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NORMAL SHOCK REFLECTION-TRANSMISSION IN RUBBER-LIKE ELASTIC MATERIAL

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Abstract

A finite transverse shock wave propagates through an unbounded medium consisting of two joined incompressible elastic half-spaces of different material properties, in the direction normal to the plane interface. A semi-inverse method is used to examine the reflection-transmission of this wave at the interface. It is found that, depending on the material properties, the reflected wave is either a simple wave or a shock; the transmitted wave is always a shock.

1. Introduction

Wright in his paper [5] on reflection of oblique finite elastic plane shocks at a plane boundary presented a semi-inverse method, based on strictly mechanical considerations, for finding the reflected waves. In this method, a reflection pattern is assumed: the reflected waves form a family of plane simple waves centred at a moving line of contact between the incident shock and the boundary. In a special case of normal reflection, the reflection pattern admits reflected waves in the form of one-parameter families of plane wavelets parallel to the boundary, propagating in the normal direction away from the boundary (cf. [1]).

We shall apply the semi-inverse method to examine the reflection-transmission problem for a plane shock wave propagating in an unbounded medium consisting of two joined elastic half-spaces of different material properties, in the direction perpendicular to the plane interface between these two

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halves. In such a composite medium, systems of additional waves can be superposed to represent the incident shock in conjunction with reflection and transmission at the interface separating the two media. The question then is what combination of additional waves is required to satisfy the condition that velocity and traction be continuous at the interface. These additional waves are called reflected and transmitted waves.

If the medium ahead of the propagating shock has a fixed state, then for a given incident shock, the region immediately behind the shock has a known fixed state. The problem now is to fit the reflected and transmitted waves so as to connect the state just fixed with some state at the boundary that is compatible with the boundary conditions. The assumption that the reflected and transmitted waves are simple waves will reduce this problem to determining the distribution and strength of the wavelets by means of ordinary differential equations. In some cases it may be necessary to modify the assumed reflection-transmission pattern, to include shocks as well.

We assume further that the elastic materials that fill the two half-spaces are of a special kind of idealised incompressible rubber, and that the normal incident wave is a plane transverse shock. Since in such cases the motion is restricted to one dimension, there are only two (nontrivial) conditions to be met at the interface; hence, the assumed reflection-transmission pattern will include a single reflected wave and a single transmitted wave only.

Section 2 contains a summary of the necessary theory, and derivation of the propagation condition for simple waves in incompressible elastic materials. Since the reflected and transmitted waves can be simple waves or shocks, we present in Section 3 differential equations for simple waves and jump conditions for shock waves. The reflection-transmission pattern is considered in Section 4 and the solution is discussed in Sections 5 and 6.

2. Basic equations

The motion of a continuum is given by $x_i = x_i(X_\alpha, t)$ where x_i and X_α are the Cartesian coordinates of a material particle in the present configuration B and the reference configuration B_R, respectively. The deformation gradient $x_{i\alpha}$, its inverse $X_{\alpha i}$ and the velocity are defined by

$$x_{i\alpha} = \frac{\partial x_i}{\partial X_\alpha}, \qquad x_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \qquad \dot{x}_i = u_i = \frac{\partial x_i}{\partial t}. \tag{2.1}$$

It is assumed that the material is homogeneous, elastic and incompressible. The incompressibility constraint requires that

$$J = \det(x_{i\alpha}) = 1. \tag{2.2}$$

The Piola-Kirchhoff stress tensor for such a material is

$$T_{i\alpha} = \rho_R \frac{\partial \sigma}{\partial x_{i\alpha}} + p X_{\alpha i}, \tag{2.3}$$

where σ denotes internal energy per unit mass in B_R, $\rho_R = \rho$ is the density and $p = p(X_\alpha)$ is an arbitrary scalar function (hydrostatic pressure).

If the stress and velocity fields are differentiable, then the equations expressing balance of momentum and moment of momentum are

$$T_{i\alpha,\alpha} = \rho \dot{u}_i, \qquad x_{i\alpha} T_{j\alpha} = x_{j\alpha} T_{i\alpha}. \tag{2.4}$$

If the functions $x_i(X_\alpha, t)$ are continuous everywhere but have discontinuous first derivatives on some propagating surface $S(X_\alpha, t) = 0$, (2.4) must be replaced by jump conditions on this surface:

$$[\![T_{i\alpha}]\!] N_\alpha = -\rho [\![u_i]\!] V, \qquad [\![x_{i\alpha}]\!] = a_i N_\alpha, \qquad [\![u_i]\!] = -a_i V. \tag{2.5}$$

Such a surface is called a shock wave. The vector **N** is a unit normal to the wave, V is the speed of propagation along **N** and **a** is the amplitude vector of the jump. The bold square brackets indicate the jump in the quantity enclosed across S; thus

$$[\![\cdot]\!] = (\cdot)^B - (\cdot)^F,$$

where the letters F and B refer to the limit values taken in the front and rear sides of S, respectively.

Simple waves [5] are defined to be regions of space-time in which all field quantities are continuous functions of a single parameter, say, $\lambda = \psi(X_\alpha, t)$. Regions of constant λ are propagating surfaces, called wavelets, with unit normal and normal velocity in B_R given by

$$N_\alpha(\lambda) = \frac{\psi_{,\alpha}}{|\nabla \psi|}, \qquad U(\lambda) = -\frac{\dot{\psi}}{|\nabla \psi|}. \tag{2.6}$$

The equation of motion (2.4) and the compatibility condition in the region of simple waves are

$$\frac{\partial T_{i\alpha}}{\partial x_{j\beta}} x'_{j\beta} \psi_{,\alpha} = \rho u'_i \dot{\psi}, \tag{2.7a}$$

$$x'_{j\beta} \dot{\psi} = u'_j \psi_{,\beta} \tag{2.7b}$$

where the prime indicates differentiation with respect to λ. If $\dot{\psi} \neq 0$, (2.7) may be written as

$$(Q_{ij} - \rho U^2 \delta_{ij}) u'_j = 0, \tag{2.8a}$$

$$U x'_{j\beta} + u'_j N_\beta = 0 \tag{2.8b}$$

where

$$Q_{ij} = \frac{\partial T_{i\alpha}}{\partial x_{j\beta}} N_\alpha N_\beta \tag{2.9}$$

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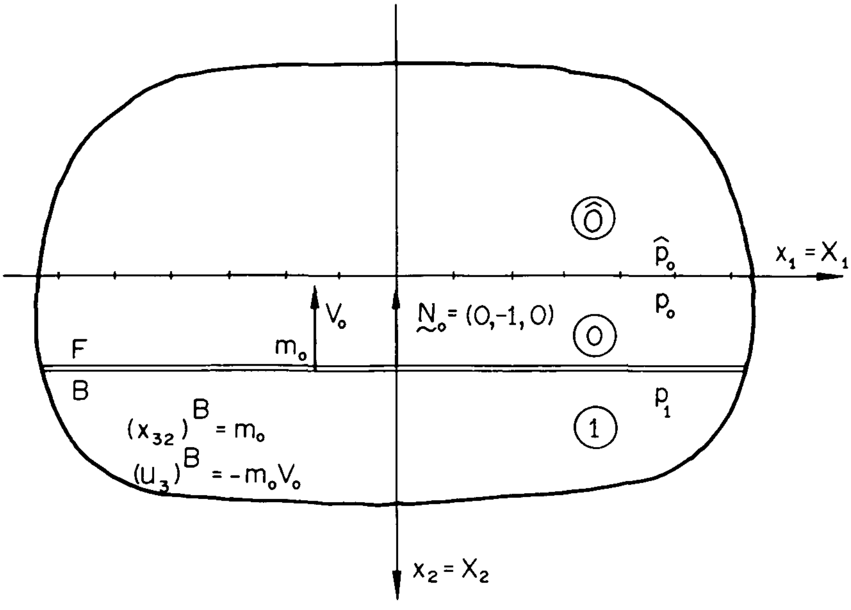

FIGURE 1. Incident shock and assumed reflection-transmission pattern.

is the acoustic tensor. For an incompressible material, substitution of (2.3) into (2.8), and the identity $X_{\alpha i,\alpha} = 0$, leads to the equation

$$(\tilde{Q}_{ij} - \rho U^2\delta_{ij})u'_j - p_{,\alpha}X_{\alpha l}U(|\nabla\psi|)^{-1} = 0,$$

or, since in the region of simple wave $p_{,\alpha} = p'|\nabla\psi|N_\alpha$, to the equation

$$(\tilde{Q}_{lj} - \rho U^2\delta_{ij})u'_j - p'UX_{\alpha i}N_\alpha = 0. \tag{2.10}$$

We denote here

$$\tilde{Q}_{ij} = \rho\sigma_{i\alpha j\beta}N_\alpha N_\beta, \qquad \sigma_{i\alpha j\beta} = \frac{\partial^2\sigma}{\partial x_{i\alpha}\partial x_{j\beta}}. \tag{2.11}$$

From the incompressibility condition (2.2) we have

$$J_{,\beta} = X_{\alpha i}x_{i\alpha}\psi_{,\beta} = 0.$$

Using this equation, together with (2.8) and the relation (cf. [4])

$$Un_i = uN_\alpha X_{\alpha i} \tag{2.11}$$

where n_i is a unit normal and u the speed of propagation of the wave in B, the scalar p' can be eliminated to obtain

$$(Q^*_{ij} - \rho U^2\delta_{ij})u'_j = 0. \tag{2.12}$$

The tensor

$$Q^*_{ij} = \tilde{Q}_{ij} - \tilde{Q}_{kj}n_kn_i \tag{2.13}$$

is called the reduced acoustic tensor.

3. Incident shock

Consider an unbounded medium consisting of two elastic half-spaces of different material properties, joined rigidly along the plane $x_2 = 0$. Suppose that a plane transversely polarised shock wave of strength m_0 propagates in the half-space $x_2 > 0$ with speed V_0, in the direction perpendicular to the interface of the two half-spaces (Figure 1). It is convenient to assume that the amplitude vector $\mathbf{a}_0$ of the shock is parallel to the x_3-axis. Such a wave has displacement components in the x_3-direction only. Thus, this propagating discontinuity surface belongs to a one-parameter family of parallel planes, with normals

$$\mathbf{N}_0 = (0, -1, 0). \tag{3.1}$$

It is assumed that both material solids are isotropic and incompressible, and are characterised by the constitutive equations

$$W(I_1, I_2) = \rho\sigma(I_1, I_2) = C_1(I_1-3)+C_2(I_2-3)+C_3(I_1^2-9), \qquad x_2 > 0, \tag{3.2}$$

$$\hat{W}(\hat{I}_1, \hat{I}_2) = \hat{\rho}\hat{\sigma}(\hat{I}_1, \hat{I}_2) = \hat{C}_1(\hat{I}_1-3)+\hat{C}_2(\hat{I}_2-3)+\hat{C}_3(\hat{I}_1^2-9), \qquad x_2 < 0, \tag{3.$\hat{2}$}$$

proposed by Zahorski [7], where $I_1 = B_{ii}$, $I_2 = (B_{ii}B_{jj} - B_{ij}B_{ij})/2$ are the invariants of the left Cauchy-Green strain tensor B_{ij}. The sets of values for C_1, C_2, C_2, for three kinds of rubber, are given in [8]. The symbol ˆ serves here to label the field quantities and the field equations in the half-space $x_2 < 0$.

Approximation (3.2) of the strain energy function W is valid for rubber-like materials under moderate strain. Experimental investigations indicate that the constant C_3, important in the following discussion, is positive.

Since the medium in front of the shock is unstrained and at rest, the jump conditions (2.5b) become now

$$[\![u_3]\!] = (u_3)^B = -m_0 V_0, \qquad [\![X_{32}]\!] = (X_{32})^B = -m_0 \tag{3.3}$$

where $m_0 = |\mathbf{a}_0|$ is the shock strength. Substituting (3.1) and (3.3) into (2.5a) we obtain the equation relating the shock speed V_0 and the shock strength m_0, [1, 2],

$$V_0^2 = c^2(1 + \eta m_0^2)/\rho \tag{3.4}$$

where

$$c^2 = 2(C_1 + C_2 + 6C_3), \qquad \eta = 4C_3/c^2. \tag{3.5}$$

The state behind the propagating shock wave (region 1) is now completely specified by the shock strength m_0. Equations (3.3) determine the deformation gradient and its inverse

$$(x_{i\alpha}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & v & 1 \end{bmatrix}, \qquad (X_{\alpha i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -v & 1 \end{bmatrix} \tag{3.6}$$

and the particle velocity

$$\mathbf{u} = (0, 0, u) \tag{3.7}$$

in this state. We denote here $v = (x_{32})^B$, $u = (u_3)^B$.

The components of $T_{i\alpha}$ and the elasticities $\sigma_{i\varphi j\beta}$ required in this paper are then evaluated in region 1:

$$T_{22} = 2\rho(\sigma_1 + 2\sigma_2) + p,$$

$$T_{32} = 2\zeta(\partial_1 + \partial_2)v, \qquad T_{12} = 0; \tag{3.8}$$

$$\sigma_{3232} = 2(\sigma_1 + \sigma_2) + 4\partial_{11}v^2 \tag{3.9}$$

where

$$\sigma_1 = \frac{\partial\sigma}{\partial I_1} = \frac{1}{\rho}(C_1 + 2C_3 I_1), \qquad \ddot{\sigma}_2 = \frac{\partial\sigma}{\partial I_2} = \frac{C_2}{\rho}$$

$$\sigma_{11} = \frac{\partial^2\sigma}{\partial I_1^2} = \frac{2C_3}{\rho}, \qquad I_1 = I_2 = 3 + v^2.$$

4. Reflection-transmission pattern

The constraint of incompressibility restricts the propagating waves to transverse waves only. In general, the reflection-transmission problem may have no solution in terms of simple waves, as there are at most two possible families of reflected waves and two families of transmitted waves; this means that there are four free parameters, with six interfacial continuity conditions for velocity and traction to be met. However solutions may exist for some types of incompressible materials, with a particular symmetry and deformation. In this paper we examine such special cases.

When the incident shock wave strikes the interface $x_2 = 0$, part of it is reflected and part transmitted across the interface, in the form of reflected and transmitted waves. We assume that both the reflected and the transmitted waves are single simple plane waves, travelling in the direction of the x_2-axis, away from the interface (region 2 and $\hat{2}$, Figure 2). The wavelets $\lambda =$ constant of the reflected wave are parallel planes with normals $\mathbf{N} = (0, 1, 0)$; the wavelets $\mu =$ constant of the transmitted wave are planes with normals $\mathbf{N} = (0, -1, 0)$. The reflected wave propagates into the just-fixed state (region 1, Figure 2); the transmitted wave propagates into the "zero" state (region $\hat{0}$).

The problem now is to fit these waves so as to connect the states at the interface (region 3 and $\hat{3}$) that are compatible with the interfacial conditions, with the states of region 1 and $\hat{0}$.

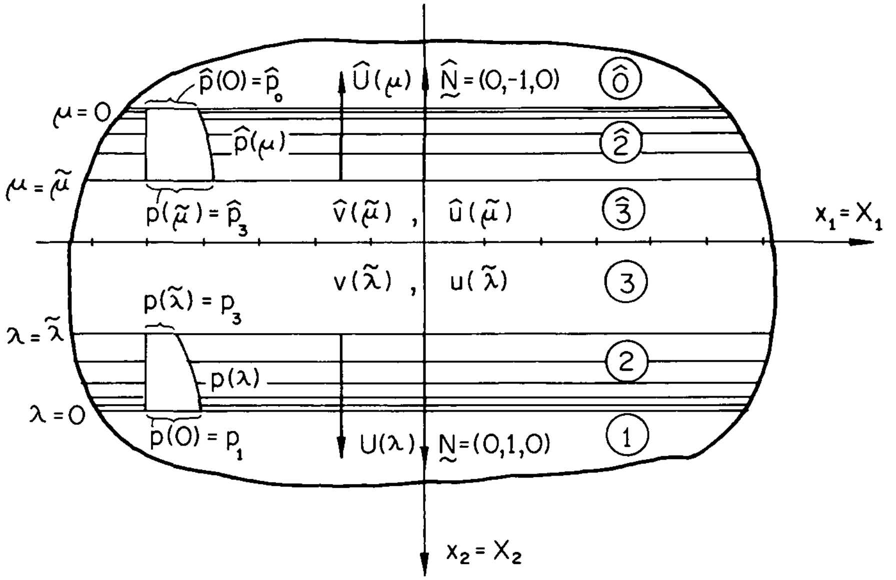

FIGURE 2. Normal incidence: reflection-transmission pattern.

The deformation gradient and velocity in the regions of the reflected and transmitted waves are similar to (3.6) and (3.7) in form. Since the components n_i (and $\hat{n}_i$) of the normals referred to the present configuration B remain the same ($n_i = N_i$), the acoustic tensor (2.13) assumes a simpler form:

$$Q^*_{ij} = \tilde{Q}_{ij} = \rho\sigma_{i2j2} \quad \text{for } i \neq 2, \qquad Q^*_{2j} = 0 \tag{4.1}$$

which, together with (2.11), (3.2) and (3.9), leads to the propagation condition (2.12) for simple waves reduced to a single equation

$$(\sigma_{3232} - U^2)u' = 0. \tag{4.2}$$

The characteristic root $U = \sqrt{\sigma_{3232}}$ (see (3.9)) is a real single-valued function of v, and it represents the speed of a simple wave

$$U = c\{\rho^{-1}(1 + 3\eta v^2)\}^{1/2}; \tag{4.3}$$

the corresponding characteristic vector function $\mathbf{u}'$ is given by

$$\mathbf{u}' = (0, 0, f) \tag{4.4}$$

where f is an arbitrary function of the wave parameter. Any particular choice of f affects only the parameterisation of the field quantities.

The differential equation relating the particle velocity and the deformation gradient in the region of a simple wave is obtained from the compatibility conditions (2.8b). We have in region 2

$$Uv' + u' = 0 \tag{4.5}$$

and in region $\hat{2}$

$$\hat{U}\hat{v}' - \hat{u}' = 0 \tag{4.$\hat{5}$}$$

where U is given by (4.3).

The geometrical significance of the above relations is evident. Differentiating $\psi(x_2, t) = \lambda$ along the line of constant λ we obtain (see (2.6))

$$dx_2/dt = U. \tag{4.6}$$

The curves given by (4.6) are the characteristics of the differential system (2.13). The trajectories of the wavelets in the (X_2, t) plane are given by the characteristics of the equation of motion in the region of the simple wave. The changes of the field quantities in this region are governed by the ordinary differential equations (4.4) and (4.5).

It is convenient to assume $f = -U$ and $\hat{f} = \hat{U}$. From (4.4), (4.5) and (4.$\hat{5}$) it follows then that in region 2

$$u' = -U, \qquad v' = 1, \tag{4.7}$$

and in region $\hat{2}$

$$\hat{u}' = \hat{U}, \qquad \hat{v}' = 1. \tag{4.$\hat{7}$}$$

The deformation gradient and velocity are assumed to be continuous throughout regions $1, 2, 3$ and throughout regions $\hat{0}, \hat{2}, \hat{3}$ (Figure 2). Thus the initial values for differential equations that describe region 2 are the constant values of region 1, and the initial values for these equations in region $\hat{2}$ are the constant values of region $\hat{0}$.

The deformation gradient and velocity in region 1 are given by (3.3); the material region $\hat{0}$ is unstrained and at rest. Hence, the initial conditions for (4.7) and (4.$\hat{7}$) are $u(0) = -m_0 V_0$, $v(0) = -m_0$ and $\hat{u}(0) = 0$, $\hat{v}(0) = 0$, respectively. The incident shock speed V_0 is given by (3.4) and m_0 is the incident shock strength. Integrating (4.7) and (4.$\hat{7}$) we obtain

$$u(\lambda) = -\int_0^\lambda U(\lambda)\, d\lambda - m_0 V_0, \qquad v(\lambda) = \lambda - m_0, \tag{4.8}$$

$$u(\mu) = \int_0^\mu U(\mu)\, d\mu, \qquad v(\mu) = \mu \tag{4.$\hat{8}$}$$

Substitution into (4.3) gives the wave speed as a function of the wave parameter.

The other field quantity required here, the static pressure $p(x_\alpha)$ (see (2.3)), is given in the region of the simple wave by

$$p(\lambda) = -4C_3 v^2(\lambda) + p_0 \tag{4.9}$$

where p_0 is the pressure in region 0 (cf. [1, 2]). Substitution of (4.9) into (3.8) shows that the stress component T_{22} is independent of the deformation gradient:

$$T_{22} = 2(C_1 + 2C_2 + 6C_3) + p_0. \tag{4.10}$$

All field quantities in the region of simple waves are constant along the wavelets λ = constant (μ = constant). The values on the leading wavelet $\lambda = 0$ ($\mu = 0$) are the constant values of the region into which the wave propagates; the values on the trailing wavelets $\lambda = \tilde{\lambda}$ ($\mu = \tilde{\mu}$) are the constant values of the region at the interface $x_2 = 0$. The problem now is to find the pair $(\tilde{\lambda}, \tilde{\mu})$ of the final values of wave parameters so that the constant states at the interface are compatible with the continuity requirement for velocity and traction.

There are three conditions for stresses and one for velocity to consider at $x_2 = 0$:

$$T_{i2} = \hat{T}_{i2}, \quad i = 1, 2, 3, \qquad \text{and} \qquad u = \hat{u}. \tag{4.11}$$

Substitution of (3.8) into (4.11) leads to two nontrivial equations involving $\tilde{\lambda}$ and $\tilde{\mu}$:

$$u(\tilde{\lambda}) = \hat{u}(\tilde{\mu}), \qquad c^2\{1 + \eta v^2(\lambda)\}v(\lambda) = \hat{c}^2\{1 + \hat{\eta}\hat{v}^2(\tilde{\mu})\}\hat{v}(\tilde{\mu}); \tag{4.12}$$

the third equation $T_{22} = \hat{T}_{22}$ relates the pressures p_0 and $\hat{p}_0$ of regions 0 and $\hat{0}$ across the interface (see (4.10)):

$$p - \hat{p}_0 = 2(\hat{C}_1 - C_1) + 4(\hat{C}_2 - C_2) + 12(\hat{C}_3 - C_3).$$

Inspection of (4.8), (4.$\hat{8}$) and (4.12) indicates that for equations (4.12) to have a solution (λ, μ) it is necessary that both $v(\tilde{\lambda})$ and $\hat{v}(\tilde{\mu})$ are negative.

Suppose that $(\tilde{\lambda}, \tilde{\mu})$ is a solution of (4.12). The requirement that $v(\tilde{\lambda})$ is negative falls into two cases: (a) $-m_0 < v(\tilde{\lambda}) < 0$ or (b) $v(\tilde{\lambda}) < -m_0$. The corresponding inequalities for region 2 are then

$$-m_0 \le v(\lambda) \le v(\tilde{\lambda}) < 0 \quad \text{for } 0 \le \lambda \le \tilde{\lambda} < m_0, \tag{4.13a}$$

$$v(\tilde{\lambda}) \le v(\lambda) \le -m_0 \quad \text{for } \lambda_0 < \tilde{\lambda} \le \lambda \le 0 \tag{4.13b}$$

where λ_0 is the root of $u(\lambda) = 0$, and by (4.8) λ_0 is negative. The counterparts of (4.13) for the transmitted wave are

$$\hat{v}(\tilde{\mu}) \le \hat{v}(\mu) \le 0 \quad \text{for } \tilde{\mu} \le \mu \le 0. \tag{4.1$\hat{3}$}$$

Substitution of relations (4.8), with conditions (4.13) (or their counterparts), into (4.3) completes the solution in the assumed form.

Whether (4.3) actually represents a simple wave depends on the speed distribution across region 2. We recall that each of the forward-propagating wavelets is identified by a fixed value of the wave parameter λ changing from 0 to its final value $\tilde{\lambda}$. It follows that a wavelet 'λ' precedes the wavelet '$\lambda + d\lambda$'. Consequently, if the wave speed U is a decreasing function of λ changing from 0 to $\tilde{\lambda}$, the wavelet '$\lambda + d\lambda$' propagates at a lower speed than the wavelet 'λ' and the reflected wave is a simple wave. If U is increasing with λ, the wavelet '$\lambda + d\lambda$' travels faster than the wavelet 'λ' and in due course a shock is formed. It may happen that U is not a monotone function of λ. If such is the case, the reflected wave may be formed by a combination of a shock and a simple wave, and it is the sign of the local value of $dU/d\lambda$ which determines how this wave is composed.

In (4.13a) the deformation gradient $v(\lambda)$ decreases in absolute value across region 2, and the wave speed $U = c\rho^{-1/2}\{1 + 3\eta v^2(\lambda)\}^{1/2}$ decreases with λ. Hence region 2 is a simple wave propagating into a deformed material. Since this wave decreases the existing strain level at the point it has just traversed, it is of the unloading type. The region of the reflected wave (Figure 3) is given by $c\rho^{-1/2}\{1 + 3\eta v^2(\tilde{\lambda})\}^{1/2} t \leq x_2 \leq c\rho^{-1/2}\{1 + 3\eta m_0^2\}^{1/2} t$.

In (4.13b) the deformation gradient $v(\lambda)$ increases in absolute value across region 2, and the wave speed increases with λ. Hence region 2 does not represent a simple wave.

We modify the solution pattern assuming now that the reflected wave is a shock propagating in direction $\mathbf{N} = (0, 1, 0)$. Equations of motion (2.4) are now replaced by jump conditions (2.5) connecting the corresponding quantities in region 1 and 3 across the wave. The constant state ahead of the wave is given by (3.3). We denote the constant values of the region behind the wave by u^B, v^B. Thus, the jumps of the deformation gradient and velocity across the wave are

$$[\![v]\!] = v^B + m_0, \qquad [\![u]\!] = u^B + m_0 V_0; \tag{4.16}$$

it follows from (4.13b) that $[\![v]\!] < 0$. Denoting by $\mathbf{a} = (0, 0, a_3)$ the amplitude vector of the shock, by V its speed and by $m = |\mathbf{a}|$ its strength, we obtain from (2.5b) and (4.16)

$$v^B = -m - m_0, \qquad u^B = -mV - m_0 V_0. \tag{4.17}$$

The continuity conditions (4.11) at the interface $x_2 = 0$ retain their (4.12) form, with the final field values of the simple wave replaced by the constant values u^B and v^B of region 3, and m assuming the role of wave parameter; equations (4.5) are replaced by (4.17).

Eliminating the velocity jump $[\![u]\!]$ from (2.5) we obtain the equation for the shock speed: $[\![T_{32}]\!] = \rho [\![v]\!] V^2$, or

$$v^2 = c^2 \rho^{-1}\{1 + \eta((v^B)^2 - v^B m_0 + m_0^2)\} \tag{4.18}$$

Substitution of v^B, calculated from the continuity conditions (4.12), into (4.18) completes the solution in the assumed form.

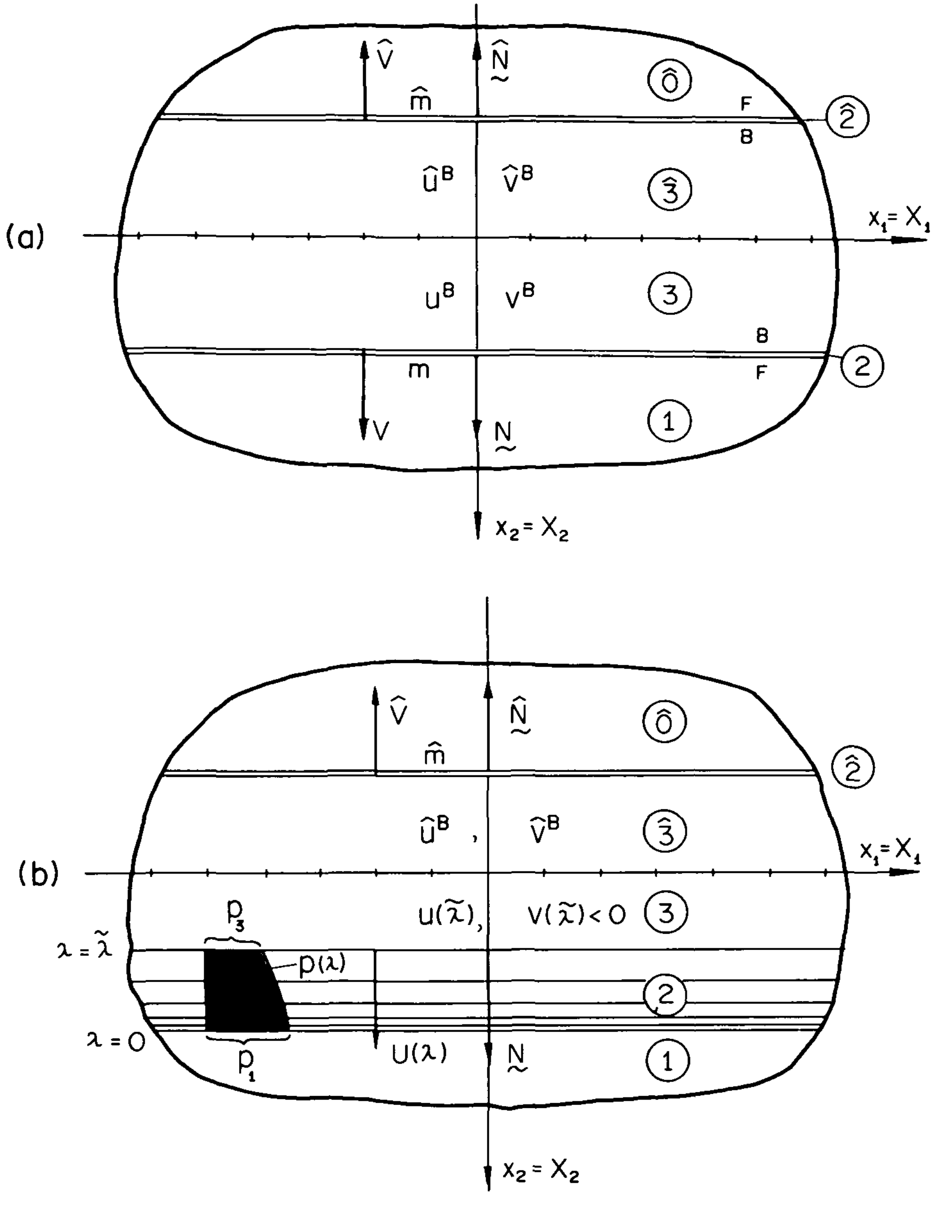

FIGURE 3. (a) Reflected simple wave and transmitted shock. (b) Reflected and transmitted shock.

Equation (4.18) represents a weak solution of the reflection problem considered here. As such, this solution does not possess the uniqueness property of smooth solutions. According to Lax [3], for (4.18) to represent an admissible shock it must also satisfy a stability criterion:

$$U^B \geq V \geq U^F \tag{4.19}$$

where U^B and U^F are the characteristic (acoustic) speeds (4.3) in the material region behind and ahead of the shock, respectively.

The speeds U^B and U^F can be calculated from (4.3) by substituting for v the value v^B and m_0. Thus

$$U^B = U(v^B) = c\rho^{-1/2}\{1 + 3\eta(v^B)^2\}^{1/2},$$

$$U^F = U(m_0) = c\rho^{-\frac{1}{2}}\{1 + 3\eta m_0^2\}^{1/2}.$$

Also, since $v^B < -m_0$ and $m_0 > 0$, we have $3(v^B)^2 > (v^B)^2 - v^B m_0 + m_0^2 > 3m_0^2$ and the stability criterion (4.19) is satisfied for an arbitrary value of the incident shock strength m_0. The reflected wave is a shock propagating into a deformed material, increasing the strain level. The region representing this wave is defined by the equation: $x_2 = Vt$ (Figure 3).

An analogous analysis for region $\hat{2}$ shows that the transmitted wave cannot be a simple wave. The assumption that region $\hat{2}$ is a shock propagating in direction $\hat{\mathbf{N}} = (0, -1, 0)$ into region $\hat{0}$ of "zero" state leads to the following equations

$$\hat{v}^B = -\hat{m}, \qquad \hat{u}^B = -\hat{m}\hat{V}, \tag{4.20}$$

$$\hat{V}^2 = \hat{c}^2\hat{\rho}^{-1}\{1 + \hat{\eta}(\hat{v}^B)^2\} \tag{4.21}$$

where $\hat{V}$ is the shock speed and $\hat{m}$ is the shock strength. Since the characteristic speed calculated in region $\hat{3}$ and $\hat{0}$ is $\hat{U}^B = \hat{c}\hat{\rho}^{-1/2}\{1 + 3\hat{\eta}(\hat{v}^B)^2\}^{1/2}$ and $\hat{U}^F = \hat{c}\hat{\rho}^{-1/2}$, the stability criterion (4.19) is satisfied. The transmitted wave is a shock propagating into an unstrained material region, loading the material; it is given by the equation: $x_2 = -\hat{V}t$. (Figure 3).

5. Reflection-transmission solution

The results presented in Section 4 were obtained under the assumption that (4.12) has a solution. The question to be considered now is whether a combination of the parameters defining the two materials and the incident shock is possible for which such a solution exists. A numerical analysis is conducted for some media composed of different kinds of rubber, [8], (Table 1).

TABLE 1

elastic constants (kG/cm^2) \ material	I	II	III
c_1	0.64	2.14	3.52
c_2	0.09	0.13	0.00
c_3	0.07	0.04	0.23
c^2	2.3	5.02	9.8
η	0.122	0.0318	0.0938

5.1 Reflected and transmitted waves are shocks

Let us consider case (4.13b) when both waves are shocks. In this case we have for the reflected wave (Figure 3):

$$u^B = [\![u]\!] - m_0 V_0 = -mV - m_0 V_0, \qquad v^B = [\![v]\!] - m_0 = -m - m_0, \quad (5.1)$$

and for the transmitted wave

$$\hat{u}^B = [\![\hat{u}]\!] = -\hat{m}\hat{V}, \qquad \hat{v}^B = [\![\hat{v}]\!] = -\hat{m}., \quad (5.\hat{1})$$

Substituting (5.1) and (5.$\hat{1}$) into the continuity conditions (4.12) we obtain two equations for two wave parameters $\hat{m}$ and m:

$$\hat{m}\hat{V} = mV + m_0 V_0, \quad (5.2a)$$

$$\hat{c}^2\{1 + \hat{\eta}\hat{m}^2\}\hat{m} = c^2\{1 + \eta(m + m_0)^2\}(m + m_0). \quad (5.2b)$$

Equation (5.2b) can be rewritten as

$$\hat{m}^3 + 3p\hat{m} - 2q = 0 \quad (5.3)$$

where $p = (3\eta)^{-1}$, $q = (c/\hat{c})^2(3\eta)^{-1}\{1 + \eta(m + m_0)^2\}$ $(m + m_0)$, and both coefficients p and q are positive. From the theory of algebraic equations it is known that in such a case, (5.3) has one only real solution

$$\hat{m} = 2(3\eta)^{-1/2}\sinh(\varphi/3) \quad (5.4)$$

where $\sinh\varphi = q/p^{3/2}$, and this solution is positive. Substitution of (5.4) into (5.2a) gives an equation for the reflected shock wave parameter m. Further numerical calculations show that for each of the following material combinations (Table 1): $A = \text{I} - \hat{\text{I}}\text{I}$, $B = \text{II} - \text{I}\hat{\text{I}}\text{I}$, $C = \text{I} - \text{I}\hat{\text{I}}\text{I}$, this equation has a unique real positive solution m as a function of m_0. The results are displayed in Figure 4(a).

Since the material region behind the propagating shock wave should remain elastic, the discontinuity jumps cannot be arbitrary, and the appropriate estimates for the shock strength should be established. In this paper we use the estimation: $m_0 < 2.66$ (cf. [2]).

Components $\hat{v}^B$ and v^B of the deformation gradient in the regions behind the transmitted and reflected shocks are plotted in Figure 4(a) as functions of the incident shock parameter m_0, for combinations A, B and C. The graphs show that the curves $\hat{v}^B = \hat{v}^B(m_0)$ (and $v^B = v^B(m_0)$) intersect for some value of m_0. Thus, different material combinations are possible for which a given incident shock may generate transmitted shocks (or reflected shocks) of the same strength.

5.2 Reflected simple wave and transmitted shock

In (4.3a) the reflected wave is a simple wave. The terminal values of this wave are (see (4.8)):

$$u(\tilde{\lambda}) = -\int_0^{\tilde{\lambda}} U(\lambda)\, d - m_0 V_0, \qquad v(\tilde{\lambda}) = \tilde{\lambda} - m_0 \quad \text{and} \quad 0 < \tilde{\lambda} < m_0; \quad (5.5)$$

the constant values of region $\hat{3}$ behind the transmitted shock wave (Figure 3) are given by (5.$\hat{1}$). Substituting (5.$\hat{1}$) and (5.5) into (4.12) we obtain two equations for two parameters $\hat{m}$ and $\tilde{\lambda}$:

$$\hat{m}\hat{V} = \int_0^{\tilde{\lambda}} U(\lambda)\, d\lambda + m_0 V_0, \tag{5.6a}$$

$$\hat{c}^2\{1 + \hat{\eta}\hat{m}^2\}\hat{m} = c^2\{1 + \eta(\tilde{\lambda} - m_0)^2\}(m_0 - \tilde{\lambda}). \tag{5.6b}$$

Equation (5.6b) is equivalent to (5.2b), with m replaced by $-\tilde{\lambda}$. Its solution for $\hat{m}$ is of the form of (5.4). Equation (5.6a), after integration, can be written as

$$w(\tilde{\lambda}, \hat{m}) = \frac{c}{2}\rho^{-1/2}(3\eta)^{1/2}\left\{m_0(\psi - \phi) - \tilde{\lambda}\psi + \frac{1}{3\eta}\ln\left|\frac{\phi - m_0}{\psi + \tilde{\lambda} - m_0}\right|\right\}$$
$$+ \hat{m}\hat{c}\rho^{-\frac{1}{2}}\{1 + \hat{\eta}\hat{m}^2\} - m_0 c\rho^{-1/2}\{1 + \eta m_0^2\} = 0, \tag{5.7}$$

where $\phi(m_0) = (m_0 + 1/3\eta)^{1/2}$, $\psi(\tilde{\lambda}, m) = ((\tilde{\lambda} - m_0)^2 + 1/3\eta)^{1/2}$. Substitution of (5.4) into (5.7) gives an equation for the final value $\tilde{\lambda}$ of the reflected wave parameter. Numerical calculations show that for each of the following material combinations: $D = \text{II} - \hat{\text{I}}$, $E = \text{III} - \text{I}\hat{\text{I}}$, $F = \text{III} - \hat{\text{I}}$, this equation has a unique real positive solution $\tilde{\lambda}$, as a function of m_0.

Components $\hat{v}^B$ and $v(\tilde{\lambda})$ of the deformation gradient in the regions behind the transmitted shock and reflected simple wave are plotted in Figure 4(b) as functions of m_0, for combinations D, E and F. The graphs show that the curves $\hat{v}^B = \hat{v}^B(m_0)$ (and $v = v(\tilde{\lambda}; m_0)$) intersect for some value of m_0. Thus, different material combinations are possible for which the transmitted (or reflected) waves are characterised by the same parameter, while the corresponding reflected (or transmitted) waves have different parameters.

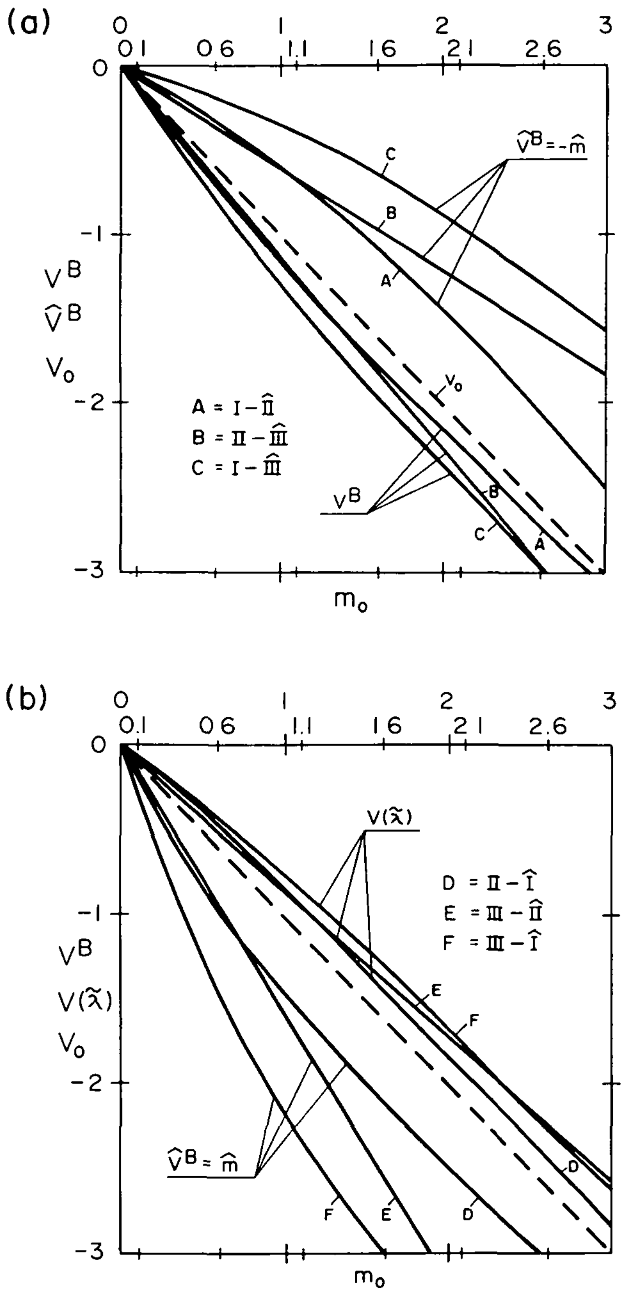

FIGURE 4. Deformation gradient as function of m_0, for material combinations:
(a) $A = \mathrm{I} - \widehat{\mathrm{II}}$, $B = \mathrm{II} - \widehat{\mathrm{III}}$, $C = \mathrm{I} - \widehat{\mathrm{III}}$;
(b) $D = \mathrm{II} - \hat{\mathrm{I}}$, $E = \mathrm{III} - \widehat{\mathrm{II}}$, $F = \mathrm{III} - \hat{\mathrm{I}}$; $\zeta = \hat{\zeta}$.

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5.3 Transmitted shock only

The shock is completely transmitted if $m = 0$ (or $\tilde{\lambda} = 0$) and $\hat{m} \neq 0$. Equations (5.3) are then reduced to the form (see (3.4) and (4.18))

$$\hat{\rho}\hat{V} = \rho V_0, \qquad \hat{\rho} m_0 = \rho \hat{m}, \tag{5.8}$$

and the problem has a solution $\hat{m} = (\hat{\rho}/\rho)m_0$, provided the incident shock and the combined materials satisfy the following condition

$$m_0^2 = \frac{\hat{\rho}^3(\hat{\rho}\hat{c}^2 - \rho c^2)}{\rho(\rho^2\eta c^2 - \hat{\rho}^2\hat{\eta}\hat{c}^2)}. \tag{5.9}$$

In the numerical examples considered here we assume that $\rho = \hat{\rho}$. Substituting $(3.5)_2$ into (5.9) we obtain

$$m_0^2 = (\hat{c}^2 - c^2)/(4(C_3 - \hat{C}_3)). \tag{5.9a}$$

Inspection of data in Table 1 indicates that (5.9a) can be satisfied only by the material combination $A = \mathrm{I} - \hat{\mathrm{I}}\mathrm{I}$. Further numerical analysis shows however that the corresponding value of the incident shock strength is $m_0 = 4.74$, well outside of the admissibility interval $(0, 2.66)$. We conclude that in the particular cases (Table 1) considered here, transmission is always associated with reflection.

5.4 Reflected shock only

The incident shock is completely reflected if $\hat{m} = 0$ and $m \neq 0$ (or $\tilde{\lambda} \neq 0$). According to condition (4.13) it must be that $m > 0$ for the reflected shock, and $\tilde{\lambda} > 0$ for the reflected simple wave. It is obvious that neither (5.3) nor (5.5), now reduced to form (5.10) and (5.11), respectively:

$$c^2\{1 + \eta(m + m_0)^2\}(m + m_0) = 0, \qquad mV + m_0V_0 = 0; \tag{5.10}$$

$$c^2\{1 + \eta(\tilde{\lambda} - m_0)^2\}(\tilde{\lambda} - m_0) = 0, \qquad \int_0^{\tilde{\lambda}} U(\lambda)\, d\lambda + m_0V_0 = 0, \tag{5.11}$$

can be satisfied for $m > 0$ and $\tilde{\lambda} > 0$. We conclude that reflection without transmission is not possible.

If, however, the half-space $x_2 < 0$ is a vacuum, there is, of course, a reflected wave only. In this case the interfacial conditions (4.12) are reduced to a single equation $v = 0$, with the particle velocity component u constrained by the compatibility condition only. The case is equivalent to 'free boundary' conditions at $x_2 = 0$. The reflection solution (cf. [1]) is a simple wave

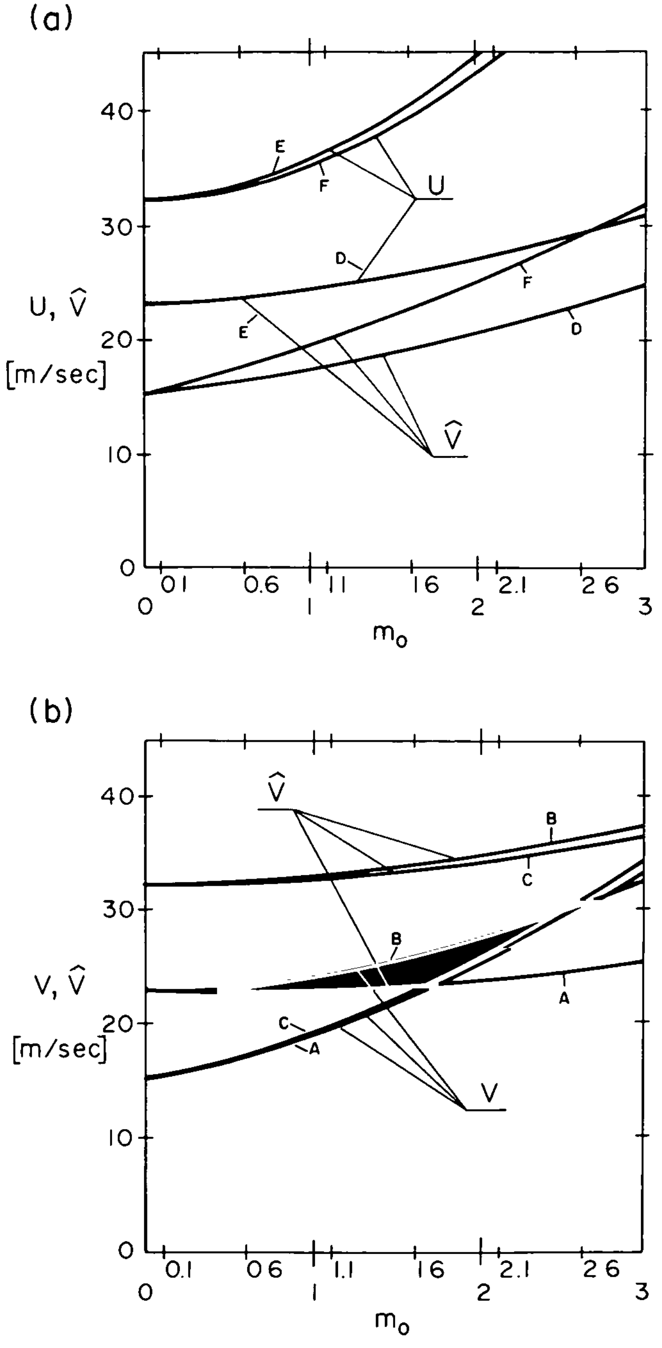

FIGURE 5. Wave speeds in case (a) reflected simple wave and transmitted shock, (b) reflected and transmitted shock.

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propagating into the just-deformed material region, unloading it to the state of zero strain.

6. Concluding remarks

Numerical solutions obtained for some combinations of rubber-like materials (Table 1) and incident shock confirm the conclusion that the transmitted waves are always shocks, while the reflected waves are either shocks or simple waves. The type of reflected wave depends on the material properties of the composite medium, and through which of the two materials the incident shock propagates. In the combinations A, B and C (Figure 4(a)) the reflected wave is a shock. In the reversed combinations D, E and F (Figure 4(b)) the reflected wave is a simple wave.

The continuity condition for velocity at $x_2 = 0$ implies an energy type relation between the amplitudes and speeds of the incident, transmitted and reflected waves: $\hat{m}\hat{V} = mV + m_0V_0$. The numerical results indicate that the term mV is small in comparison with m_0V_0. This means that the major part of the incident shock energy is used to form the transmitted shock. The graphs in Figure 4 also show that $\hat{m} > m_0$ when the reflected wave is a shock, and $\hat{m} < m_0$ when the reflected wave is a simple wave. The strength m of the reflected shock is comparatively small, and the range of variation of the reflected simple wave is also small.

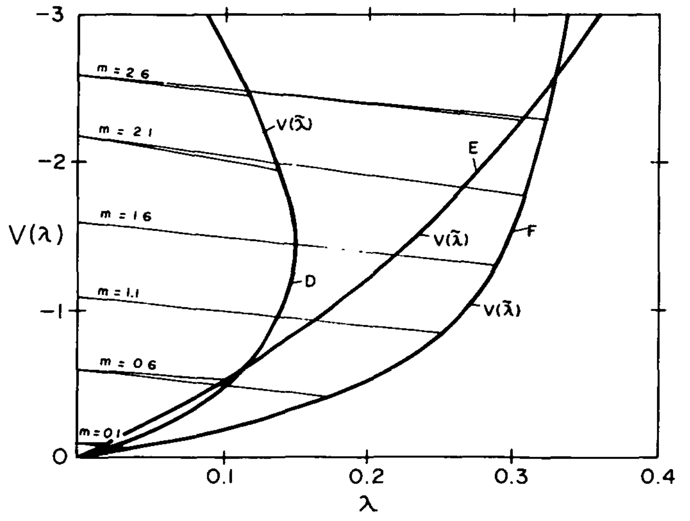

FIGURE 6. Deformation gradient in region of reflected simple wave.

The speeds of the reflected and transmitted waves for all six combinations are plotted in Figure 5, as functions of the incident shock strength m_0. Figure 6 shows the linear variation of the deformation gradient in the region of a simple wave (see (4.8)), for various values of $m_0 \in (0, 2.66)$, as a function of λ. The envelopes $v(\tilde{\lambda})$ for the final values of the wave parameter are also plotted. In combination D the range of variation of λ is decreasing with parameter m_0 changing from 1.6 to 2.66.

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