthood $Pxy \equiv Cyx \wedge Hxy$

to be read as "x is part of y if and only if y contains x and x and y are homogeneous". To clarify parthood we thus need to clarify the concept of homogeneity.[33]

3.1.1. Homogeneity and Archimedean quantities. In the early modern age, 'homogeneity' was the term used to denote comparable magnitudes, i.e., magnitudes that stand in ratio. The obvious reference was Euclid's *Elements* book V, where Definition 3 stated that "A ratio is a certain type of condition with respect to size of two magnitudes of the same kind". Here 'same kind' indicates what the modern mathematicians called homogeneity. Definition 4 then explicated the notion of comparability via the so-called Archimedean property: "(Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another" ([22, p. 129]).

In contemporary fashion, this amounts to the following condition:

Archimedean Property: If F is an ordered field, then given any positive x and y in F there is an integer $n > 0$ such that either $nx > y$ or $ny > x$.

Leibniz agrees with this use of the term as witnessed by the following passages:

> There is ratio only between homogeneous quantities, and this is clear by definition (GM VII 34).
>
> Among all the relations, the simplest is that which is called ratio or proportion, i.e., a relation between two homogeneous quantities, which originates only from them, without assuming a third [quantity]. (GM VII 23).

On some occasions, he even characterizes homogeneous magnitudes as those that satisfy the Archimedean property:

> I consider that comparables are only homogeneous magnitudes, of which the product of one of them for a number, I mean a finite number, can surpass the other. (GM V 322; see also A III 69)

However, this is not Leibniz's standard definition of homogeneity. Leibniz defines homogeneity in terms of two more basic relations, similarity and equality. It is this definition that we now analyze.

3.1.2. Similarity and equality. Similarity is conceived of as the principle of comparability: "similitude is the principle of homogeneity, or of comparability, since

[33] For a brief history of the notion of homogeneity in Leibniz see [18, p. 160, footnote 31]. Note that I am reading Pxy as x is part of y, while Cxy as x contains y. Therefore, Pxy implies Cyx (and not Cxy).

comparable are those things that can be made similar" ([32, p. 156]). Similars are those things that share all attributes or qualities, and for this reason they can be discerned only through comparison: "similars are those things whose attributes are the same, or which are of the same ultimate species, or which cannot be distinguished one by one" (A VI 4 872). In turn, these definitions rely on the notion of quality which is introduced in contrast to the notion of quantity:

> Quantity or magnitude is that in things which can be known only through their simultaneous compresence—or by their simultaneous perception. [...] Quality, on the other hand, is what can be known in things when they are observed singly, without requiring any compresence" (GM VII 18–19).

Both the notions of quality and quantity are considered to be primitive notions (by the late Leibniz), and are characterized in phenomenological terms.[34]

Similarity is thus thought of as qualitative identity, and can be formalized as follows (in the following $Ql(x)$ is used as a third-order predicate indicating the property of being a quality, and $S(x, y)$ means that x and y are similar):

Similarity $S(x, y) \equiv \forall P(Ql(P) \rightarrow (P(x) \leftrightarrow P(y)))$

Similarity indicates qualitative indiscernibility, and it is clearly an equivalence relation (this trivially follows from the fact that the bi-conditional is an equivalence relation). Notice that the only way to distinguish two similar things is by appealing to their quantitative difference or to their position: "Similar are those things that can be discerned only by their magnitude" ([29, p. 563]/ A VI 4 418). A case in point of similar things are line-segments.

Quantitative comparability of similar things is thus embedded in the same definition of similarity. Since similar things are qualitatively indiscernible, we need to compare them in order to distinguish them, and by this comparison we get to know their quantity or magnitude. For example, if we compare two line-segments by superimposing one over the other, and as a result we cannot discern them anymore, we can conclude that they have the same magnitude, namely they are equal. Here the idea is that *we see* when two similar things are equal or not (indeed, this is the meaning of the claim that the knowledge of quantity requires co-perception).

Equality is conceived of as a quantitative identity: "Equals have the same quantity" (GM VII 19; in what follows, $Qt(x)$ is used as a third-order predicate for properties, and $E_P(x, y)$ means that x and y are equals with regard to property P):

Equality $E_P(x, y) \equiv \exists P(Qt(P) \wedge (P(x) \leftrightarrow P(y)))$

Like similarity, equality too is an equivalence relation.

We followed [9] for the formalization of the notion of similarity; however, we modified [9]'s formalization of equality by restricting it to a specific property. The

34 At first Leibniz had characterized equality starting from another key notion of his *Analysis Situs*, namely congruence. For instance ([29, p. 563]/ A VI 4 418), he had written that "Congruent things are those that are not discernible by themselves" and "Equals are those things that can be decomposed into different parts such that each one is congruent with a part of the other" (the translation from Latin is mine). The late Leibniz would reverse the relation between these notions, and would define congruence as the conjunction of similarity and equality.

reason is that when two objects have more than one quantitative property (i.e., volume and weight), they may be equal with regard to one, but not the other. Therefore, when working with a specific case, we should always specify which property we are interested in. However, in many cases the context makes it clear which property is at stake and so—following Leibniz's usage—we may safely talk of equality in general. In the following, I shall use $E(x, y)$ to denote equality. It is, however, understood that the notion is restricted to the pertinent property.

But what about comparability of non-similar things? For example, how can we compare a curve and a line-segment? Here superimposing one over the other is clearly not enough to establish whether they are equal or not. Leibniz thus needs a way of generalizing comparability from similar to non-similar things. He does this via the notion of homogeneity:

> Two entities are *homogeneous* to which two other entities can be assigned which are equal to them and similar to each other. Given A and B; if L is taken equal to A, and M equal to B, and L and M are similar, we call A and B homogeneous. Hence I usually say also that homogeneous entities are those which can be made similar to each other by means of transformation, like curves and straight lines. That is if A is transformed into its equal L, it can be made similar to B, or to its equal M into which B is assumed to have been transformed. (*Initia Rerum Mathematicarum Metaphysica*, GM VII 19; translation from [35, p. 667])

Formally, we have:

Homogeneity $Hxy \equiv \exists v \exists w (E(x, v) \wedge E(y, w) \wedge S(v, w))$

This definition is more complex than what may appear at first. Notice that at least one of the two pairs (x, v) and (y, w) is not made up of similar things. Otherwise, due to the similarity of v and w, x and y would be similar and the definition would collapse to the case of comparability of similar things. But then the definition requires that we already know when two non-similar things are equal, and this presupposes that we already have a way to compare non-similar things. In this case, the definition of homogeneity would turn out to be useless, if not circular (recall that we need this notion to generalize comparability beyond the cases of similarity). This problem also emerges in other presentations of homogeneity, where Leibniz makes explicit that the truth of clauses such as $E(x, y)$ when x and y are not similar means that there is a transformation of x into y:

> Homogeneous things are those which can be made similar without affecting magnitude [*salva magnitudine*] (Quoted from [6, p. 366]).
>
> Homogeneous things are those which are similar or can be transformed into similar things. (A VI 872)
>
> Magnitude is what is preserved in transformation, or the only thing that can be discerned in similar things. (A VI 873)

The idea is that x and y are homogeneous if and only if we can transform them into two other things v and w, which are respectively equal to x and y (i.e., they have the same quantity) and are similar to one another. An arc of curve C is homogeneous to a straight line L (and can thus be compared to L), because we can transform C into a straight line L' (by rectification) and compare L and L'. For the sake of the explanation, let us suppose that L is smaller than L': $L \leq L'$. Since $E(C, L')$, we can substitute L' with C in $L \leq L'$, *salva quantitate*. In this way, we conclude that $L \leq C$. Therefore, the notion of homogeneity allows us to extend a comparison between similar things to a comparison between things which are not similar.

The key property of such a transformation is that it preserves quantity (and thus it must preserve topological dimensionality as well).[35] However, the problem above remains: how to understand the preservation of quantity in a non-circular way?

Here we may follow Leibniz's suggestion of thinking about the quantity-preserving transformation in the following terms: if two objects are such that there is a decomposition of them into disjoint parts (namely parts that have nothing in common) which are congruent one by one (each part of one object is congruent with one part of the other, and no part remains outside this relation), then we can say that they are equal. But this proposal is circular, because this characterization uses the notion of part, which is defined via the same notion of homogeneity! Another proposal that can be found in GM VII 273[36] consists in exploiting the resources of the infinitesimal calculus. According to this proposal, the transformation/rectification of a curve C into a straight line L' requires us to consider the curve C as if it were composed of infinitesimal straight lines. These infinitesimal straight lines work as if they were a common measure between C and L'; in fact, they are parts of C congruent with parts of L'. That we can feign the existence of a common measure depends on the possibility of taking them as small as we please, i.e., in making the error between the curve and its approximations smaller than any given, and thus null. In this way, we obtain an explanation of the quantity-preserving feature of a transformation that is based on the notion of congruence between fictional parts of the transformed object. However, the circularity expounded above is still present. Indeed, the text clearly treats these infinitesimal lines as parts of C ("a line is not composed of points [...]; rather, a line is

[35] The idea is simply that we can compare lengths with lengths, areas with areas, volumes with volumes, but it does not make much sense to say that an area is greater than the length of a line, or less than the volume of a solid. In other terms, there cannot be any proper comparison (or common measure) between figures of different dimensions: "we must not say, unless in the broad sense Euclid employed, that a line as long as one pleases is smaller than a surface as small as one pleases: in fact, they can by no means be compared to one another" (quoted from [18, p. 201]). For this reason different authors have stressed that transformation is akin to topological homeomorphism (or local homeomorphism). But since it preserves quantity, transformation must be stronger than just (local) homeomorphism, being some isometry of some kind.

[36] Here is the full passage: "Also the method of indivisibles and infinities, or rather of the infinitely small or infinitely large, or of infinitesimals and infinituples, is of the utmost use. For it contains a certain resolution as if into a common measure, albeit one that is smaller than any given quantity, or a means by which it is shown that by neglecting other things which make an error smaller than any given, and thus null, one of the two things that are comparable can be transformed into the other by a transposition. It must be recognized, however, that a line is not composed of points, nor a surface of lines, nor a body of surfaces; rather, a line is composed of indefinitely small linelets, a surface of indefinitely small surfacelets, and a body of indefinitely small corpuscles; that is, it can be shown that two extensa can be compared by resolving them into equal or congruent particles as small as we please, as if into a common measure, and the error will always be smaller than any one of these particles [...]" (GM VII 273; translation from R.T.W. Arthur).

composed of indefinitely small linelets"—where the difference is that a point is not a part of the line, while a small linelet is) which requires again homogeneity.

Despite the difficulties in providing a non-circular definition of homogeneity, it is quite clear which things are homogeneous with each other and which are not. For example, the following are homogeneous objects: two circles with the same radius (they are equals and similar); two circles with different radii (they are similar, but not equals); a curve and a straight line have the same dimensions (they are homogeneous, but not similar); a curve and its rectification are equals (but not similar). On the contrary, a 2-dimensional circle and a 2-dimensional infinite plane, despite having the same dimensions, are neither similar, nor homogeneous[37]. As a consequence, while homogeneous things have the same dimensions, objects with the same dimensions may not be homogeneous.

3.1.3. Metaphysical definitions of homogeneity. The definition of homogeneity given in the previous paragraph is a mathematical definition that mainly applies to geometrical objects.[38] Indeed, it cannot apply to actual things (i.e., non-abstract things) since Leibniz holds that actual things always differ from each other for some intrinsic quality, and so they are never (perfectly) similar (i.e., they never share all qualities). But parthood, which is based on homogeneity, is also applied outside the mathematical realm (for example when Leibniz says that bodies have infinitely many parts) and this requires a metaphysical notion of homogeneity. Leibniz gives this characterization in different places:

> Homogeneous are those things that agree, even though in different ways, in some form or nature intelligible in itself. (A VI 3 483—cited from [44, p. 59])
>
> All things are homogeneous with regard to the essence or reality, such as all bodies are homogeneous with regard to mass. (A VI 4a 26)
>
> I call basis that by which many things are homogeneous, or similar and they differ for a modification of it, like space for figures, matter for bodies, time for hours, movements for its parts. (A VI 4a 278—1679).[39]

The idea behind these definitions is that things are homogeneous because they share a nature or form, like all bodies share matter, all figures share space, all instants share times, etc. Even though the metaphysical definitions are not equivalent to the mathematical definition, they clearly share the idea that homogeneity is grounded on the sharing of some qualities (natures, essences or forms are collections of qualities). Since similarity cannot have a place among actual existing things, we need to widen

37 If A is an infinite area (for instance an area under a certain curve) and we subtract a finite area B from it, we would obtain the same A, as if the subtraction had no effect. This seems to violate the idea that the whole is always greater than the parts, and for this reason—Leibniz concludes—the infinite is not a whole, but only a fiction. While a 2D finite circle can be a whole, a 2D infinite plane cannot, and for this reason, even though they have the same dimensionality, they are not homogeneous. On this argument see [2, §2].

38 Leibniz is fully aware of the limitation of this definition even within the mathematical context. See A VI 4a 933-934.

39 See [44, pp. 58–60] for more characterizations of homogeneity.

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the notion of homogeneity such that now things that only share some quality can be said to be homogeneous. Alternatively, we may change the same notion of similarity and say that two things are ψ-similar if they share the quality ψ. The mathematical notion of similarity can thus be seen as a limit case of a wider range of cases: from the weakest one where two things are said to be similar with regard to only one quality, to stronger and stronger cases where the objects in question share more and more qualities. Properly speaking, Leibniz does not have a unique concept of similarity (and so of homogeneity), but a fuzzy range of concepts that go from sharing only one quality to sharing all qualities. The central idea of all these concepts stems from the fact that any quantitative comparison is made with regard to some qualitative aspects (we measure heights, weights, etc. which are all qualitative aspects of objects). This is the basic idea behind this intricate net of definitions.

Not only do the mathematical and the metaphysical definitions of homogeneity share the same root, i.e., the fact that quantitative comparison is always done with regard to some quality, but the mathematical definition can be seen as a limit case of the metaphysical one. The reason is that the notion of similarity that enters into the mathematical definition of homogeneity can be seen as a limit case of ψ-similarity (i.e., the case in which we consider *all* intrinsic properties of the objects, and not only some of them). For these reasons, from now on I will not come back to this difference and will operate as if Leibniz had a univocal notion of homogeneity. In particular, while looking at the logical features of Leibniz's notion of part, I shall presuppose the mathematical definition, the only one sharp enough to allow us to study the parthood relation. However, it is important to keep in mind that different contexts may require slightly different characterizations of homogeneity.

3.2. Restricting containment through homogeneity. The notion of part is defined via containment and homogeneity: $Pxy \equiv Cyx \wedge Hxy$. As mentioned before, Leibniz never presented a formal theory of parthood; he limited himself to saying that the theorems of the RA calculus remain valid when we restrict containment via homogeneity. To capture Leibniz's mereological system, I propose to extend the RA system with the following axiom:

Homogeneity Axiom: $\forall x \forall y (Cxy \rightarrow Hxy)$

This has the effect of imposing that the containment relation that we are interested in while doing mereology is only that among homogeneous things.[40]

Definition of the system M: RA + Homogeneity Axiom.

Since M extends RA, it straightforwardly follows that all theorems of RA are also theorems of M. Leibniz clearly envisaged this fact:

[40] It is here that the choice made in §2.3.1 concerning the way out of the problem raised by the term *Nihil* is useful. In fact, the option that we chose implies that *Nihil* is never contained in other objects. Both the alternative options allow for the existence of a theorem according to which *Nihil* is contained in everything. But then, if we want the *Nihil* term to be within the mereological calculus, the homogeneity axiom would force *Nihil* to be a part of every object. The transitivity of the homogeneity relation would immediately entail that everything is homogeneous to everything else, which is absurd. A possible way out is to deny that *Nihil* is homogeneous to any other object. Therefore, *Nihil* would not be part of anything. Our choice completely avoids this problem, delivering the same conclusion. Clearly, since *Nihil* is not contained in anything, it won't be a part of anything.

> [...] if A and B are homogeneous, and A contains B, A will be the whole, and B a part, so that the demonstrations that I gave in another place regarding the containment relation [*de continente et contento seu inexistente*] can be transferred to the whole and the part. (GM VII 274)

Homogeneity is an equivalence relation, since it has been defined through similarity and equality, which are equivalence relations.[41] The notion of part that we obtain in this way is the so-called notion of improper part: since both containment and homogeneity are reflexive, it follows that for all x we have both Cxx and Hxx, which means that Pxx is valid as well: everything is part of itself. Reasoning in a similar way it is straightforward to show that the parthood relation is transitive and antisymmetric (since the C-relation is transitive and antisymmetric); as such it is a partial order.[42]

Even though it is Leibniz himself who introduces the relation of 'part' in this precise way, when he speaks of 'part' he usually has in mind the notion of proper part, i.e., a notion that does not contemplate the case in which something is part of itself:

$PPxy \equiv Pxy \wedge \neg(x = y)$[43]

If $PPxy$ is the case, by Subtraction 2, we have that $\exists z(PPzy \wedge \neg(z \circ x))$. This is simply the result of subtracting x from y, and substituting the containment relation with parthood in Subtraction 2, since we are working within M. If I subtract a proper part of something A, I always get a remainder different from A. This is the feature that allows Leibniz to define the 'smaller than' ($<$) relation via (proper) parthood:[44]

Definition of 'smaller than': $a < b \equiv \exists x E(a, x) \wedge PP(x, b)$

41 Here is the proof: 1) $H(x, x)$ follows from the fact that $\forall x S(x, x)$ is true. So homogeneity is reflexive. 2) Suppose that $H(x, y)$. Then there is a v and a w such that $E(x, v)$ and $E(y, w)$ and $S(v, w)$. By symmetry of the similarity relation, the latter is equivalent to $S(w, v)$. So we have $E(y, w)$ and $E(x, v)$ and $S(w, v)$, which means $H(y, x)$. Homogeneity is a symmetric relation. 3) Suppose now that $H(x, y)$ and $H(y, z)$ is the case. From the former we have $E(x, v)$ and $E(y, w)$ and $S(v, w)$; from the latter, for some q, we have $E(y, w)$ and $E(z, q)$ and $S(w, q)$. By transitivity of symmetry, we have that $S(v, q)$. So we have that $E(x, v)$ and $E(z, q)$ and $S(v, q)$, which means $H(x, z)$. Homogeneity is therefore transitive.

42 Here is the proof: 1) reflexivity has been proved in the main text; 2) suppose Pxy and Pyz. By definition, this means $C(y, x)$ and $H(x, y)$, moreover $C(z, y)$ and $H(y, z)$. By transitivity of the C-relation, we have $C(z, x)$. By transitivity of the homogeneity relation, $H(x, z)$. So Pxz. Therefore Parthood is transitive; 3) Suppose Pxy and Pyx. The former implies $C(y, x)$. The latter implies that $C(x, y)$. By the antisymmetry of the C-relation, we have $x = y$. Parthood is antisymmetric.

43 Sometimes Leibniz is explicit on this point: "If B is in A and is homogeneous to the same A, but they do not coincide with each other, then B will be the part, A the whole" ([31, p. 109]). See also A VI 4 1001: "If the constituting things are different from each other, A, B, etc., they are called parts and L the whole". In some writings probably dating back to 1700 (see [47]), Leibniz even introduced a further relation—the *intra* relation—which differs from the *inesse*/containment relation insofar as it is irreflexive: nothing is 'intra' itself. Then he proceeded to define x as a part of y if and only if x is *intra* y, and is homogeneous to it. Clearly, this definition gives us the notion of proper part. From a logical point of view, the two ways of defining proper parthood are equivalent; from a historical point of view, the latter casts doubt on the fact that Leibniz has always considered mereology as a sub-part of the containment calculus. However, in *Initia Rerum Mathematicarum Metaphysica* of 1714, Leibniz gives the standard definition of parthood based on the containment relation.

44 "Smaller is what is equal to a part of the other (the Bigger)" (A VI 4 418).

From this it immediately follows that each (proper) part is less than the whole in which it is contained.

Part-Whole Principle: *The whole is always bigger than each of its proper parts.*

Proof suppose $PP(a, b)$ is the case. For all a, we have that $E(a, a)$. By the definition of smaller than, we conclude that $a < b$.[45] □

Since it extends RA (and not RA^{∞}) M is weaker than CEM. The system M has only a finite composition principle to the effect that any pair of objects can be summed together to form a whole. This is in line with Leibniz's rejection of infinite wholes. Leibniz's commitment to the idea that the whole is always bigger than each of its (proper) parts led him to reject infinite wholes on pain of contradiction. Moreover, notice that the possibility of defining the 'smaller than'-relation via parthood rests on the idea that all quantities have a mereological structure, i.e., all quantities are wholes with parts. This is in fact an explicit assumption made by Leibniz (that we shall call Part-Quantity axiom in the next section).

A couple of things should be noted here. First, FCP is compatible with a finite whole being decomposed into infinitely many parts. What we need to guarantee is that there is at least one finite decomposition of the whole. In this case the whole can be obtained by applying FCP. But this feature is guaranteed by homogeneity. In fact, a decomposition into parts of a finite whole requires that the parts are homogeneous to the whole, and so each part will be finite (i.e., it will have a finite magnitude). Second, it has been noted that a principle like FCP may clash with the Subtraction 2, since the remainder z is the fusion of all parts of x disjoint from y, and there is nothing that guarantees that there are only finitely many of these parts (see [16, p. 164]). However, again, homogeneity rules out this possibility, because the remainder z is always a part of x, and since x is finite, by homogeneity, z must be finite too.

3.3. Leibniz's mereology: the system LM. The Part-Whole Principle is a straightforward consequence of the definition of the $<$-relation via the notion of parthood. The latter is based on homogeneity, which indicates quantitative comparability: there is always a ratio among homogeneous things. Mereology is from the very beginning a theory that constitutes the bones of Leibniz's theory of measurement (of quantities). Indeed, it is straightforward to show that, from the definition of parthood, it follows that if x is a whole with proper parts, then x has quantity. In other words, 'being a whole (with proper parts)' implies 'having quantity'. This is the case because (proper) parts are homogeneous to their wholes, which means that a (proper) part is always quantitatively comparable with the whole in which it inheres (i.e., there is always a ratio between a part and the whole in which it inheres). As a consequence, they both have quantity. If we denote the property 'having quantity' as $\lambda x.Qt(x)$, we may express it as follows:

$$\exists w PP(w, a) \rightarrow [\lambda x.Qt(x)](a)$$

Notice that we have formulated this claim with reference to proper parthood, and not simply to parthood. The reason is that since everything contains itself and is

[45] Since there is always a ratio among homogeneous things, the Trichotomy law holds for the $<$-relation (see GM VII 34). Moreover the $<$-relation is irreflexive and transitive since proper parthood is irreflexive and transitive. Therefore it is a linear-order.

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homogeneous to itself, everything is a part of itself. This implies that everything has quantity, a claim that Leibniz cannot accept. In addition, since Leibniz usually has in mind the proper parthood relation, it is therefore natural to restrict the claim to the proper parthood relation.

What about the converse? Does 'having quantity' imply 'being a whole with parts'? Leibniz has a positive answer to this:

> We can define the quantity of a thing as the property of the whole insofar as it has all its parts (GM VII, 30/A VI 4a 418).
>
> [...] And more precisely those things which are found within a whole, but in such a way that they are homogeneous to the whole itself are called *parts* by the mathematicians who proceed meticulously [...]. It is considered that only those wholes have quantity [nec quantitatem habere censetur nisi ea tota] within which there are many [parts] homogeneous to the whole itself. (LH XXXV 1, 9, fol. 1-4; [47, pp. 206–208])[46]

The claim that only wholes have quantity is equivalent to the claim that 'having quantity' implies 'being a whole with parts'. This is presented as an axiomatic principle, which allows Leibniz to apply mereology in the measurement of any kind of quantities. Notice that this principle—that I shall call Part-Quantity—cannot be literally interpreted as a definition of quantity, without incurring a vicious circularity. In fact, the notion of part and whole is defined via homogeneity, which requires the notion of equality, and so of quantity. Therefore, the notion of part and whole presupposes the notion of quantity, and cannot be exploited to define it. If we denote the property 'having quantity' as $\lambda x.Qt(x)$, we may express it as follows:

Part-Quantity Axiom: $[\lambda x.Qt(x)](a) \rightarrow \exists w PP(w, a)$

This says that if a has quantity, then it has (proper) parts, namely it is a whole with parts. Clearly, we can strengthen the claim into an equivalence, since we have seen that the right-to-left direction follows from the definition of parthood. By adding Part-Quantity Axiom to the system M we get Leibniz's Mereology LM:

Definition of the system LM: M + Part-Quantity Axiom.

This is the final system that translates in a formal language Leibniz's own mereology.

There are two things to notice here. First, this conception of quantity presupposes that a quantity is something positive, different from zero. Since quantity is a property of a whole, if something has quantity zero, then—properly speaking—it has no quantity at all, which implies that it is not a whole with parts. For example, a point does not have quantity, and in fact it does not have parts;[47] or a simple substance does not have

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46 I translated the double negation with an affermative sentence to make the translation more comprehensible. On the notion of quantity in Leibniz see [12], [14], and [49]. The acronym LH stands for 'Leibniz Handschriften' (Leibniz's manuscripts); the series XXXV contains mathematical works. Leibniz's manuscripts are accessible in digital format from the website of the GWL Bibliothek of Hannover at the following link: https://digitale-sammlungen.gwlb.de/start

47 In the early work *Theoria motus abstractis* (1672) Leibniz had followed Hobbes in claiming that a non-extended point has parts. If this is the case, then this notion of point allows points to have a correspondent quantity. However, Leibniz would soon reject this early view.

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parts, and so it does not have a correspondent quantity. Second, one may question the presence of Part-Quantity among the axioms of Leibniz's mereology. Such an axiom seems extraneous to a proper mereological theory, at least as it is conceived today, since it seems to establish a connection between mereology (the M system) and something external to it, i.e., the theory of quantity. However, this objection fails to appreciate the deep connection between mereology and quantity in Leibniz. Mereology works as a background theory for the theory of measurement. This is already clear from the same definition of parthood via homogeneity. By claiming that 'having quantity' implies 'being a whole with parts', Part-Quantity guarantees that any kind of quantity can be treated mereologically, assuring that there is a unique theory that can then be applied to different fields. Finally, as we will see, Part-Quantity plays a important role in the justification of some formal features of the parthood relation.[48]

3.4. A Gunky notion of part. Within the LM the following theorem is derivable:

Part-Hereditary: $\forall x \forall y \exists w \exists z (H(x, y) \wedge PPwx \rightarrow PPzy)$.

In words: if x and y are homogeneous, and x has proper parts, so does y. Why is this principle true? Let us start by considering the special case in which two homogeneous objects—say a and b—are homogeneous in virtue of being similar. Having (proper) parts is a quality, not a quantitative property (compare 'having parts' with 'having n-parts', i.e., a determined number of parts, which is a quantitative property). Then it directly follows that if one between a and b has parts, so the other. This is because similar things share all qualitative properties, and having parts is a qualitative property. Therefore, if a and b are similar, and one has parts, so the other.[49]

Let us now consider the most general case, where a and b are homogeneous (but not similar), a is transformed into v, b is transformed into w, and v and w are similar. We want to show that if a has parts, so does b. To do that we need to appeal to Part-Quantity (and its converse). Suppose that a has parts. Since a is transformed into v, a and v are equals, i.e., they have the same quantity. In particular, v has a quantity, and so, by Part-Quantity, it is a whole with parts. v is similar to w, which implies that also w is a whole with parts. But w is equal to b, so they have the same quantity, which implies that b has a quantity, and so, by Part-Quantity, it is a whole with parts. Therefore, if a and b are homogeneous, and one has parts, so the other.

The fact that the notion of part is defined by means of homogeneity implies that a part is always homogeneous to the whole of which it is a part. If the whole is a 1-dimensional object (a line with a length), then each of its proper parts will be 1-dimensional lines; if the whole is a material extended body, then each of its parts will be material extended bodies. Once Part-Hereditary has been established, it is easy to derive another interesting consequence that we shall label PP-Non-Well-Foundedness, namely the claim that everything which is a part is composed of further parts. Clearly this implies that the parts of the parts will be composed of further parts, and so on

48 One may claim that Leibniz's mereology is a kind of megethology, i.e., a theory of quantity. The term megethology comes from [41]; a formal treatment of Euclid's mereology and megethology (on which Leibniz's own work is strongly indebted) is [50].

49 I am here working with the mathematical notion of similarity that requires that similar objects share all qualities. As stressed in §3.1.3, in metaphysics, Leibniz often relax this constraint and speaks of similarity of two substances when they only agree on some but not all (qualitative) properties.

without an end. The upshot is that, with regard to the part-whole relation, we never reach a point where we find things without proper parts. In other words, within LM parthood is gunky:

PP-Non-Well-Foundedness $\forall x \forall y \exists z (PPyx \rightarrow PPzy)$

Proof Suppose that the antecedent is true; since parthood requires homogeneity, this means that we have $H(x, y) \wedge PPyx$. But this is exactly the antecedent of Part-Hereditary, where w has been replaced by y. By *modus ponens* with Part-Hereditary, we can infer that there is a z such that $PPzy$ is the case, i.e., y itself has its own proper parts. □

This principle amounts to saying that the PP-relation is not well-founded: given a part we can always find an infinitely descending chain of parts. One has to notice that we have formulated the principle with regard to *PP* and not to *P*: since the latter is reflexive, the existence of an infinitely descending chain is trivial (just consider the chain starting from a whose links are always a: a is a part of a, which is a part of a, and so on).[50]

It is important to stress that Leibniz explicitly recognizes this feature of the notion of part. In particular focusing on the case of similar objects, he writes that "because in the line [recta] the part is similar to the whole, it is evident that in each part there will again be another part, and so given any line we can take a smaller line, since the part is smaller and it is still a line" (GM V 206, my translation). In fact, from PP-Non-Well-Foundedness, the converse of Part-Quantity and the definition of the $<$-relation it immediately follows that given any quantity there is always a smaller quantity.

3.5. What about atoms? Within the LM calculus, we may define an atom as an object without proper parts:

Definition of Atom: $Ax \equiv \neg \exists z PPzx$

This corresponds to Leibniz's notion of simplicity, which is the key feature of monads. Therefore, the question of atoms within Leibniz's mereology is the question of the relationship between monads and compounds that have (proper) parts.

First, it is straightforward to see that a universe that contains only one object (which will thus be an atom) is a model for the LM calculus. Indeed, PP-Non-Well-Foundedness is a conditional statement to the effect that if something has proper parts, then its proper parts have proper parts too. As such it does not exclude the existence of atoms. FCP is satisfied (in virtue of the idempotence property), Part-Quantity and its converse are trivially satisfied since there is no quantity and nothing has proper parts; finally Subtraction 2 is satisfied since we can subtract the atom from itself and obtain *Nihil*. But *Nihil* is not, properly speaking, an object, but rather an empty term (this is the rationale behind the claim that *Nihil* is not contained in anything: you must be something to be contained in something else). Therefore, the LM calculus does not—by itself—exclude atoms. Moreover, notice that in virtue of Part-Quantity,

50 We do not agree with [8] when they impose a principle of well-foundedness to the PP-relation to account for the idea that monads are simple entities in composite aggregates. Their mistake is not technical, but rather philosophical: they misinterpret the relation between monads and composite as a case of part-whole relation, while it is clear that monads are contained in bodies, but are not part of them.

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it follows that an atom has no quantity, which is exactly what Leibniz claims about monads: "where there are no parts, there is no extension, nor figure nor divisibility" (Monadology §3, GP VI 607).

However, if we assume that there are at least two distinct objects, then no atom can exist within the LM calculus. In other words, the following holds:

Week Atomless: $\exists x \exists y \neg(x = y) \rightarrow \neg \exists w A w$

Proof. Suppose that there are two different homogeneous[51] objects a and b, and that a is an atom. By FCP there exists c such that c is the mereological sum of a and b. By definition of the parthood relation, a is a proper part of c; by PP-Non-Well-Foundedness, a has proper parts as well. But this contradicts the fact that a has been assumed to be an atom. Therefore, a is not an atom. On the assumption that there are at least two distinct objects, we need to reject the existence of atoms. □

This latter result depends on PP-Non-Well-Foundedness. It is this principle that does not allow atoms to be (proper) parts of other objects. The principle is a direct consequence of PP-Hereditary, which in turn depends on the notion of homogeneity. As such, an atom cannot be a proper part of a composite object, because it is not homogeneous to it. As a consequence, mereology does not apply to them, in the precise sense that any model of the LM calculus with at least two distinct objects will not have any atom.

However, what about atoms in the wider context of the RA (or RA^{∞}) calculus? Here there is no problem for mereological atoms (as defined above) to be contained in aggregates or compounds (without being a part of these compounds). Indeed, this captures well Leibniz's often repeated claim that monads are in bodies, but not parts of bodies, as we will see in the next paragraph.

The notion of atom that we have worked with so far has been defined with regard to the parthood relation, but nothing prevents us from defining a different notion of atom with regard to containment. Indeed, we find a definition of 'atom' that we may formalize as follows: $A_c x \equiv \neg \exists z C x z \wedge \neg(x = z)$ (where $A_c(x)$ indicates the property of being an atom with regard to the containment, and not the parthood relation). Examples of atoms with regard to the containment relation are points in space and instants in time.[52] Clearly, $A_c x$ implies Ax (if nothing distinct from a is contained in a, then *a fortiori* nothing is a (proper) part of a), but the inverse is not true: something might be simple with regard to the parthood relation (so it won't have proper parts) but there might be something contained in it.[53]

51 Since we are working within LM, we are only considering homogeneous objects.

52 Here is the text: "I call atom [*ultimum inesistens*] what is in [something] so that nothing is in it, or if L is an atom, and we assume that A+B=L, then A=B=L. For example the point is in space, the instant in time" (A VI 822, my translation). Thanks to Wolfgang Lenzen for bringing my attention to this passage.

53 A non-trivial model for the LM theory is provided by mereological sums of finite sets of open intervals in the real numbers (including the empty set $\emptyset$), where mereological sum of a set A is interpreted as the interior of the closure of the union of A. Parthood is interpreted as the subset relation ($\subseteq *$) between these mereological sums (which are open intervals) with the restriction that if $A \subseteq B \rightarrow \neg(A = \emptyset)$. This is necessary since *Nihil* is interpreted as the empty set, and in LM it is not part of any other objects. Subtraction 2 is interpreted as set difference and the overlapping relation corresponds to intersection. The homogeneity relation is interpreted as the property 'being an interval of real numbers' (notice that we are

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§4. Applications of the containment-parthood distinction. The aim of this section is to scrutinize some of the applications of the containment-parthood distinction. The fruitfulness of the distinction will provide us with an (*a posteriori*) justification of it; moreover, we shall see that in some cases Leibniz needs an infinitary Real Addition operation.

4.1. Monads are in bodies, but not parts of bodies. We have shown that, provided there are at least two objects, there can be no atom according to the LM calculus. Since an atom is defined as something with no proper part, their metaphysical counterparts are monads. In this paragraph, with Ax we shall indicate that x is a monad (or simple substance). The conclusion is that monads are not parts of any compound: in particular, monads are not parts of bodies. Indeed, monads are not homogeneous to bodies, since the former are immaterial and not extended, the latter material and extended. However, monads "enter into compounds", i.e., they are (contained) in bodies. In the Monadology we read:

> 1. The monad which we are to discuss here is nothing but a simple substance which enters into compounds. *Simple* means without parts. (*Theodicy*, Sec. 10).
> 2. There must be simple substances, since there are compounds, for the compounded is but a collection or an *aggregate* of simples. (GP VI 607 - [35, p. 643])

In the *Principles of Nature and Grace* it is clarified that these compounds are bodies:

> Compounds, or bodies [Les composés ou les corps] are pluralities, and simple substances—lives, souls, and spirits—are unities. (GP VI 598; [35, p. 636])

This latter thought was echoed in the 1690s as follows:

> A body is not a substance but an aggregate of substances. (Leibniz 1690: A VI, 4, 1668)

A body is an aggregate or a compound of simple substances. Notice that Leibniz does not say that a body is a whole whose parts are the monads; indeed, he is very clear that this is not the case:

> Moreover, even if the body of an animal or my own body is composed, in turn, of innumerble substances, they are not parts of the animal or of me. (GM III 537—translation from [34, p. 167])

That bodies are aggregates of substances means that substances are (contained) in bodies. Here Leibniz is simply applying the definition of containment of the RA calculus: if x is in y, then x plus something else is identical to y, which means that y is an aggregate (a container) of which x is one of the ingredients.

Remarkably, in order for this distinction to work, real addition must be an infinitary operation along that of RA^{∞}: each body is constituted by infinitely many substances, i.e., each is an infinite aggregate of monads. If we admitted only a finitary operation

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not requiring that homogeneity preserves open intervals, since set difference among open intervals does not always yield back open sets).

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(like that of RA), then we could not interpret the relationship between monads and bodies via the containment relation, and so it would become totally mysterious which relations are here in play.[54]

Working within RA^{∞}, we can be more specific on the relationship among bodies and monads. The gist of Leibniz's theory can be summed up in 6 propositions:

1. There are monads, i.e., substances with no proper parts: in formulas, $\exists x Ax$;
2. There are bodies, i.e., concrete objects with proper parts: $\exists x Bx$, where $Bx \equiv Conc(x) \land \exists w PPwx$ ($Conc(x)$ means the x is concrete, i.e., not ideal or abstract);
3. What exists is either a monad or a body;
4. Each part of a body is a body as well, and so it has its own proper parts: $\forall x(Bx \to \forall y(PPyx \to \neg Ay))$;
5. There are monads in bodies: $\forall x(Bx \to \exists y(Ay \land Cxy))$;
6. Bodies are aggregate of monads (only), i.e., bodies results from monads only: $\forall x(Bx \to \Sigma_{\phi} x)$, where ϕ is the condition $Ay \land Cxy$.

Propositions 1 and 2 simply state the existence of, respectively, bodies and monads. Proposition 3 states that if something exists, either it is a monad or a body. Together with Proposition 6, this entails that what exists is either a monad or an aggregate of monads. Proposition 4 follows from PP-Non-Well-Foundedness applied to bodies: since bodies have parts, by PP-Non-Well-Foundedness, these parts have further parts and so on. So no atom (i.e., no monad) is a part of a body. Proposition 5 states that monads are contained in bodies. Notice that it does not explicitly say that all that there is to bodies are monads. This latter claim is in fact Proposition 6, which says that bodies are aggregates of monads. In other words, bodies are the sum (real addition) of all and only the monads they contain. So, if b is a body, then b is the sum of all monads y (Ay) contained in it (Cby). By applying UCP (where ϕ is the condition $Ax \land Cbx$) we obtain: $B(b) \to \Sigma_{\phi} b$. 6 is the generalization of this. Given Leibniz's nominalistic attitude toward aggregates (and sums) this amounts to the idea that the reality of bodies just is the reality of the monads in them.

At this point one might worry about the mutual consistency of propositions 4 and 6. The latter claims that bodies are aggregates of monads only: the idea is that we do not need anything else apart from monads to obtain bodies; when God creates a body he simply created the correspondent monads. But Proposition 4 says that there are parts in bodies, and we know that these parts are not monads.

To meet this concern the first thing to notice is that, given the RA^{∞} calculus, Propositions 5 and 6 are equivalent.

Theorem: 5 $\leftrightarrow$ 6

Proof: From right to left. That 6 implies 5 is straightforward: if $B(b)$ is the case for an arbitrary b, then $\Sigma_{\phi} b$ is the sum of all monads in b. In order for the sum to exist, ϕ must be satisfied, i.e., there must be a y such a that Ay and Cby. Therefore, we have $B(b) \to \exists y(Ay \land Cby)$. Generalizing we obtain 5.

From left to right. The other direction is more complex, and here we adopt (a slightly modified version of) a proof from [17, p. 146]. First, notice that in virtue of the definition of $\Sigma_{\phi} x$ proposition 6 amounts to the conjunction of the following two propositions:

54 For the relationship between monads and bodies see the illuminating pages of [4].

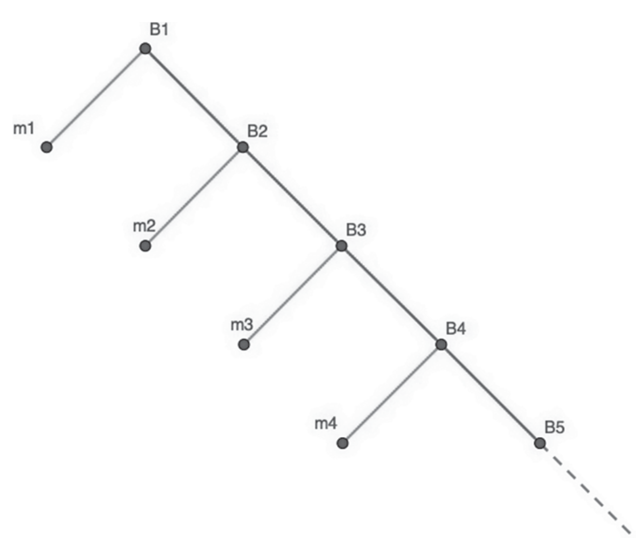
6A) $\forall x(Bx \to (\forall z(Az \wedge Cxz) \to Cxz))$,
6B) $\forall x(Bx \to \forall y(\forall z(Az \wedge Cxz) \to Cyz) \to Cyx)$.

Since 6A is a logical truth, it is enough to show that 5 implies 6B. To do that we shall use our Subtraction 2 axiom, according to which, given any x and y, $x - y$ is always defined as the only object z such that Cxz and Dzy (when Cyx or $x = y$, the result is *Nihil*). The proof goes as follows.

Suppose that $B(a)$. Moreover, suppose that $\neg C(b,a)$. By Subtraction 2, there is a z such that $\forall w(C(z,w) \leftrightarrow (C(a,w) \wedge D(w,b)))$. Thus $z = a - b$. In particular, we have that there is a w such that $C(a,w) \wedge D(w,b)$. By Proposition 3, w is a monad or a body. If it is a body, by Proposition 5, it contains a monad d (and so Ad). If it is a monad, then Aw. In this case we can put $w = d$. In each case, we have that there is a d such that Ad and $C(w,d)$. By transitivity of containment, we obtain $C(a,d)$. We thus have $Ad \wedge C(a,d)$.

From $D(w,b)$ and $C(w,d)$, we get $\neg C(b,d)$. Therefore, we have $Ad \wedge C(a,d) \wedge \neg C(b,d)$. By existential generalization, we obtain $\exists z(Az \wedge C(a,z) \wedge \neg C(b,z))$. Since this depends on having assumed $\neg C(b,a)$, we can introduce the implication: $\neg C(b,a) \to \exists z(Az \wedge C(a,z) \wedge \neg C(b,z))$. Contraposing and by propositional equivalences, this amounts to $\forall z(Az \wedge C(a,z) \to C(b,z)) \to C(b,a)$. By introducing the implication and generalization, we obtain $\forall x(Bx \to \forall y(\forall z(Az \wedge C(x,z) \to C(y,z)) \to C(y,x)))$, i.e., 6B. □

The equivalence of Propositions 5 and 6 means that we just need to assume that there are monads in bodies to derive that bodies are nothing else than (non-mereological) aggregates of monads. We are now in a position to see why Propositions 4 and 6 are consistent. The following model clearly validates Propositions 4 and 5, but since 5 is equivalent to 6, it validates 6 as well.



In the picture, the lines represent the containment relation. So, $m1$ is contained in $B1$, $m2$ is contained in $B2$, and $B2$ is contained in $B1$, etc. $m1, m2$, etc. represent monads, while $B1$, $B2$, etc. represent bodies. The lines connecting the B-terms with the

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m-terms indicate that the containment relation holds among non-homogeneous things, while the lines connecting the B-terms with other B-terms are for containment between homogeneous things, i.e., parthood. Proposition 4 holds, because all (proper) parts of a body ($B1, B2$, etc.) are not atoms, namely they have further (proper) parts. At the same time Proposition 5 holds, since each body has at least one monad contained in it. But then also Proposition 6 holds, and indeed every object in the model is ultimately decomposable into monads. In particular, the sum of all monads is identical to $B1$.[55]

It can be easily seen that the present model is not a model of LM or RA^{∞}. For example, there is no sum of $m1$ and $m2$, which implies that there is no remainder of $B1 - B3$. And this may be seen as problematic, since the existence of remainders is guaranteed by Subtraction 2, which we have used in the proof of the equivalence of 5 and 6. But a closer look at the proof will reveal that we have not used all the strength of Subtraction 2. In fact the proof only relies on a weaker principle, i.e., $\neg C(y,x) \rightarrow (\exists z C(x,z) \wedge D(z,y))$. And this principle is valid in the model above. The point of the model is to show that the two claims at the heart of Leibniz's theory of substances (bodies are aggregates of substances and at the same time they are divisible without end into further composite bodies) are consistent with each other. The model can thus be seen as an oversimplified picture of the structure of a Leibnizian corporeal substance. Clearly the distinction containment-parthood is essential to it.

4.2. Multiplicity in the simple. In the Monadology, Leibniz famously claimed that there is a multiplicity (of states) in each simple substance:

> 12. But besides the principle of change there must be a particular detail of what changes, which constitutes the specific nature and the variety, so to speak, of simple substances.
> 13. This detail must enfold a multitude in the unity or the simple. For every natural change takes place by degrees—something changes and something remains—and as a result there must be a plurality of affections and of relations in the simple substance, even though it has no parts. (translation from [35, p. 644], slightly modified).

Leibniz here introduces a multiplicity of different states within each monad; multiplicity which is required for the explanation of change. This is Leibniz's reinterpretation of the traditional claim that substances have a plurality of accidents. But how is it possible that what is simple, i.e., what has no parts, hosts a multiplicity in itself? Clearly, these states cannot be parts of the substance, otherwise we would have a contradiction with the attribute of simplicity. Here, again, the distinction between containment and parthood solves the problem. States are (contained) in the substance, but are not part of it. Substances are simple with regard to the parthood relation: on the contrary, with regard to the containment relation, the substance is something complex, exactly in the sense that there is a multiplicity of states in it. Each state is in the substance, which means that in the substance there is an aggregate of states.

---

[55] An easy way to see that this is the case is to consider an example from [21, p. 75] (who first introduced this model). Interpret the lines as representing the subset relation; then take $B1$ to be the set of positive integers $\{1,2,3,...\}$; $B2$ the set $\{2,3,4,...\}$; $B3$ the set $\{3,4,5,...\}$ etc. Moreover take $m1$ to be $\{1\}$; $m2$ $\{2\}$, etc. Then it is clear that the set $B1$ is the union of all elements $m1, m2$, etc.

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It goes without saying that this multiplicity of states is infinite. Here again we find an example of an (implicit) use of an infinitary Real Addition operation. But there is more: each state represents the entire world, which—as we are going to see—is an infinite aggregate of substances. Therefore, not only is there an infinite aggregate of states in each substance, but each one of these states is an infinite aggregate of representations (of everything that exists). Again, with only a finitary Real Addition, we could not apply the containment relation, and the relation in play here would be mysterious. These applications both require the RA^{∞} calculus.

4.3. The world. The last application of the containment-parthood distinction that I want to scrutinize is Leibniz's conception of the created world. Here again we find what appear to be incompatible statements about it. On the one hand, Leibniz is clear that the world constitutes an actual infinite:

> Created things are actually infinite. For any body whatever is actually divided into several parts, since any body whatever is acted upon by other bodies. And any part whatever of a body is a body by the very definition of body. So bodies are actually infinite, i.e., more bodies can be found than there are unities in any given number. (A VI 4, 1393/ [25, p. 235])
>
> There is an infinity of creatures in the smallest particle of matter, because of the actual division of the continuum to infinity. (GP VI 232/ [33, §195])

Since there is an actual infinity of bodies within each body, the world, i.e., the aggregate of every created thing, is an actual infinite as well. The world is sometimes described as the aggregate of all bodies ("The aggregate of all bodies is called the world"—A VI 4, 1509), other times it is described as the aggregate of all (created) substances (since what really exists are substances). In both cases, the infinite in play is actual, showing once again that Leibniz was using an infinitary real addition operation (indeed, from the claim that the world is the aggregate of all bodies or all substances, by the definition of *inesse* we can infer that each body and each substance is (contained) in the world).

However, there are other passages that suggest what may appear as a rather different view:

> Thus, we may indeed call all bodies together "the world", but in reality the world is not some one thing, but this alone can be said: for any given body, there is some larger one in the world and we never reach a finite body that includes all [bodies]. Nor, however, is there such an infinite body. (A VI 4, 1469 - about 1683-85).
>
> Yet M. Descartes and his followers, in making the world out to be indefinite so that we cannot conceive of any end to it, have said that matter has no limits. They have some reason for replacing the term 'infinite' by 'indefinite', for there is never an infinite whole in the world, though there are always wholes greater than others *ad infinitum*. As I have shown elsewhere, the universe itself cannot be considered to be a whole. (A VI, 6, 151/ translation from [36, pp. 150–151],)

One may be tempted to interpret these claims as suggesting the view that the world constitutes a potential infinite (given finitely many bodies, always more can be founded). But this would immediately give us a contradiction with the claim that the world is an actually infinite aggregate. Something cannot be at the same time and under the same respect potential and actual.

Again, the contradiction can be solved by appealing to the distinction between containment and parthood. When we look at the world with the "glasses" of the containment relation, the world is an infinite aggregate, but it is not an infinite whole ("infinity itself is nothing, i.e., that it is not one and not a whole"—A VI 3, 168/[25, p. 9]). Recall the nominalistic reading of aggregates: to say that the world is an infinite aggregate of substances/bodies simply means that there are actually infinitely many substances/bodies. However, when we look at it with the "glasses" of the parthood relation, what we see are finite wholes (here finite means that they have a finite magnitude), and given a whole a more comprehensive whole can always be found. Notice that the actual infinity of the universe is what guarantees that "there are always wholes greater than other ad infinitum". The idea is that FCP allows us to add together more and more things into bigger and bigger wholes. Suppose that A and B are two different things, possibly overlapping, but such that one is not contained in the other. Let us compose them via FCP into a whole C. Then A and B will both be proper parts of C: in virtue of the definition of the $<$-relation, since each of them are proper parts of C, they both are smaller than C. If we now compose C with a further thing D, we will obtain a bigger whole E. It seems that in this way, by applying FCP repeatedly, we can get bigger and bigger wholes. This only holds if one of the two things that we sum is not contained in the other. But what guarantees that, for any thing a, we can find a b, not contained in a? It is the fact that the universe is infinite that guarantees that the application of FCP can always result in bigger and bigger wholes. Because if we have infinitely many disjoint things at disposal, then given a certain thing a, there is always a distinct thing b (not contained in a) such that their composition results in a whole bigger than each of them. The picture is thus as follows: the universe is an actual infinity of substances, but not a whole.[56] There are only finite wholes, and given an arbitrary whole, there is always a bigger whole that can be obtained via FCP. This implies that Leibniz's mereology is not only gunky, but junky as well, meaning that each whole is a proper part of another whole.

§5. Conclusion. In this paper we presented a logical reconstruction of both Leibniz's Real Addition and his mereological calculus. Compared to other reconstructions of the former (such as [39, 52]), we interpreted the Real Addition calculus not as pertaining to logic conceived as a theory of (valid) inferences, but as a "mereological" theory (here I am using the adjective 'mereological' as referring to contemporary mereology). In doing this we followed the suggestion of [44].

A consequence of this interpretation is that the Real Addition calculus—RA (or RA^{∞})—corresponds to what we nowadays call mereology. When we think of mereology as a general calculus of individuals that can be applied to any object whatsoever, then, in Leibniz's own terms, we are dealing with the Real Addition

[56] On the metaphysical consequences of the claim that the world is not a whole see [15].

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calculus. It would thus be a mistake to 'translate' contemporary mereology into Leibniz's mereology (the LM system) without taking into account Real Addition.

The LM system is an extension of RA that corresponds to a restriction of containment to homogeneous things. It is a theory mainly conceived for mathematical purposes, as witnessed by the Part-Quantity Axiom which establishes a direct link between mereology and *Mathesis Universalis* conceived as the theory of (the measurement of) quantity. If we keep this in mind, it should not be surprising that the parthood relation turns out to be both junky and gunky. Indeed, if objects equipped with a quantitative aspect are wholes with parts, and "given a magnitude, it is possible to find a bigger or a smaller magnitude" (GM V 206), then the same should be true for parts and wholes. Indeed, it is the fact that the parthood relation is junky and gunky that grounds the fact that, given a magnitude or quantity, there are always bigger and smaller magnitudes:

> Let us take a straight line, and extend it to double its original length. It is clear that the second line, being perfectly similar to the first, can be doubled in its turn to yield a third line which is also similar to the preceding ones; and since the same principle is always applicable, it is impossible that we should ever be brought to a halt; and so the line can be lengthened to infinity (A VI, 6, 158/ [36, p. 158]).

Even though within this mereological framework, Leibniz defines key mathematical concepts (number, measure, infinite and infinitesimal, etc.), the theory should not be interpreted in foundational terms. Mereology is not what logic was meant to be—for example—for Frege. It is not the case that we can derive the whole of mathematics from the axioms of mereology; as a matter of fact, Leibniz was fully aware of the limits of mereology. His calculus is blind to the notion of structure, which has a central role in his mathematics, as witnessed by the *Analysis situs*. The fact that the notion of *situs* must be taken as primitive shows that Leibniz never meant to reduce mathematics to mereology, but rather mereology provides us with a general framework that should work together with other notions to produce more complex theories.

Acknowledgments. I would like to thank Vincenzo De Risi, Richard T.W. Arthur, David Rabouin, Wolfgang Lenzen, Jeffrey Elawani, Jacopo Giraldo and Fabio De Martin Polo for helpful comments on previous versions of this paper. This paper benefited from very helpful comments from two anonymous referees for *The Review of Symbolic Logic* (plus two further referees from another journal). Finally, I would like to thank the participants at the conferences *Philosophical Roots of Mathematical Logic* (Università di Torino, 4-6 March 2022) and *XI. Internationaler Leibniz Kongress* (Leibniz Universität Hannover, 31 July-4 August 2023), where I presented parts of the paper and received helpful feedback.

Funding. This paper is part of the REALE project that has received funding from the European Union's Horizon 2020 Research and Innovation Program under the Marie Skłodowska-Curie grant agreement no. 890812. The paper reflects only the author's view and the Research Executive Agency is not responsible for any use that may be made of the information it contains.

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https://doi.org/10.1017/S1755020325000012 Published online by Cambridge University Press