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# Enlarged Inclusion of Subdifferentials

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*Abstract.* This paper studies the integration of inclusion of subdifferentials. Under various verifiable conditions, we obtain that if two proper lower semicontinuous functions $f$ and $g$ have the subdifferential of $f$ included in the $\gamma$-enlargement of the subdifferential of $g$, then the difference of those functions is $\gamma$-Lipschitz over their effective domain.

## 1 Introduction

The famous Rockafellar integration result (see [13, 14]) concerning the subdifferentials of convex functions states that the inclusion

$$\partial f(x) \subset \partial g(x) \quad \text{for all } x \in E, \tag{1.1}$$

entails that the functions $f$ and $g$ are equal up to an additive constant whenever $E$ is a Banach space and the functions $f$ and $g$ from $E$ to $\mathbb{R} \cup \{+\infty\}$ are proper, convex, and lower semicontinuous. This result has been also established in [1, 3, 4, 15], for some classes of locally Lipschitz functions such as the directionally regular, semismooth, and separately regular real-valued functions. Poliquin [11] provided the first strong extension of (1.1) for nonconvex and non locally Lipschitz extended real valued functions. He showed that the integration result of (1.1) still holds whenever E is finite dimensional and the functions $f$ and $g$ are primal lower nice, a notion that he introduced in [12]. Note that those functions found several applications in the theory of second order epi-derivatives and proto-differentiability of nonsmooth functions.

Later, we extended in [19] the Poliquin theorem recalled above to the class of convexly subdifferentially similar functions defined over a Banach space. We also showed that this class includes the primal lower nice functions relative to infinite dimensional Hilbert spaces and also the differences of convex functions. The key point in that extension is the following statement of our Theorem 2.1 in [19]: If for some $\gamma \geq 0$ and for two lower semicontinuous functions $f, g$ from an open convex set $X$ of a Banach space $E$ into $\mathbb{R} \cup \{+\infty\}$ one has

$$\partial f(x) \subset \partial g(x) + \gamma \mathbb{B} \quad \text{for all } x \in X, \tag{1.2}$$

then for all $u \in X$ and $v \in X \cap \operatorname{dom} g$ one has

$$g(u) - g(v) - \gamma\|u - v\| \leq f(u) - f(v) \leq g(u) - g(v) + \gamma\|u - v\| \tag{1.3}$$

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whenever the function $g$ is convex on $X$. Since that paper, two articles have been devoted to the inclusion (1.2). Geoffroy, Jules and Lassonde [6] studied the inclusion (1.2) for some classes of subdifferentials, the function $g$ being still convex. Very recently, Ngai, Luc and Théra [10] showed that the result above concerning (1.2) still holds whenever $g$ is approximate convex. We also refer the reader to the paper [8] by Ivanov and Zlateva where the integration of subdifferentials of semi-convex functions is studied.

The purpose of this paper is to introduce a general class of extended real valued (not necessarily convex) functions $g$ for which the conclusion (1.3) above is preserved. After recalling some preliminaries, we define the functions of that class in section 3 with the name of *subdifferentially* and *directionally stable* functions. We show a stability result for the sum of two functions and we prove that convex functions, directionally regular functions and approximate convex functions are all subdifferentially and directionally stable functions. Then, we establish in the last section that the inclusion (1.2) entails the inequalities (1.3) whenever $g$ is subdifferentially and directionally stable. So, on the one hand, we provide a unified proof of several results concerning (1.2) and on the other hand, we bring to light, by the stability result for the sum and by Corollary 3.1, new classes for which (1.2) implies (1.3). The study of (1.2) with functions in the line of primal lower nice functions and differences of convex functions requires in addition a much longer development. It will then appear in a forthcoming paper.

## 2 Preliminaries

Throughout this paper, $E$ will be a real Banach space. For a function $f$ from $E$ into $\mathbb{R} \cup \{+\infty\}$, it is usual to denote by $\operatorname{dom} f$ the effective domain of $f$, *i.e.*, $\operatorname{dom} f = \{x \in E : f(x) < +\infty\}$.

We will consider in this paper a general presubdifferential operator. Recall that a presubdifferential operator $\partial$ associates with each function $f$ from $E$ into $\mathbb{R} \cup \{+\infty\}$ and with each point $x \in E$ a subset $\partial f(x)$ of the topological dual space $E^\star$ such that the following properties hold:

(1) $\partial f(x) = \varnothing$ if $x \notin \operatorname{dom} f$;
(2) $\partial f(x) = \partial g(x)$ whenever $f$ and $g$ coincide on a neighborhood of $x$;
(3) $\partial f(x)$ is equal to the subdifferential in the sense of convex analysis whenever $f$ is convex;
(4) $0 \in \partial f(x)$ whenever $x$ is a local minimum point of $f$;
(5) for $f$ lower semicontinuous near $x$ and $g$ convex and continuous on a neighborhood of $x$

$$\partial(g + f)(x) \subset \partial g(x) + \limsup_{y \to_f x} \partial f(y).$$

Here limsup denotes the weak-star sequential limit superior and $y \to_f x$ means $y \to x$ and $f(y) \to f(x)$. The effective domain of the set-valued mapping $\partial f$ is the set $\operatorname{dom} \partial f = \{x \in E : \partial f(x) \neq \varnothing\}$.

All usual subdifferentials or presubdifferentials (see [19]) over appropriate Banach spaces are subdifferentials in the sense above.

When $f\colon S \to \mathbb{R} \cup \{+\infty\}$ is just defined on a subset of $E$ and $x \in S$, we define $\partial f(x)$ as the presubdifferential at $x$ of the function extending $f$ to all of $E$ and equal to $+\infty$ at all points outside of $S$.

One of the virtues enjoyed by a presubdifferential operator is the following mean value theorem by Zagrodny [20] (see also [5]). In [20] the theorem has been stated with the Clarke subdifferential. However, it is not difficult to see (as it is made clear in [18]) that the theorem still holds for any presubdifferential satisfying the natural properties above. We use for $a \neq b$ in $E$ the notation $[a, b[ := \{(1-t)a + tb : t \in [0, 1[\}$.

***Theorem 2.1*** *Let $f\colon E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $a, b \in \operatorname{dom} f$ with $a \neq b$. Then there exist $x_n \to_f c \in [a, b[$, $x_n^\star \in \partial f(x_n)$ such that:*

(a) $f(b) - f(a) \leq lim_{n\to\infty} \langle x_n^\star, b - a\rangle$;

(b) $\frac{\|b-c\|}{\|b-a\|}(f(b) - f(a)) \leq lim_{n\to\infty} \langle x_n^\star, b - x_n\rangle$;

(c) $\|b - a\|(f(c) - f(a)) \leq \|c - a\|(f(b) - f(a))$.

Two important presubdifferentials will be considered in this paper: the Fréchet and the Clarke subdifferentials. Let $x \in \operatorname{dom} f$. The Fréchet subdifferential $\partial^F f(x)$ of $f$ at $x$ is the set of all $x^\star \in E^\star$ for which for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $y \in x + \delta\mathbb{B}$

$$\langle x^\star, y - x\rangle \leq f(y) - f(x) + \varepsilon\|y - x\|. \tag{2.1}$$

Here $\mathbb{B}$ denotes the closed unit ball around the origin. All properties (1)–(5) hold for $\partial^F f$ whenever the Banach space $X$ is an Asplund space (see *e.g.*, [9]).

Supposing that $f$ is lower semicontinuous, the Clarke subdifferential $\partial^C f(x)$ is equal to the set of all $x^\star \in E^\star$ such that for all $h \in E$

$$\langle x^\star, h\rangle \leq f^\uparrow(x; h)$$

where

$$f^\uparrow(x; h) = \sup_{\eta>0} \limsup_{\substack{y \to_f x \\ t\downarrow 0}} \inf_{v \in h + \eta\mathbb{B}} t^{-1}[f(y + tv) - f(y)] \tag{2.2}$$

(see [2] for properties (1)–(5) ). Observe that the obvious monotonicity property with respect to $\eta$ allows us to see that $f^\uparrow(x; h)$ is also equal to the expression above with $\lim_{\eta\downarrow 0}$ in place of $\sup_{\eta>0}$.

When $f$ is Lipschitz near $x$, then (see [2]) the expression of $f^\uparrow(x; h)$ takes the following simpler form

$$f^\uparrow(x; h) = \limsup_{\substack{y \to x \\ t\downarrow 0}} t^{-1}[f(y + th) - f(y)]. \tag{2.3}$$

## 3 Subdifferentially and Directionally Stable Functions

In this section we introduce a general class of functions for which the $\gamma$-enlargement of inclusion of subdifferentials entails that the difference of the functions is $\gamma$-Lipschitz over their effective domain. Then we study several examples of such functions. The enlarged subdifferential inclusion result above will be established in section 4 via the mean value theorem recalled in the preliminaries.

Recall that the usual directional derivative $g'(x;h)$ of a function $g : E \to \mathbb{R} \cup \{+\infty\}$ at a point $x \in \operatorname{dom} g$ in the direction $h$ is given by the limit

$$g'(x;h) = \lim_{t \downarrow 0} t^{-1}[g(x+th) - g(x)]$$

when the latter exists in $\mathbb{R} \cup \{-\infty, +\infty\}$.

***Definition 3.1*** Let $X$ be a nonempty open convex subset of $E$ and $g \colon E \to \mathbb{R} \cup \{+\infty\}$ be a function that is lower semicontinuous on $X$ with $X \cap \operatorname{dom} g \neq \varnothing$. We say that the function $g$ is $\partial$-*subdifferentially* and *directionally stable* (*sds* for short) on $X$ provided that for all $v \in X \cap \operatorname{dom} \partial g$ and $u \in X \cap \operatorname{dom} g$ with $u \neq v$ the following properties hold:

(i) the restriction to $[0,1]$ of the function $t \mapsto g(v + t(u-v))$ is finite and continuous;

(ii) for any $t \in [0,1[$ the directional derivative $g'(v + t(u-v); u-v)$ exists and is less than $+\infty$;

(iii) for each fixed $y \in [v,u[$ and for each real number $\varepsilon > 0$, there exists some $r_0 \in ]0,1[$ such that for any $w = y + r(u-y)$ with $r \in ]0, r_0]$ and for every sequence $(x_n, x_n^\star) \in \partial g$ with $x_n \to x_0 \in \big[y, w\big[$

$$\liminf_{n \to \infty} \langle x_n^\star, w - x_n \rangle \leq g'(y; w - x_0) + \varepsilon \|w - x_0\|. \tag{3.1}$$

***Remark 3.1*** (a) One may replace lim inf in (3.1) by lim sup.

Indeed choose a subsequence (with an infinite subset $K \subset \mathbb{N}$) such that

$$\limsup_{n \to \infty} \langle x_n^\star, w - x_n \rangle = \lim_{k \in K} \langle x_k^\star, w - x_k \rangle.$$

So, by (3.1) one has

$$\lim_{k \in K} \langle x_k^\star, w - x_k \rangle \leq g'(y; w - x_0) + \varepsilon \|w - x_0\|$$

which finishes the verification.

(b) If $g$ is $\partial$-sds on $X$, then it is obviously $\partial'$-sds on $X$ for any subdifferential $\partial'$ satisfying $\partial' g \subset \partial g$.

We first observe the preservation of the sds property under the sum of a finite number of functions.

***Proposition 3.1*** *Let $X$ be a nonempty open convex subset of $E$ and let $g_1, g_2 \colon E \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and sds over $X$ with $X \cap \operatorname{dom} g_1 \cap \operatorname{dom} g_2 \neq \varnothing$. Assume that for any $x \in X$*

$$\partial(g_1 + g_2)(x) \subset \partial g_1(x) + \partial g_2(x). \tag{3.2}$$

*Then the function $g_1 + g_2$ is sds over $X$.*

**Proof** Putting $g := g_1 + g_2$ one obviously has $\operatorname{dom} g = \operatorname{dom} g_1 \cap \operatorname{dom} g_2$ and hence $X \cap \operatorname{dom} g \neq \varnothing$. Fix any $v \in X \cap \operatorname{dom} \partial g$ and $u \in X \cap \operatorname{dom} g$. By assumptions $v \in X \cap \operatorname{dom} \partial g_i$ and $u \in X \cap \operatorname{dom} g_i$ for $i = 1, 2$. Consequently, conditions (i) and (ii) of Definition 3.1 easily are seen to hold for the function $g$ because they hold by assumption for $g_1$ and $g_2$. Fix now any $y \in [v, u[$ and any real number $\varepsilon > 0$. Choose some $r_0$ by the assumption of condition (iii) for $g_1$ and $g_2$ with $\varepsilon/2$ in place of $\varepsilon$ and with the limit superior according to Remark 3.1. Take any $r \in ]0, r_0]$ and put $w := y + r(u - y)$. Consider also any sequence $(x_n, x_n^\star) \in \partial g$ with $x_n \to x_0 \in \big[y, w\big[$. By (3.1), for $i = 1, 2$, there exist $x_{i,n}^\star \in \partial g_i(x_n)$ with $x_n^\star = x_{1,n}^\star + x_{2,n}^\star$. Due to the choice of $r_0$ we have for $i = 1, 2$

$$\limsup_{n\to\infty} \langle x_{i,n}^\star, w - x_n \rangle \leq g_i'(y; w - x_0) + \frac{\varepsilon}{2} \|w - x_0\|. \tag{3.3}$$

Since by (ii) and the definition of the directional derivative one has

$$g'(y; w - x_0) = g_1'(y; w - x_0) + g_2'(y; w - x_0),$$

additioning the two inequalities (3.3) yields after small rearrangement

$$\liminf_{n\to\infty} \langle x_n^\star, w - x_n \rangle \leq g'(y; w - x_0) + \varepsilon \|w - x_0\|,$$

*i.e.*, condition (iii) for $g$. ∎

***Remark 3.2*** Similar preservations also hold with similar proofs under several sorts of compositions (precomposition or postcomposition) with smooth functions.

We proceed now to illustrate the class of sds functions with several concrete examples. Let us start by establishing that any convex function is sds.

***Proposition 3.2*** *Let $X$ be a nonempty open convex subset of $E$. Then any function $g \colon E \to \mathbb{R} \cup \{+\infty\}$ that is convex, proper, and lower semicontinuous on $X$ is sds over $X$.*

**Proof** It is well-known that, for $u, v \in X \cap \operatorname{dom} g$, the convexity of $g$ ensures that (i) in Definition 3.1 holds and that, for $t \in [0, 1[$, the directional derivative $g'(v + t(u - v); u - v)$ exists and, for $s \in ]0, 1 - t]$,

$$g'(v + t(u - v); u - v) \leq s^{-1}[g(v + (s + t)(u - v)) - g(v + t(u - v))] < +\infty.$$

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So, it remains to show (iii) in Definition 3.1. Fix $v, u$ and $y$ as in the statement of Definition 3.1 and fix also $v^\star \in \partial g(v)$. The function $\varphi \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ with

$$\varphi(s) = \begin{cases} +\infty & \text{if } s \in \,]1, +\infty[\,, \\ g(v) + s\langle v^\star, u - v\rangle & \text{if } s \in \,]-\infty, 0[\,, \\ g(v + s(u - v)) & \text{if } s \in [0, 1] \end{cases}$$

is convex and finite on $]-\infty, 1[$ and hence it is locally Lipschitz on $]-\infty, 1[$ (see [2]). The right derivative $\varphi'$ of $\varphi$ is then finite on $]-\infty, 1[$ and it is also upper semicontinuous on $]-\infty, 1[$ because

$$\varphi'(s) = \inf_{\lambda > 0} \lambda^{-1}[\varphi(s + \lambda) - \varphi(s)]$$

and hence the restriction of the function $s \mapsto g'(v + s(u - v); u - v)$ is finite and upper semicontinuous on $[0, 1[$. Fix any real number $\varepsilon > 0$ and choose $s_0$ satisfying $y = v + s_0(u - v)$. As $g'(y; u - v)$ is finite, there exists some $\lambda \in \,]0, 1 - s_0[$ such that for all $s \in \,]s_0, s_0 + \lambda]$

$$\|u - v\|^{-1} g'(v + s(u - v); u - v) \le \|u - v\|^{-1} g'(y; u - v) + \varepsilon. \tag{3.4}$$

Put $r_0 := (1 - s_0)^{-1}\lambda \in \,]0, 1[$ and for any fixed $r \in \,]0, r_0]$ consider $w$ and $(x_n, x_n^\star)$ as in the statement of Definition 3.1. Then, by convexity of $g$ one has

$$\langle x_n^\star, w - x_n\rangle \le g(w) - g(x_n)$$

which entails by the lower semicontinuity and the convexity of $g$

$$\liminf_{n \to \infty} \langle x_n^\star, w - x_n\rangle \le g(w) - g(x_0) \le g'(w; w - x_0). \tag{3.5}$$

But (3.4), with $s := s_0 + r(1 - s_0) \in \,]s_0, s_0 + \lambda]$, entails that (because $w = v + s(u - v)$)

$$\|w - x_0\|^{-1} g'(w; w - x_0) \le \|w - x_0\|^{-1} g'(y; w - x_0) + \varepsilon,$$

that is,

$$g'(w; w - x_0) \le g'(y; w - x_0) + \varepsilon\|w - x_0\|.$$

This inequality and (3.5) complete the proof. ■

The next example concerns locally Lipschitz functions that are regular in the sense of Clarke (see [2]). Recall that a locally Lipschitz function $g$ from an open subset $X$ of $E$ into $\mathbb{R}$ is Clarke directionally regular at a point $x \in X$ provided for all $h \in E$ the expression $g^\uparrow(x; h)$ coincides with the Dini lower directional derivative,

$$d^- g(x; h) = \liminf_{\substack{v \to h \\ t \downarrow 0}} t^{-1}[g(x + tv) - g(x)]. \tag{3.6}$$

It is well-known that this amounts (thanks to the Lipschitz property) to the existence of the usual directional derivative $g'(x; h)$ and to the equality

$$g^\uparrow(x; h) = g'(x; h).$$

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***Proposition 3.3*** *Let $X$ be a nonempty open convex subset of $E$ and $g\colon X \to \mathbb{R}$ be a locally Lipschitz function. Suppose that $\partial g$ is included in the Clarke subdifferential of $g$. Then $g$ is $\partial$-sds over $X$ provided it is Clarke directionally regular on $X$, and the converse also holds whenever* $\operatorname{dom} \partial g = X$.

**Proof** Suppose that $g$ is Clarke directionally regular. Obviously, (i) and (ii) in Definition 3.1 hold. Let $v$, $u$, and $y := v + s_0(u - v)$ with $0 \le s_0 < 1$ as in the statement of Definition 3.1 and let $\varepsilon > 0$. As the function $x \mapsto g^{\uparrow}(x; u - v)$ is finite and upper semicontinuous (by (2.3)) on $X$, we may choose some $\lambda \in \,]0, 1 - s_0[$ such that for all $s \in \,]s_0, s_0 + \lambda]$

$$\|u - v\|^{-1} g^{\uparrow}(v + s(u - v); u - v) \le \|u - v\|^{-1} g^{\uparrow}(y; u - v) + \varepsilon. \tag{3.7}$$

Putting $r_0 := (1 - s_0)^{-1}\lambda$ in $]0, 1[$, and for any fixed $r \in \,]0, r_0]$ considering $w$ and $(x_n, x_n^\star)$ as in the statement of Definition 3.1, one obtains by the upper semicontinuity of $g^{\uparrow}(\,\cdot\,;\,\cdot\,)$ (because of the Lipschitz property of $g$)

$$\liminf_{n\to\infty} \langle x_n^\star, w - x_n\rangle \le \liminf_{n\to\infty} g^{\uparrow}(x_n; w - x_n) \le g^{\uparrow}(x_0; w - x_0). \tag{3.8}$$

Choosing $\delta \in [0, r[$ such that $x_0 = y + \delta(u - y)$, we also have for $s := s_0 + \delta(1 - s_0)$

$$x_0 = v + s(u - v) \quad \text{along with} \quad s_0 < s \le s_0 + \lambda,$$

and hence taking (3.7) into account we get

$$\|u - v\|^{-1} g^{\uparrow}(x_0; u - v) \le \|u - v\|^{-1} g^{\uparrow}(y; u - v) + \varepsilon,$$

which is equivalent to

$$g^{\uparrow}(x_0; w - x_0) \le g^{\uparrow}(y; w - x_0) + \varepsilon\|w - x_0\|,$$

because $\|u - v\|^{-1}(u - v) = \|w - x_0\|^{-1}(w - x_0)$. The property (iii) in Definition 3.1 then follows from this last inequality, from (3.8), and from the Clarke directional regularity of $g$.

Let us now prove the converse. Fix $v \in X$ and a nonzero vector $h \in E$ and take any $\varepsilon > 0$. Observe first by (ii) in Definition 3.1 that $g'(v; h)$ exists. Choose by (2.3) a sequence $(z_n)$ in $X$ with $z_n \to v$ and a sequence $t_n \downarrow 0$ with $t_n < 1$ such that

$$g^{\uparrow}(v; h) = \lim_{n\to\infty} t_n^{-1}[g(z_n + t_n h) - g(z_n)].$$

The mean value theorem gives $x_{n,k} \underset{k}{\to} u_n \in [z_n, z_n + t_n h[$ and $x_{n,k}^\star \in \partial g(x_{n,k})$ such that

$$t_n^{-1}[g(z_n + t_n h) - g(z_n)] \le \lim_{k\to\infty} \langle x_{n,k}^\star, h\rangle.$$

It is not difficult to find a sequence of integers $(k_n)$ satisfying $x_{n,k_n} \to v$ along with

$$\lim_{k\to\infty} \langle x_{n,k}^\star, h\rangle \le \langle x_{n,k_n}^\star, h\rangle + t_n.$$

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Putting $x_n := x_{n,k_n}$ and $x_n^\star := x_{n,k_n}^\star$, applying (iii) in Definition 3.1 with $u = v + \rho h \in X$ and $\rho \in \,]0,1[$ and with $y = v \in \operatorname{dom} \partial g$ one obtains some $r \in \,]0,1[$ such that for $w = v + r(u - v)$, one has (because $(\|x_n^\star\|)_n$ is bounded)

$$g^\uparrow(v;h) \le \liminf \langle x_n^\star, h\rangle = (r\rho)^{-1} \liminf \langle x_n^\star, w - x_n\rangle \le g'(v;h) + \varepsilon\|h\|.$$

As this holds for any $\varepsilon > 0$, it follows that $g^\uparrow(v;h) \le g'(v;h)$, which means that $g$ is Clarke directionally regular at $v$. The proof is then complete. ■

The third example is the class of approximate convex functions recently introduced by Ngai–Luc–Thera [10]. They thoroughly studied that class and proved in particular that our Theorem 2.1 in [19] still holds provided one replaces our convexity assumption for the function $g$ by the assumption of approximate convexity. We proceed to showing that approximate convex functions belong to the class of Definition 3.1. So, the result recalled above will be included in Theorem 4.1.

Recall that a function $g\colon E \to \mathbb{R} \cup \{+\infty\}$ is approximate convex at a point $\bar{x} \in \operatorname{dom} g$ provided for each $\rho > 0$ there exists a real number $\delta > 0$ such that for all $x', y' \in \bar{x} + \delta\mathbb{B}$ and $\lambda \in \,]0,1[$

$$g(\lambda x' + (1-\lambda)y') \le \lambda g(x') + (1-\lambda)g(y') + \rho\lambda(1-\lambda)\|x' - y'\|. \tag{3.9}$$

As in the convex case, we observe the following monotone-like properties. Let $x \in \bar{x} + \frac{1}{2}\delta\mathbb{B}$, $h \in E$ and $t > 0$ with $t\|h\| < \delta/2$. Take any $s \in \,]0,t[$ and any $r > 0$ with $r\|h\| < \delta/2$. Then writing

$$x = s(r+s)^{-1}(x - rh) + r(r+s)^{-1}(x + sh)$$

and applying (3.9) with $\lambda = s(r+s)^{-1}$ we obtain

$$(r+s)g(x) \le sg(x - rh) + rg(x + sh) + \rho rs\|h\|$$

and hence for $x \in \operatorname{dom} g$

$$-r^{-1}[g(x - rh) - g(x)] \le s^{-1}[g(x + sh) - g(x)] + \rho\|h\|. \tag{3.10}$$

In the same way, with $\lambda = s/t,\ x + th$ in place of $x'$, and $x$ in place of $y'$, we obtain

$$g(x + \lambda th) \le \lambda g(x + th) + (1-\lambda)g(x) + \rho t\lambda(1-\lambda)\|h\|$$

and hence, taking the inequality $1 - \lambda \le 1$ into account, we deduce

$$s^{-1}[g(x + sh) - g(x)] \le t^{-1}[g(x + th) - g(x)] + \rho\|h\|. \tag{3.11}$$

Observe that any strictly differentiable function is easily seen to be approximate convex and that the class is obviously stable with respect to sum and maximum of finite families.

Consider now the case of approximate convex functions over the real line. Let $I$ be an interval in $\mathbb{R}$ and $g\colon I \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous and approximate convex at each point in $I$. As for convex functions, the restriction of $g$ to $[r, s]$ is continuous for any interval $[r, s] \subset \operatorname{dom} g$ with $r < s$. Let us prove, for example, that $g$ is continuous on the right at $r$. Fix $\delta > 0$ given by (3.9) for $\bar{x} = r$ and $\rho = 1$. For $t \in ]r, \sigma[$ where $\sigma := \min(s, r + \delta)$ one has

$$g(t) \leq \frac{t-r}{\sigma - r} g(\sigma) + \frac{\sigma - t}{\sigma - r} g(r) + \frac{(t-r)(\sigma - t)}{(\sigma - r)^2} |\sigma - r|$$

and hence $\limsup_{t \downarrow r} g(t) \leq g(r)$. As $g$ is lower semicontinuous, we obtain that $g$ is continuous on the right at $r$. Similar arguments show that $g$ is continuous on the left at $s$ and continuous at each point in $]r, s[$. (More generally, it can be proved that $g$ is locally Lipschitz on $]r, s[$, see [10]).

We begin now by showing in Proposition 3.4 some properties that will be needed next. They can also be found in [10] where an $\varepsilon$-approximation in some sense by a convex function $g_y$ is used to derive the properties (see [10, Theorems 3.4 and 3.6]). For the convenience of the reader, we give direct proofs. For the description of subdiferentials of other convex-like or paraconvex functions, we refer the reader to [7, 16].

***Proposition 3.4*** *Let $g\colon E \to \mathbb{R} \cup \{+\infty\}$ be approximate convex at $x \in \operatorname{dom} g$.*

(a) *Then $g'(x; \cdot)$ exists and is a positively homogeneous convex function.*

(b) *Further, if $g$ is lower semicontinuous at $x$, then*

$$\partial^c g(x) = \partial^F g(x) = \{x^\star \in E^\star : \langle x^\star, h\rangle \leq g'(x; h),\ \forall h \in E\}.$$

**Proof** (a)Let $h$ be a vector in $E$ and let $\rho$ be any positive number. Choose $\delta > 0$ given by (3.9) and fix $\sigma > 0$ with $\sigma\|h\| < \delta/2$. Fix also $0 < s < t < \sigma$. It follows from (3.11) that we have

$$s^{-1}[g(x + sh) - g(x)] \leq t^{-1}[g(x + th) - g(x)] + \rho\|h\|.$$

Fixing $t$ and taking $s \downarrow 0$, it gives

$$\limsup_{s \downarrow 0} s^{-1}[g(x + sh) - g(x)] \leq t^{-1}[g(x + th) - g(x)] + \rho\|h\|$$

and this entails that

$$\limsup_{s \downarrow 0} s^{-1}[g(x + sh) - g(x)] \leq \liminf_{t \downarrow 0} t^{-1}[g(x + th) - g(x)] + \rho\|h\|.$$

As this last inequality holds for all $\rho > 0$, we obtain that $g'(x; h)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$ .

The positive homogeneity being obvious, it remains to prove the convexity of $g'(x; \cdot)$. Fix any $(h, \alpha)$ and $(h', \beta)$ in $E \times \mathbb{R}$ and satisfying $g'(x; h) < \alpha$ and

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$g'(x;h') < \beta$. Choose some $\gamma > 0$ satisfying $2\gamma(\|h\| + \|h'\|) < \delta$ and such that for all $0 < t < \gamma$,

$$(2t)^{-1}[g(x + 2th) - g(x)] < \alpha \quad \text{and} \quad (2t)^{-1}[g(x + 2th') - g(x)] < \beta.$$

For every $t \in ]0, \gamma[$, it follows, from (3.9) with $y' = x + 2th'$ and with $x + 2th$ in place of $x'$, that one has

$$g(x + th + th') \leq \frac{1}{2}g(x + 2th) + \frac{1}{2}g(x + 2th') + \frac{1}{2}t\rho\|h - h'\|,$$

and this is equivalent to

$$\begin{aligned} t^{-1}[g(x + th + th') - g(x)] \leq (2t)^{-1}&[g(x + 2th) - g(x)] \\ &+ (2t)^{-1}[g(x + 2th') - g(x)] + \frac{1}{2}\rho\|h - h'\| \end{aligned}$$

and that entails

$$t^{-1}[g(x + th + th') - g(x)] \leq \alpha + \beta + \frac{1}{2}\rho\|h - h'\|.$$

Therefore, we obtain

$$g'(x; h + h') \leq \alpha + \beta + \frac{1}{2}\rho\|h - h'\|$$

and hence

$$g'(x; h + h') \leq \alpha + \beta.$$

This completes the proof of assertion (a).

(b)Fix $x^\star \in \partial^c g(x)$ and fix any $\rho > 0$. Take $\delta > 0$ given by (3.9) and choose $\sigma > 0$ with $4\sigma < \delta$. Fix any vector $h$ with $\|h\| \leq \sigma$. For $\eta \in \left]0, \delta/4\right[$, $h' \in E$ with $\|h' - h\| < \eta$, $t \in ]0, 1[$, and for

$$|y - x|_g := \|y - x\| + |g(y) - g(x)| < \delta/3$$

we have by (3.11), with $t$ in place of $s$ and 1 in place of $t$,

$$\begin{aligned} t^{-1}[g(y + th') - g(y)] &\leq g(y + h') - g(y) + \rho\|h'\| \\ &\leq g(y + h') - g(y) + \rho(\eta + \|h\|). \end{aligned}$$

Hence

$$\begin{aligned} &\limsup_{\substack{y \to_g x \\ t \downarrow 0}} \inf_{h' \in h + \eta\mathbb{B}} t^{-1}[g(y + th') - g(y)] \\ &\qquad \leq \rho(\eta + \|h\|) + \limsup_{y \to_g x} \inf_{h' \in h + \eta\mathbb{B}} [g(y + h') - g(y)] \\ &\qquad = \rho(\eta + \|h\|) - g(x) + \limsup_{y \to_g x} \inf_{h' \in h + \eta\mathbb{B}} g(y + h'). \end{aligned}$$

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Observe that the last term of the last member above is a non increasing function of $\eta$ over the open interval $]0, +\infty[$. The inequality $\langle x^\star, h\rangle \le g^\uparrow(x; h)$ thus entails

$$\langle x^\star, h\rangle \le \rho\|h\| - g(x) + \lim_{\eta\downarrow 0}\limsup_{y\to_g x}\inf_{h'\in h+\eta\mathbb{B}} g(y+h'). \tag{3.12}$$

For any real number

$$\beta < \sup_{\eta>0}\inf_{\lambda>0}\sup_{|y-x|_g<\lambda}\inf_{h'\in h+\eta\mathbb{B}} g(y+h'),$$

there exists some $\eta \in \left]0, \delta/4\right[$ such that

$$\beta < \inf_{\lambda>0}\sup_{|y-x|_g<\lambda}\inf_{h'\in h+\eta\mathbb{B}} g(y+h') \le \sup_{|y-x|_g<\eta}\inf_{h'\in h+\eta\mathbb{B}} g(y+h'),$$

which yields the existence of some $y_\eta$ with $|y_\eta - x|_g < \eta$ and for which

$$\beta < \inf_{h'\in h+\eta\mathbb{B}} g(y_\eta + h') \le g(y_\eta + h + (x - y_\eta)) = g(x+h).$$

Taking (3.12) into account we get

$$\langle x^\star, h\rangle \le g(x+h) - g(x) + \rho\|h\|$$

and hence

$$x^\star \in \partial^F g(x).$$

So, $\partial^c g(x) \subset \partial^F g(x)$ and the equality follows because the converse inclusion always holds (as it is easily seen).

It remains to prove that any $x^\star$ satisfying $\langle x^\star, h\rangle \le g'(x; h)$ for all $h \in E$ belongs to $\partial^F g(x)$ (because the converse property obviously holds). Fix such an $x^\star$ and choose for any $\rho > 0$ a real number $\delta > 0$ such that (3.9) holds for $\bar{x} = x$. For any $y \in x + \frac{1}{2}\delta\mathbb{B}$ we have according to (3.11) with $t = 1$ and $s \downarrow 0$ and according to the first assertion of the proposition,

$$\langle x^\star, y - x\rangle \le g'(x; y - x) \le g(y) - g(x) + \rho\|y - x\|$$

and this ensures $x^\star \in \partial^F g(x)$. The proof is then complete. ■

The following properties of approximate convex functions need to be observed and they will be used in the next proposition. Let $g\colon E \to \mathbb{R}\cup\{+\infty\}$ be approximate convex at $\bar{x} \in \operatorname{dom} g$ and let for $\rho > 0$ a positive number $\delta$ for which (3.9) holds. Consider $x \in (\bar{x} + \frac{1}{2}\delta\mathbb{B}) \cap \operatorname{dom} g$ and suppose that $g$ is approximate convex at the point $x$ too. Then, as it is easily seen in the proof above, (3.11) and the first assertion of Proposition 3.4 imply for $t > 0$ with $t\|h\| < \delta/2$

$$g'(x; h) \le t^{-1}[g(x+th) - g(x)] + \rho\|h\|. \tag{3.13}$$

Further, if $x$ belongs in fact to $\bar{x} + \frac{1}{4}\delta\mathbb{B}$, then for all $y \in \bar{x} + \frac{1}{4}\delta\mathbb{B}$ and $x^\star \in \partial^c g(x)$ the description of $\partial^c g(x)$ in Proposition 3.4 in terms of $g'(x; \cdot)$ and the inequality (3.13) with $t = 1$ yield

$$\langle x^\star, y - x\rangle \le g(y) - g(x) + \rho\|y - x\|. \tag{3.14}$$

***Proposition 3.5*** *Let $X$ be a nonempty open convex subset of $E$ and $g\colon X \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function that is approximate convex at any point in* dom $g$, *with* dom $g$ *being convex. Then $g$ is $\partial$-sds over $X$ provided $\partial g(x) \subset \partial^c g(x)$ for all $x \in X$.*

**Proof** The analysis just in front of Proposition 3.4 says that Definition 3.1(i) holds. Further, (ii) of the same definition follows from Proposition 3.4 and from (3.13). Let us prove (iii). Let $v$ and $u$ as in the statement of Definition 3.1 and let $\varepsilon > 0$. Fix any $y \in [v, u[$. Choose $\delta > 0$ such that (3.9) holds over $y + \delta\mathbb{B}$ with $\frac{\varepsilon}{4}$ in place of $\rho$. For any $t > 0$ with $t\|u - v\| < \delta/2$ we obtain from (3.13),

$$g'(y+t(u-v);u-v) \le t^{-1}[g(y+2t(u-v)) - g(y+t(u-v))] + \frac{\varepsilon}{4}\|u-v\|. \tag{3.15}$$

Now observe on the one hand that $g'(y; u - v) < +\infty$ because by (3.13)

$$g'(y; u - v) \le t^{-1}[g(y + t(u - v)) - g(y)] + \frac{\varepsilon}{4}\|u - v\|$$

and $y + t(u - v) \in \operatorname{dom} g$ for $t$ small enough. On the other hand, if $y \neq v$ we have $g'(y; u - v) > -\infty$ because by (3.10)

$$g(y) - g(v) \le \gamma(g'(y; u - v) + \frac{\varepsilon}{4}\|u - v\|)$$

for some real number $\gamma > 0$, and if $y = v$ we still have $g'(y; u - v) > -\infty$ taking the nonemptiness of $\partial g(v)$ into account. So, $g'(y; u - v)$ is finite. Further, using Proposition 3.4 it is easy to see that the equality

$$\lim_{t\downarrow 0} t^{-1}[g(y + 2t(u - v)) - g(y + t(u - v)] = g'(y; u - v) \tag{3.16}$$

holds. It then follows from (3.15) and (3.16) that there exists some $r_0 \in\, ]0, 1[$ with $r_0\|u - v\| < \delta/4$ such that for all $t \in\, ]0, r_0]$

$$g'(y + t(u - v); u - v) \le g'(y; u - v) + \frac{\varepsilon}{2}\|u - v\|. \tag{3.17}$$

For any fixed $r \in\, ]0, r_0]$, put $w := y + r(u - y)$ and consider any sequence $(x_n, x_n^\star) \in \partial g$ with $(x_n)$ converging to some $x_0 \in [y, w[$. Then (3.14) says that we have for $n$ large enough

$$\langle x_n^\star, w - x_n\rangle \le g(w) - g(x_n) + \frac{\varepsilon}{4}\|w - x_n\|$$

and hence using the lower semicontinuity of $g$

$$\liminf_{n\to\infty}\langle x_n^\star, w - x_n\rangle \le g(w) - g(x_0) + \frac{\varepsilon}{4}\|w - x_0\|.$$

Taking (3.10) into account, the latter yields

$$\liminf_{n\to\infty}\langle x_n^\star, w - x_n\rangle \le g'(w; w - x_0) + \frac{\varepsilon}{2}\|w - x_0\|. \tag{3.18}$$

As $w = y + r(u - y) = y + s(u - v)$ for some $s \in \,]0, r]$, (3.17) gives

$$g'(w; u - v) \leq g'(y; u - v) + \frac{\varepsilon}{2}\|u - v\|$$

and this inequality may be written in the form

$$g'(w; w - x_0) \leq g'(y; w - x_0) + \frac{\varepsilon}{2}\|w - x_0\|. \tag{3.19}$$

According to (3.18) and (3.19), we conclude

$$\liminf_{n\to\infty}\langle x_n^\star, w - x_n\rangle \leq g'(y; w - x_0) + \varepsilon\|w - x_0\|. \qquad \blacksquare$$

The three examples of functions $g$ above given by Propositions 3.2, 3.3, and 3.5 were already known, with separate different proofs, to satisfy the integration property (1.1). Actually, the class of sds functions is much larger as the following corollary says.

***Corollary 3.1*** Let $X$ be a nonempty open convex subset of $E$ and $g_1\colon X \to \mathbb{R}\cup\{\infty\}$ be a lower semicontinuous function which is approximate convex with dom $g_1$ being convex. Let $g_2\colon X \to \mathbb{R}$ be locally Lipschitz and Clarke directionally regular on $X$. Assume that $\partial$ is the Clarke subdifferential. Then $g_1 + g_2$ is sds over $X$.

**Proof** We know by Theorem 2.9.8 in [2] that assumption (3.2) is satisfied. Thus the corollary is a direct consequence of Propositions 3.1, 3.3, and 3.5. $\blacksquare$

There are Clarke directionally regular functions which are not approximate convex. To see this, consider the example of Spingarn [17, p. 83] of any even function $f\colon \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}$ such that

(i) $f(0) = 0$ and $f(\frac{1}{n}) = \frac{1}{n} - \frac{1}{n^2}$ for all integers $n \geq 2$;

(ii) for each integer $n \geq 2$, the usual derivative $\nabla f$ exists and is continuous and decreasing on the open interval $\left]\frac{1}{n+1}, \frac{1}{n}\right[$;

(iii) $f'_+(\frac{1}{n+1}) = 1$, and $f'_-(\frac{1}{n}) = 0$ for all integers $n \geq 2$, where $f'_+$ and $f'_-$ denote the right and left derivatives respectively.

For such a function $f$, one has $|x| - x^2 \leq f(x) \leq |x|$ for all $x \in \left]-\frac{1}{2}, \frac{1}{2}\right[$ and hence $f'(0; v) = |v|$ for all $v \in \mathbb{R}$. So $\partial^F f(0) = [-1, 1]$ and the Lipschitz function $f$ is Clarke directionally regular on $\left]-\frac{1}{2}, \frac{1}{2}\right[$. Observing that $f'(\frac{1}{n}; -1) = -f'_-(\frac{1}{n}) = 0$ we may choose for each $n$ some $t_n \in \left]0, \frac{1}{n}\right[$ such that

$$\frac{f\left(\frac{1}{n} - t_n(\frac{1}{n} - \frac{1}{n+1})\right) - f(\frac{1}{n})}{t_n(\frac{1}{n} - \frac{1}{n+1})} \geq -\frac{1}{4}. \tag{3.20}$$

As for

$$q_n := \frac{f(\frac{1}{n+1}) - f(\frac{1}{n})}{\frac{1}{n} - \frac{1}{n+1}},$$

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we have $\lim_{n\to\infty} q_n = -1$; we see that $\lim_{n\to\infty}(q_n + \frac{1}{2}(1-t_n)) = -\frac{1}{2}$ and hence for every integer $n$ sufficiently large

$$-\frac{1}{4} > q_n + \frac{1}{2}(1-t_n).$$

Combining the latter inequality and (3.20), we get

$$f\Big(\frac{1}{n} - t_n\Big(\frac{1}{n} - \frac{1}{n+1}\Big)\Big) > \frac{1}{2}t_n(1-t_n)\Big(\frac{1}{n} - \frac{1}{n+1}\Big) + (1-t_n)f\Big(\frac{1}{n}\Big) + t_n f\Big(\frac{1}{n+1}\Big)$$

which contradicts the approximate convexity of $f$ at 0 (the number $\rho$ in (3.9) cannot be smaller than $1/2$). So the Lipschitz function $f$ is Clarke directionally regular but not approximate convex at 0.

It is also obvious for a function $f$ as above that if $g$ is approximate convex but not locally Lipschitz, then the function $f + g$ may be neither locally Lipschitz nor approximate convex. This means that the class of functions which can be written in the form $g_1 + g_2$ of the above corollary is actually larger than the ones of Propositions 3.3 and 3.5.

In the next section we will establish the extended integration property related to (1.2) for sds functions. So we may point out that the fact that such a property holds for functions $g_1 + g_2$ as provided by Corollary 3.1 is new. Indeed, even in Hilbert space such functions may be non primal lower nice in the sense of Poliquin.

There are also sds functions $g$ which cannot be written in the form $g_1 + g_2$ as in Corollary 3.1. Indeed, as it can be checked, the function $g$ defined on $\mathbb{R}^2$ by

$$g(x, y) = \begin{cases} \sqrt{y} - 2\sqrt{x} & \text{if } 0 \le x \le y \\ \infty & \text{otherwise} \end{cases}$$

is sds but it is not in the class of functions of Corollary 3.1.

## 4 $\gamma$-Inclusion of Subdifferentials

This section is devoted to showing how some techniques developed in our paper [19] can be adapted to the study of enlarged inclusion of subdifferentials of functions in the class of sds functions introduced in the previous section. The famous Rockafellar integration result (see [13, 14]) states that the inclusion $\partial f(x) \subset \partial g(x)$ for every $x \in E$ and for $f$ and $g$ convex and lower semicontinuous ensures that the functions $f$ and $g$ are equal up to an additive constant. Theorem 2.1 in [19] extends the Rockafellar result in the sense that the convexity assumption is required only for $g$ and the inclusion above is replaced by the inclusion $\partial f(x) \subset \partial g(x) + \gamma\mathbb{B}$ for some $\gamma \ge 0$, and it proves that the difference of the functions $f$ and $g$ is $\gamma$-Lipschitz over their effective domains which must be identical. Here we generalize this result establishing that it still holds if we replace the convexity assumption of $g$ by just its sds property. In so doing, we provide via Proposition 3.1 and Corollary 3.1 some new concrete functions for which the integration property holds.

***Theorem 4.1*** *Let $X$ be an open convex subset of $E$ and $\partial_1$ and $\partial_2$, two presubdifferential operators. Let $\gamma$ be a real non negative number. Let $g\colon E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function that is $\partial_2$-sds on $X$ and let $f\colon E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function with $X \cap \operatorname{dom} f \neq \varnothing$.*

(a) *If*

$$\partial_1 f(x) \subset \partial_2 g(x) + \gamma\mathbb{B} \quad \text{for all } x \in X, \tag{4.1}$$

*then $X \cap \operatorname{dom} f = X \cap \operatorname{dom} g$ and for all $u \in X$ and $v \in X \cap \operatorname{dom} g$ the following inequalities hold*

$$g(u) - g(v) - \gamma\|u - v\| \leq f(u) - f(v) \leq g(u) - g(v) + \gamma\|u - v\|.$$

(b) *Further, the converse also holds provided $\partial_1 f(x) \subset \partial^c f(x)$ and $\partial_2 g(x) \supset \partial^c g(x)$ for all $x \in X$.*

**Proof** (a) Note first that Theorem 2.1 ensures that $X \cap \operatorname{dom} \partial_1 f \neq \varnothing$. Fix $v \in X \cap \operatorname{dom} \partial_1 f$ and $u \in X$. Thanks to assumption (4.1) we have $v \in \operatorname{dom} \partial_2 g$ and hence $g(v) \in \mathbb{R}$. We are going to prove

$$f(u) - f(v) \leq g(u) - g(v) + \gamma\|u - v\|. \tag{4.2}$$

The inequality obviously holds for $u = v$ or for $g(u) = +\infty$. So, suppose $u \in \operatorname{dom} g$ along with $u \neq v$ and fix any real number $\varepsilon > 0$.

Consider any $v' \in [v, u[ \cap \operatorname{dom} f$ with $v' \neq u$. Choose any $r \in ]0, 1[$ satisfying property (iii) in Definition 3.1 with $y = v'$ and with $\varepsilon/2$ in place of $\varepsilon$ and such that, by definition of $g'(v'; u - v')$ and by property (ii) of Definition 3.1, for all $t \in ]0, r]$

$$g'(v'; u - v') \leq t^{-1}[g(v' + t(u - v')) - g(v')] + \frac{\varepsilon}{2}\|u - v'\|.$$

So, for $w := v' + r(u - v')$ we get

$$g'(v'; w - v') \leq g(w) - g(v') + \frac{\varepsilon}{2}\|w - v'\|. \tag{4.3}$$

We proceed now to show that the following inequality holds

$$f(w) - f(v') \leq g(w) - g(v') + (\gamma + \varepsilon)\|w - v'\|. \tag{4.4}$$

Fix any integer $k \in \mathbb{N}$ and put $f_k(x) = f(x)$ if $x \neq w$, $f_k(x) = f(w)$ if $f(w) \in \mathbb{R}$, and $f_k(x) = k$ otherwise. The function is lower semicontinuous and hence the mean value theorem (Theorem 2.1) says there are $x_0 \in \big[v', w\big[$, $x_n \to x_0$, and $x_n^\star \in \partial_1 f_k(x_n) = \partial_1 f(x_n)$ such that

$$\|w - v'\|^{-1}(f_k(w) - f(v')) \leq \|w - x_0\|^{-1} \lim_{n\to\infty} \langle x_n^\star, w - x_n\rangle. \tag{4.5}$$

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By assumption (4.1), we have $x_n^\star = z_n^\star + e_n^\star$ with $z_n^\star \in \partial_2 g(x_n)$ and $\|e_n^\star\| \le \gamma$, and hence

$$\liminf_{n\to\infty}\langle x_n^\star, w - x_n\rangle \le \liminf_{n\to\infty}\langle z_n^\star, w - x_n\rangle + \gamma\|w - x_0\|. \tag{4.6}$$

Further, the choice of $r$ above and (3.1) give

$$\liminf_{n\to\infty}\langle z_n^\star, w - x_n\rangle \le g'(v'; w - x_0) + \frac{\varepsilon}{2}\|w - x_0\|.$$

As $\|w - x_0\|^{-1}(w - x_0) = \|w - v'\|^{-1}(w - v')$, using (4.3), (4.5) and (4.6) we obtain

$$\|w - v'\|^{-1}(f_k(w) - f(v')) \le \|w - v'\|^{-1}(g(w) - g(v') + \frac{\varepsilon}{2}\|w - v'\|) + \frac{\varepsilon}{2} + \gamma$$

which entails

$$f_k(w) - f(v') \le g(w) - g(v') + (\gamma + \varepsilon)\|w - v'\|.$$

So, (4.4) is true because the last inequality holds for all $k \in \mathbb{N}$.

Now, as in [19] we put

$$\sigma := \sup\bigl\{t \in\, ]0,1] : f(v+t(u-v)) - f(v) \le g(v+t(u-v)) - g(v) + (\gamma+\varepsilon)\|t(u-v)\|\bigr\}$$

and $y := v + \sigma(u - v)$. By (4.4) with $v$ in place of $v'$, the set defining $\sigma$ is nonempty. So, the continuity of the restriction of $g$ to the segment $[u, v]$ (see (i) in Definition 3.1) and the lower semicontinuity of $f$ ensure that the supremum above is attained. Further, according to Definition 3.1(i) one has $g(y) < +\infty$ and hence $f(y) < +\infty$. We claim $y = u$. Otherwise $y \in [v, u[$ and $\sigma \in [r, 1[$. Choose for this element $y$ some $\rho \in\, ]0, 1[$ satisfying Definition 3.1(iii) with $\varepsilon/2$ in place of $\varepsilon$ and such that, by definition of $g'(y; \cdot)$ and by Definition 3.1(ii), for all $t \in\, ]0, \rho]$

$$g'(y; u - y) \le t^{-1}[g(y + t(u - y)) - g(y)] + \frac{\varepsilon}{2}\|u - y\|.$$

For $w := y + \rho(u - y)$ we get

$$g'(y; w - y) \le g(w) - g(y) + \frac{\varepsilon}{2}\|w - y\|.$$

Using (4.4) with $y$ in place of $v'$, we obtain

$$f(w) - f(y) \le g(w) - g(y) + (\gamma + \varepsilon)\|w - y\|. \tag{4.7}$$

As the supremum defining $\sigma$ is attained, we also have

$$f(y) - f(v) \le g(y) - g(v) + (\gamma + \varepsilon)\|y - v\|.$$

Adding that inequality and (4.7) yields

$$f(w) - f(v) \le g(w) - g(v) + (\gamma + \varepsilon)\|w - v\|.$$

This contradicts the definition of $\sigma$ and hence $y = u$, which entails that (4.2) holds.

Now we prove $X \cap \operatorname{dom} f = X \cap \operatorname{dom} g$. Take any $v \in X \cap \operatorname{dom} f$ and choose by Theorem 2.1 a sequence $(v_n)$ in $X \cap \operatorname{dom} \partial_1 f$ with $v_n \to v$ and $f(v_n) \to f(v)$. For every $u \in X \cap \operatorname{dom} g$ we have by (4.2)

$$f(u) - f(v_n) \le g(u) - g(v_n) + \gamma\|u - v_n\|$$

and hence $X \cap \operatorname{dom} g \subset X \cap \operatorname{dom} f$. The inequality above also ensures by the lower semicontinuity of $g$

$$f(u) - f(v) \le g(u) - g(v) + \gamma\|u - v\| \tag{4.8}$$

and this obviously implies $v \in X \cap \operatorname{dom} g$. Therefore

$$X \cap \operatorname{dom} f = X \cap \operatorname{dom} g$$

and (4.8) holds true for all $u, v \in X \cap \operatorname{dom} g = X \cap \operatorname{dom} f$.

Further, taking $u$ and $v$ in $X \cap \operatorname{dom} g$ and interchanging $u$ and $v$ in (4.8) we obtain

$$g(u) - g(v) - \gamma\|u - v\| \le f(u) - f(v)$$

and this also holds for any $u \in X$ and $v \in X \cap \operatorname{dom} g = X \cap \operatorname{dom} f$. So, we have proved

$$g(u) - g(v) - \gamma\|u - v\| \le f(u) - f(v) \le g(u) - g(v) + \gamma\|u - v\| \tag{4.9}$$

for all $u \in X$ and $v \in X \cap \operatorname{dom} g$.

(b)Let us prove now the converse. Fix any $x \in X \cap \operatorname{dom} \partial_1 f$ and any $h \in E$ and observe that (4.9) entails $y \to_f x$ if and only if $y \to_g x$. Choose $r > 0$ such that

$$(x + r\mathbb{B}) + ]0, r[\ (h + r\mathbb{B}) \subset X,$$

and fix any $\varepsilon \in ]0, r[$. Then for any $y \in (x + \varepsilon\mathbb{B}) \cap \operatorname{dom} f$, $v \in h + \varepsilon\mathbb{B}$ and $t \in ]0, \varepsilon[$, we derive from (4.9)

$$t^{-1}[f(y + tv) - f(y)] \le t^{-1}[g(y + tv) - g(y)] + \gamma\|v\|$$

and hence for any $\eta \in ]0, \varepsilon[$

$$\limsup_{\substack{y \to_f x \\ t \downarrow 0}} \inf_{v \in h + \eta\mathbb{B}} t^{-1}[f(y+tv) - f(y)] \le \limsup_{\substack{y \to_g x \\ t \downarrow 0}} \inf_{v \in h + \eta\mathbb{B}} t^{-1}[g(y+tv) - g(y)] + \gamma\|h\| + \gamma\varepsilon.$$

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We deduce from that inequality and (2.2)

$$f^{\uparrow}(x;h) \le g^{\uparrow}(x;h) + \gamma\|h\| + \gamma\varepsilon.$$

As this holds for all $\varepsilon \in ]0, r[$, we obtain

$$f^{\uparrow}(x;h) \le g^{\uparrow}(x;h) + \gamma\|h\|$$

which obviously yields, by the Moreau–Rockafellar subdifferential formula in convex analysis applied at the origin to the sum $g^{\uparrow}(x;\,\cdot\,) + \gamma\|\cdot\|$,

$$\partial^c f(x) \subset \partial^c g(x) + \gamma\mathbb{B}.$$

The proof is then complete because of the assumptions

$$\partial_1 f(x) \subset \partial^c f(x) \quad \text{and} \quad \partial^c g(x) \subset \partial_2 g(x)$$

concerning the subdifferential $\partial$. ■

Of course, the equality $\partial^c g(x) = \partial_2 g(x)$ (and hence the required inclusion in (b) of the theorem) holds whenever $\partial_2$ is the Clarke subdifferential. It also holds for any presubdifferential provided $g$ is convex. Another important example where that equality is still true is given by the inclusions $\partial^F g \subset \partial_2 g \subset \partial^c g$ and the property of approximate convexity of $g$ according to Proposition 3.4. When $g$ is locally Lipschitz, the property (see (3.6))

$$\{x^\star \in E^\star : \langle x^\star, h\rangle \le d^- g(x;h), \forall h \in E\} \subset \partial_2 g(x) \subset \partial^c g(x)$$

also obviously entails the equality $\partial^c g(x) = \partial_2 g(x)$ whenever $g$ is Clarke regular.

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