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INDECOMPOSABLE NON-HOLONOMIC $\mathcal{D}$-MODULES IN DIMENSION 2

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Abstract A module of a ring of differential operators $\mathcal{D}$ over a smooth surface has *order* 1 if it is isomorphic to a factor module of $\mathcal{D}$ by a cyclic ideal generated by an operator of order 1. Let k be a positive integer. We give conditions under which an indecomposable $\mathcal{D}$-module of order 1 is GK-critical of length k. We also give examples of indecomposable, non-critical, $\mathcal{D}$-modules whose subfactors have order 1.

Keywords: module; ring of differential operators; critical module

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1. Introduction

Holonomic modules have been the showpiece of the theory of rings of differential operators for many years. Their connections with many other areas of mathematics, and the fact that they are amenable to a complete classification, has secured them this place. However, many modules that occur in nature are not holonomic; the most conspicuous are probably those that correspond to a single partial differential equation.

In this paper we study the structure of the non-holonomic modules associated with a first-order algebraic partial differential equation in dimension 2. These modules have often been used in the construction of examples of simple non-holonomic modules (see [**2**, **7**, **8**, **20**]). This work has shown that there are two cases to consider, depending on whether the symbol of the differential operator is singular or non-singular. In this paper we deal only with the singular case.

The main results of the paper are easily summarized. Let $X \subseteq \mathbb{C}^n$ be an irreducible, smooth affine complex algebraic surface, and let $\bar{X}$ be its projective closure in $\mathbb{P}^n(\mathbb{C})$. Suppose that d is a derivation of the coordinate ring $\mathcal{O}(X)$. Denote by $\mathcal{D}(X)$ the ring of differential operators of X. We say that a $\mathcal{D}(X)$-module M has *order* 1 if there exists a derivation d and an element f of $\mathcal{O}(X)$ such that $M \cong \mathcal{D}(X)/\mathcal{D}(X)(d+f)$. The main results of the paper are summarized in the following theorem.

Theorem 1.1. *Suppose that $\bar{X}$ is an irreducible, smooth projective variety. If* $\operatorname{Pic}(X) = 0$, *then there exist*

(1) *infinitely many non-isomorphic GK-critical $\mathcal{D}(X)$-modules of length k, for every integer $k \geqslant 1$;*

(2) *infinitely many non-isomorphic indecomposable $\mathcal{D}(X)$-modules of length 2, whose simple subfactors have order 1.*

Theorem 1.1 (1) is a generalization of Theorem 3.3 of [**7**] which draws heavily on ideas from Bernstein and Lunts in [**2**, § 4] (see also [**17**]). However, all of these results apply only to modules over the Weyl algebra, and even in this case they are of a more limited scope than the results of § 3.

We prove the theorem by passing to the analytic category. As we will see, there are considerable differences between germs of non-holonomic $\mathcal{D}$-modules in the analytic and algebraic categories. Thus, modules that are simple in the algebraic category cease to be so when one passes to the analytic category.

2. Preliminary results

Throughout this section we assume that A is a Noetherian regular $\mathbb{C}$-algebra, which is a domain. The ring of differential operators $\mathcal{D}(A)$ is the subring of $\operatorname{End}_{\mathbb{C}}(A)$ generated by A and its $\mathbb{C}$-derivations. As usual, the module of $\mathbb{C}$-derivations of A will be denoted by $\operatorname{Der}_{\mathbb{C}}(A)$.

The ring $\mathcal{D}(A)$ has a natural filtration $\{C_k\}_{k\geqslant 0}$ by the order of a differential operator, where $\mathcal{C}_0 = A$, $\mathcal{C}_1 = A + \operatorname{Der}_{\mathbb{C}}(A)$ and $\mathcal{C}_k = \mathcal{C}_1^k$. The corresponding graded ring is isomorphic to the symmetric algebra of $\operatorname{Der}_{\mathbb{C}}(A)$, which we denote by $\mathcal{S}(A)$. If $\mathcal{S}_k(A)$ is the kth homogeneous component of $\mathcal{S}(A)$, the *symbol map* of order k is defined as the composition

$$\sigma_k : \mathcal{C}_k \to \mathcal{C}_k/\mathcal{C}_{k-1} \xrightarrow{\sim} \mathcal{S}_k(A),$$

where the first map is the canonical projection. An operator $P \in \mathcal{D}(A)$ has *order* k if $P \in \mathcal{C}_k \setminus \mathcal{C}_{k-1}$, and the *principal symbol* of P is $\sigma(P) = \sigma_k(P)$.

Let M be a finitely generated $\mathcal{D}(A)$-module with generators $u_1, \dots, u_s$. The assignment $\Gamma_k = \sum_{i=1}^{s} \mathcal{C}_k u_i$ defines a filtration of M. Moreover, $\operatorname{gr}^{\Gamma}(M)$ is finitely generated as an $\mathcal{S}(A)$-module and the ideal

$$\mathrm{I}(M) = \sqrt{\operatorname{ann}_{\mathcal{S}(A)}(\operatorname{gr}^{\Gamma}(M))}$$

is independent of the choice of generators for M. The variety $\operatorname{Ch}(M)$ that corresponds to $\mathrm{I}(M)$ in $\operatorname{Spec}(\mathcal{S}(A))$ is called the *characteristic variety* of M, and it is an invariant of M.

The *Poisson bracket* of $\mathcal{S}(A)$ is defined on homogeneous elements $f = \sigma_r(d)$ and $f' = \sigma_s(d')$ by the formula

$$\{f, f'\} = \sigma_{r+s-1}([d, d']).$$

When A is the coordinate ring of a smooth affine variety, the Poisson bracket defined above coincides with the one determined by the standard symplectic structure of the cotangent bundle. An ideal J of $\mathcal{S}(A)$ is *closed* with respect to the Poisson bracket if $\{J, J\} \subseteq J$. In this case we say that the variety of $\operatorname{Spec}(\mathcal{S}(A))$ defined by J is *involutive*. The characteristic variety of a finitely generated $\mathcal{D}(A)$-module is always an involutive variety, a fact that we will often use in this paper; for a proof see [**10**]. As a consequence of this fact we have that the dimension of the characteristic variety of a finitely generated $\mathcal{D}$-module over an irreducible smooth complex variety X is always greater than or equal to $\dim(X)$. *Holonomic modules* are those whose characteristic variety has dimension equal to $\dim(X)$.

An important special case occurs when A is a regular local ring. If $x_1, \dots, x_n$ is a regular system of parameters of A, then $\Omega^1(A)$ is generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$. Denoting by ∂_i the derivation of A that is dual to $\mathrm{d}x_i$, we have that $\mathcal{D}(A)$ is generated by A and by $\partial_1, \dots, \partial_n$. Hence, $\mathcal{S}(A)$ is isomorphic to the polynomial ring $A[\xi_1, \dots, \xi_n]$, where $\xi_i = \sigma_1(\partial_i)$. In particular, $\mathcal{S}(A)$ is a unique factorization domain. Note that if X is an irreducible smooth affine complex variety and $p \in X$, then the reasoning above applies to the local ring of X at p, which we denote by $\mathcal{O}_p(X)$. In this case we refer to $x_1, \dots, x_n$ as the *local coordinates* of X at p.

Let d be a derivation of A. An ideal I of A is *invariant* under d if $d(I) \subseteq I$. If I is a principal ideal generated by $a \in A$, then we say that a is *invariant* under d. A closed subvariety Y of an affine variety X is *invariant* under d if its ideal in $\mathcal{O}(X)$ is invariant under d.

Theorem 2.1. *Suppose that A is a unique factorization domain. Let d be a derivation of A and let $a \in A$ be invariant under d. Assume that a is not invertible in A and that $\sigma_1(d)$ is irreducible in $\mathcal{S}(A)$. Then, for all $f \in A$,*

$$\mathcal{D}(A)(d+f) \subsetneqq \mathcal{D}(A)(d+f) + \mathcal{D}(A)a \subsetneqq \mathcal{D}(A).$$

In particular, $\mathcal{D}(A)(d+f)$ is not a maximal left ideal of $\mathcal{D}(A)$.

Proof. Suppose that

$$\mathcal{D}(A)(d+f) + \mathcal{D}(A)a = \mathcal{D}(A),$$

and let us argue to a contradiction.

Choose an operator D_1 of smallest possible order r, for which there exists D_2 of order s such that

$$D_1 a + D_2(d+f) = 1. \tag{2.1}$$

Note that both D_1 and D_2 must be non-zero, because $d+f$ and a are non-invertible elements of $\mathcal{D}(A)$. Moreover, $r = s+1$, otherwise we would have that either

$$\sigma_r(D_1)a = 0 \quad \text{or} \quad \sigma_s(D_2)\sigma_1(d) = 0,$$

which could hold only if $D_1 = 0$ or $D_2 = 0$.

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Taking symbols of order $r = s + 1$, we get

$$\sigma_r(D_1)a + \sigma_s(D_2)\sigma_1(d) = 0.$$

But A is a regular domain, so $\mathrm{Der}_{\mathbb{C}}(A)$ is a projective A-module. Hence, $\mathcal{S}(A)$ is a unique factorization domain by [**18**, Corollaire 1, p. 243]. Since $\sigma_1(d)$ is irreducible by hypothesis, it follows that it divides $\sigma_r(D_1)$. Thus we can write

$$D_1 = C(d + f) + D'_1 \quad \text{and} \quad D_2 = -Ca + D'_2,$$

where $C, D'_1 \in \mathcal{C}_{r-1}$ and $D'_2 \in \mathcal{C}_{r-2}$. Let $d(a) = ga$, for some $g \in A$. It follows from (2.1) that

$$1 = C[d + f, a] + D'_1 a + D'_2(d + f) = (Cg + D'_1)a + D'_2(d + f).$$

But this contradicts the minimality of D_1, because $Cg + D'_1 \in \mathcal{C}_{r-1}$. □

Let d be a derivation of A and let $f \in A$. Write

$$M_A(d, f) = \frac{\mathcal{D}(A)}{\mathcal{D}(A)(d + f)}.$$

For most of the remainder of this paper we will be studying the structure theory of $M_A(d, f)$. Our first result is an immediate consequence of Theorem 2.1.

Corollary 2.2. *Let d be a derivation of A and let $f \in A$. Assume that $\sigma_1(d)$ is irreducible in $\mathcal{S}(A)$. If $M_A(d, f)$ is simple, then no non-unit of A is invariant under d.*

From now on we will assume that d is a derivation with a non-empty singularity set $\mathrm{Sing}(d)$, and that A is one of the following rings:

(1) the coordinate ring $\mathcal{O}(X)$ of an irreducible smooth affine complex surface;

(2) the localization $\mathcal{O}_p(X)$ of $\mathcal{O}(X)$ at a point $p \in X$;

(3) the ring $\mathcal{O}_2 = \mathbb{C}\{x_1, x_2\}$ of convergent power series in two variables.

For the sake of simplicity we will write $M_X(d, f)$ instead of $M_{\mathcal{O}(X)}(d, f)$.

Let d be a derivation of A which is singular at a closed point $p \in \mathrm{Spec}(A)$. Denote by λ_1 and λ_2 the eigenvalues of the 1-jet $j_p(d)$ of d at p. We will refer to them simply as the eigenvalues of d at p. Then

(1) λ_1, λ_2 are *resonant* if either $\lambda_1 = n\lambda_2$ or $\lambda_2 = n\lambda_1$, for some integer $n \geqslant 2$;

(2) λ_1, λ_2 belong to the *Poincaré domain* if $\lambda_1\lambda_2 \neq 0$ and $\lambda_1/\lambda_2 \notin \mathbb{R}^-$.

The following theorem of Poincaré [**1**, Chapter 5, § 24, p. 187] explains the importance of these conditions.

Theorem 2.3. *Let d be a derivation of $\mathcal{O}_2$ and suppose that 0 is a singularity of d whose eigenvalues are non-resonant and belong to the Poincaré domain. Then d is biholomorphically equivalent to its linear part $\lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2$.*

From now on we denote the ring of differential operators of $\mathcal{O}_2$ by $\mathcal{D}_2$. We will make frequent use of the following corollary of Theorem 2.3.

Corollary 2.4. *Let d be a derivation of $\mathcal{O}_2$ and suppose that* 0 *is a singularity of d whose eigenvalues are non-resonant and belong to the Poincaré domain. Then there exists an automorphism ϕ of $\mathcal{D}_2$ such that*

$$\phi(d) = \lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2.$$

Note that, as a consequence of Corollary 2.4, $\phi^{-1}(x_1)$ is invariant under d. Thus, by Corollary 2.2, the module $M_{\mathcal{O}_2}(d, f)$ cannot be simple when 0 is a singularity of d whose eigenvalues are non-resonant and belong to the Poincaré domain. Indeed, this holds even when the singularity does not satisfy these conditions, as the next result shows. However, it should be noted that the result of Camacho and Sad used in the proof of the corollary breaks down for dimensions greater than 2 (see [**11**]).

Corollary 2.5. *Let $d = g_1\partial_1 + g_2\partial_2$ be a derivation of $\mathcal{O}_2$, and assume that g_1 and g_2 have no common factor in $\mathcal{O}_2$. There is no choice of $f \in \mathcal{O}_2$ for which $M_{\mathcal{O}_2}(d, f)$ is a simple left $\mathcal{D}_2$-module.*

Proof. It follows from a result of Camacho and Sad that there exists a non-unit $h \in \mathcal{O}_2$ invariant under d (see [**6**]). Moreover, the hypothesis on g_1 and g_2 implies that the symbol of d is irreducible in $\mathcal{S}(A)$. The result follows from Corollary 2.2. □

Suppose now that A is the coordinate ring of a smooth affine surface X. It follows from Corollary 2.2 that for $M_X(d, f)$ to be simple d must have no invariant curves. The next result gives sufficient conditions on X for the existence of such derivations. Let $\mathrm{Sing}_0(d)$ be the set of $p \in \mathrm{Sing}(d)$ for which the eigenvalues of $j_p(d)$ are linearly independent over $\mathbb{Q}$ and belong to the Poincaré domain. In particular, such eigenvalues are non-resonant. Set $\mathrm{Sing}_1(d) = \mathrm{Sing}(d) \setminus \mathrm{Sing}_0(d)$.

Proposition 2.6. *Let k be a positive integer and let X be an algebraic irreducible complex affine surface with $\mathrm{Pic}(X) = 0$, whose projective closure $\bar{X}$ is smooth. There exist infinitely many derivations of $\mathcal{O}(X)$ whose singularity sets are pairwise distinct and such that, if d is one of these derivations, then*

(1) *d has no invariant algebraic curves; and*

(2) *$\mathrm{Sing}_0(d)$ has at least k points.*

Proof. Choose $m \geqslant 3$ such that $m^2 + m + 1 > k$. Let $\mathcal{G}$ be a generic foliation of degree m of $\mathbb{P}^2$ (see [**14**, p. 157] for definitions). By [**14**, Théoréme 1.1, p. 158], $\mathcal{G}$ has no algebraic invariant curves and the eigenvalues at each one of its singular points are linearly independent over $\mathbb{Q}$ and belong to the Poincaré domain.

Now consider a finite projection $\pi : \bar{X} \to \mathbb{P}^2$, and let $\mathcal{F}$ be the pull-back of $\mathcal{G}$ under π. We can assume, without loss of generality, that the branch locus of π does not contain any singular point of $\mathcal{G}$. It follows from [**16**, Theorem 1] that $\mathcal{F}$ leaves no algebraic curve of $\bar{X}$ invariant. Moreover, for every singularity p of $\mathcal{G}$, $\pi^{-1}(p)$ is a singular point of $\mathcal{F}$ whose eigenvalues are linearly independent over $\mathbb{Q}$ and belong to the Poincaré domain. However, $\pi^{-1}\operatorname{Sing}(\mathcal{G})$ does not account for the whole singular set of $\mathcal{F}$. Indeed, $\operatorname{Sing}_1(d)$ will always be non-empty.

Finally, since $\operatorname{Pic}(X) = 0$ and $\mathcal{F}$ is a locally free $\mathcal{O}_{\bar{X}}$-module of rank one, it follows that $\Gamma(X, \mathcal{F})$ is generated by one derivation d. Therefore, d cannot have algebraic invariant curves, and $|\operatorname{Sing}_0(d)| \geqslant m^2 + m + 1 \geqslant k$. □

An example of a family of foliations without algebraic invariant curves and whose eigenvalues satisfy the required properties can be found in [**14**, p. 157]. Of course the affine space has trivial Picard group and its projective closure is smooth, but there are many other surfaces with this property, as the next proposition shows.

Proposition 2.7. *Suppose that S is a generic complete intersection of dimension 2 in $\mathbb{P}^n$, for some $n \geqslant 3$. Let H be a hyperplane of $\mathbb{P}^n$, and write $X = S \setminus H$. Then X is a smooth irreducible affine surface whose Picard group is 0.*

Proof. Since S is a generic complete intersection, it must be irreducible and smooth. Thus so is X. We need only prove that $\operatorname{Pic}(X) = 0$. But by [**9**, Théorème 1.2, p. 328], the Picard group of S is the free abelian group generated by the hyperplane class in S. It follows that the Picard group of X is 0 by [**12**, Proposition 6.5, p. 133]. □

For a concrete example of a surface non-isomorphic to $\mathbb{A}^n$ whose Picard group is zero, see [**19**, § 4, p. 311]. We end with some results on analytic $\mathcal{D}$-modules in dimension 1. Denote by $\mathbb{C}\{x\}$ and $\mathcal{D}_1$, respectively, the ring of holomorphic functions in one variable and its ring of differential operators, and let $\partial = \mathrm{d}/\mathrm{d}x$. Although the next result is well known, we sketch the proof for lack of an adequate reference.

Proposition 2.8. *Let $f \in \mathbb{C}\{x\}$ and $\alpha, \beta \in \mathbb{C}$. Then*

(1) $\mathcal{D}_1/\mathcal{D}_1(x\partial + f) \cong \mathcal{D}_1/\mathcal{D}_1(x\partial + f(0))$;

(2) $\mathcal{D}_1/\mathcal{D}_1(x\partial + \alpha) \cong \mathcal{D}_1/\mathcal{D}_1(x\partial + \beta)$ *if and only if* $\alpha - \beta \in \mathbb{Z}$;

(3) $\mathcal{D}_1/\mathcal{D}_1 x\partial$ *is an indecomposable* $\mathcal{D}_1$*-module;*

(4) $\mathcal{D}_1/\mathcal{D}_1(x\partial + \alpha)^k$ *is indecomposable for all* $\alpha \notin \mathbb{Z}$ *and all integers* $k \geqslant 1$;

(5) $\mathcal{D}_1/\mathcal{D}_1(x\partial + \alpha)$ *is irreducible for all* $\alpha \notin \mathbb{Z}$.

Proof. If $P \in \mathcal{D}_1$ and I is any left ideal of $\mathcal{D}_1$, then $\bar{P}$ will denote the class of P in the factor module $\mathcal{D}_1/I$. Let $f = xg + f(0)$ and choose $G \in \mathbb{C}\{x\}$ such that $\mathrm{d}G/\mathrm{d}x = g$. The isomorphism in (1) maps $\bar{1} \in \mathcal{D}_1/\mathcal{D}_1(x\partial + f)$ to $\overline{\exp(-G)} \in \mathcal{D}_1/\mathcal{D}_1(x\partial + f(0))$.

The homomorphism

$$\mathcal{D}_1/\mathcal{D}_1(x\partial + \alpha) \to \mathcal{D}_1/\mathcal{D}_1(x\partial + \alpha + n)$$

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is defined by mapping $\bar{1}$ to $\bar{x}^n$, if n is a positive integer; and $\bar{1}$ to $\bar{\partial}^n$, if n is a negative integer. The converse follows from [**5**, Lemme 3, p. 318]; while (3) and (4) are proved in [**15**, Proposition 1.14, p. 39] (see also [**5**]).

Now let $M = \mathcal{D}_1/\mathcal{D}_1(x\partial + \alpha)$, for some $\alpha \notin \mathbb{Z}$. By [**15**, p. 37], the characteristic cycle of M is $T_{\mathbb{C}^n}\mathbb{C}^n + T_0\mathbb{C}^n$. But the characteristic cycle is additive over short exact sequences in the holonomic category. Thus if Q is a proper quotient of M, then $\mathrm{Ch}(Q) = T_{\mathbb{C}^n}\mathbb{C}^n$ or $\mathrm{Ch}(Q) = T_0\mathbb{C}^n$. In the first case, M has $\mathbb{C}\{x\}$ as a simple quotient by [**3**, Theorem 7.1, p. 207]. This implies that there exists a germ of holomorphic function g in the neighbourhood of the origin such that $x\,\mathrm{d}g/\mathrm{d}x = -g\alpha$, which is clearly impossible since $\alpha \notin \mathbb{Z}$. In the second case, M has $\mathbb{C}[\partial]$ as a simple quotient supported at the origin. Let $\sum_{i=0}^{r} c_i\partial^i$, $c_i \in \mathbb{C}$, be the image of $\bar{1} \in M$ under the map from M to $\mathbb{C}[\partial]$. An easy computation shows that

$$0 = (x\partial + \alpha)\left(\sum_{i=0}^{r} c_i\partial^i\right) = \sum_{i=0}^{r} c_i(\alpha - i - 1)\partial^i.$$

Since $\alpha \notin \mathbb{Z}$, this equation will hold only if $c_i = 0$ for all $0 \leqslant i \leqslant r$; which completes the proof of (5). □

3. Module structure

In this section we study the module structure of $M_X(d, f)$ when X is an affine algebraic surface. Some of the results we prove are generalizations and improvements of those in [**2**, § 4], [**7**]. Throughout the section X will denote an affine, smooth, irreducible surface over $\mathbb{C}$ whose Picard group is 0.

Proposition 3.1. *Let d be a derivation that has no proper invariant algebraic sets of dimension greater than zero and let $f \in \mathcal{O}(X)$. If Q is a proper quotient module of $\mathcal{D}(X)/\mathcal{D}(X)(d + f)$, then Q is a holonomic module supported either on the whole of X or on a finite number of points of X.*

Proof. The hypotheses imply that Q is isomorphic to $\mathcal{D}(X)/L$ for some left ideal L which contains $\mathcal{D}(X)(d + f)$ properly. Taking symbols we have that

$$\mathcal{S}(X)\sigma(d) \subsetneqq \sigma(L) \subsetneqq \mathcal{S}(X).$$

But $\sigma(d)$ has degree 1 in $\mathcal{S}(X)$. Thus it can only be factorized in the form $\sigma(d) = g\xi$, where $g \in \mathcal{O}(X)$ and ξ is an irreducible element of $\mathcal{S}_1(X)$. Now $g \in \mathbb{C}$, otherwise $g = 0$ would be a curve invariant under d, which is excluded by hypothesis. Hence, $\sigma(d)$ is irreducible in $\mathcal{S}(X)$. Since $\mathcal{S}(X)$ is a unique factorization domain by [**18**, Corollaire 1, p. 243], it follows that $\sigma(d)$ generates a prime ideal of $\mathcal{S}(X)$. In particular,

$$\emptyset \neq \mathcal{Z}(\sigma(L)) \subsetneqq \mathcal{Z}(\sigma(d)).$$

This implies that $\mathcal{Z}(\sigma(L))$ has dimension smaller than 3. But $\mathcal{Z}(\sigma(L))$ is involutive, so it is a Lagrangian variety. Thus $Q = \mathcal{D}(X)/L$ must be a holonomic module.

Since $d+f \in L$ and $\sqrt{\sigma(L)}$ is involutive, it follows that $I = \sqrt{\sigma(L)} \cap \mathcal{O}(X)$ is invariant under d. But the support of Q is equal to the set of zeros of I in X. Thus it is invariant under d. Therefore, by the hypothesis on d, the support of Q is either X or a subset of $\mathrm{Sing}(d)$. □

Lemma 3.2. *Let Y be an irreducible smooth affine variety over $\mathbb{C}$, and let M be a holonomic simple left $\mathcal{D}(Y)$-module whose support is Y. If the characteristic variety of M is not the zero section $T^*_Y Y$, then it must have an irreducible component supported at a hypersurface of Y.*

Proof. Since M is a holonomic module that has Y as its support, the characteristic variety of M must contain the zero section $T^*_Y Y$. Suppose that the other irreducible components of $\mathrm{Ch}(M)$ are supported on varieties of codimension greater than 2. Let W be the union of the supports of these components. We wish to arrive at a contradiction. Let $\mathcal{M}$ be the sheaf of $\mathcal{D}_Y$-modules that corresponds to M. Put $U = Y \setminus W$ and

$$\mathcal{D}_U \otimes_{\mathcal{D}_Y} \mathcal{M} = \mathcal{H}.$$

Hence, $\mathrm{Ch}(\mathcal{H}) = T^*_U U$. By [**4**, Chapter VI, Proposition 1.7], $\mathcal{H}$ is a locally free coherent sheaf of modules over $\mathcal{O}_U$. Thus there exists a morphism of $\mathcal{O}_U$-modules $\mathcal{H} \hookrightarrow \mathcal{O}_U^m$, for some positive integer m.

Let $i : U \hookrightarrow Y$ be the canonical embedding. Since Y is affine and W has codimension at least 2 in Y, it follows that $\Gamma(U, \mathcal{O}_U) \cong \Gamma(Y, \mathcal{O}_Y)$ (see [**13**, Theorem 2.15, p. 124]). Therefore, the sheaf-theoretic direct image $i_\star(\mathcal{O}_U)$ is isomorphic to $\mathcal{O}_Y$. Since the direct image functor is left exact, we have an injective map

$$i_\star \mathcal{H} \hookrightarrow \mathcal{O}_Y^m.$$

However, i is an open embedding, so the sheaf-theoretic direct image coincides with the $\mathcal{D}$-module-theoretic direct image (see [**4**, Chapter VI, § 5.2]).

It follows that the $\mathcal{D}_Y$-module $i_\star(\mathcal{H})$ is coherent over $\mathcal{O}_Y$. Therefore, $i_\star\mathcal{H}$ is locally free over $\mathcal{O}_Y$ by [**4**, Chapter VI, Proposition 1.7]. Thus $\mathrm{Ch}(i_\star\mathcal{H}) = T^*_Y Y$. Now consider the natural morphism $\phi : \mathcal{M} \to i_\star\mathcal{H}$. Since M is irreducible, ϕ must be injective. Therefore,

$$\mathrm{Ch}(M) = \mathrm{Ch}(\mathcal{M}) \subseteq \mathrm{Ch}(i_\star\mathcal{H}) = T^*_Y Y.$$

Hence, $\mathrm{Ch}(M) = T^*_Y Y$, a contradiction; and the proof is complete. □

Let d be a derivation of $\mathcal{O}(X)$. We denote by $\Lambda_p(d)$ the lattice of $\mathbb{C}$ generated by the eigenvalues of the 1-jet $j_p(d)$.

Lemma 3.3. *Let d be a derivation of $\mathcal{O}(X)$ and let $f \in \mathcal{O}(X)$. Suppose that S is a finite subset of X for which $S \cap \mathrm{Sing}_1(d) = \emptyset$. The module $M_X(d,f)$ has a factor Q supported at S if and only if*

(1) $S \subseteq \mathrm{Sing}(d)$;

(2) $f(p) = \mathrm{tr}(j_p(d))$ *for all* $p \in S$.

Moreover, under these hypotheses, Q has length $|S|$.

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Proof. Suppose that $M_X(d,f)$ has a factor module Q supported at a finite set S of points. (1) is clear. By [**4**, Theorem 10.6, p. 296], Q is completely reducible, and each direct summand is supported at a point $p \in S$. Thus, we may consider one point of S at a time. Let Q_p be the localization of Q at $p \in S$. Without loss of generality we may assume that p is the origin.

Passing to the analytic category and denoting by $\mathcal{D}_2$ the ring of germs of differential operators at p, we have

$$\mathcal{D}_2 \otimes M_X(d,f) \cong \frac{\mathcal{D}_2}{\mathcal{D}_2(d+f)}. \tag{3.1}$$

By Corollary 2.4 there exists an automorphism ϕ of $\mathcal{D}_2$ such that $d' = \phi^{-1}(d) = \lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2$. Twisting both sides of (3.1) with ϕ we get

$$M' = (\mathcal{D}_2 \otimes M_X(d,f))_\phi \cong \frac{\mathcal{D}_2}{\mathcal{D}_2(d'+f')},$$

where $f' = \phi^{-1}(f)$ and $f(0) = f'(0)$. Thus M' has $(\mathcal{D}_2 \otimes Q_p)_\phi$ for a factor module. Since ϕ preserves the origin, it follows that this last module is supported on the origin. Hence, by [**15**, Lemma 2.1, p. 41],

$$(\mathcal{D}_2 \otimes Q_p)_\phi \cong B_p^r,$$

where B_p is the unique simple $\mathcal{D}_2$-module supported at the origin p. The result will follow if we show that $f(p)$ is the sum of the eigenvalues of d at p and that $r = 1$.

However, x_1 is invariant under d', and the origin is contained in $x_1 = 0$, so that B_p^r is also a quotient of

$$N = \frac{\mathcal{D}_2}{\mathcal{D}_2(\lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2 + f') + \mathcal{D}_2 x_1} = \frac{\mathcal{D}_2}{\mathcal{D}_2(\lambda_2 x_2 \partial_2 + f' - \lambda_1) + \mathcal{D}_2 x_1}.$$

Let $i : \mathbb{C} \to \mathbb{C}^2$ be the embedding defined by $i(x_2) = (0, x_2)$. It follows from Kashiwara's equivalence (in the analytic category) that

$$N \cong i_+\left(\frac{\mathcal{D}_1}{\mathcal{D}_1(\lambda_2 x_2 \partial_2 + f'(0,x_2) - \lambda_1)}\right),$$

where $\mathcal{D}_1$ is the ring of differential operators of the ring of germs of holomorphic functions on x_2. Taking into account that $f(0,0) = f'(0,0)$, we have, by Proposition 2.8, that

$$N \cong i_+\left(\frac{\mathcal{D}_1}{\mathcal{D}_1(x_2 \partial_2 + (f(0,0) - \lambda_1)/\lambda_2)}\right).$$

Since we are assuming that N has B_p^r as a factor module, it follows from Proposition 2.8 that $(f(0,0) - \lambda_1)/\lambda_2$ must be an integer. Thus $f(0,0) = \lambda_1 + n\lambda_2$, for some integer n. Replacing x_1 with x_2 in the argument above, we conclude that $f(0,0) = m\lambda_1 + \lambda_2$, for some integer m. Since λ_1 and λ_2 are linearly independent over $\mathbb{Q}$ by hypothesis, it follows that $f(0) = \lambda_1 + \lambda_2$. But, in this case,

$$N \cong i_+\left(\frac{\mathcal{D}_1}{\mathcal{D}_1 x_2 \partial_2}\right)$$

by Proposition 2.8. In particular, N cannot have a holonomic quotient supported at 0 of length greater than 1. Hence, $r = 1$.

Conversely, suppose that (1) and (2) hold and that $p \in S$. We may assume, without loss of generality, that $p = (0,0)$ and that x_1, x_2 are local coordinates at the origin. If $d = g_1\partial_1 + g_2\partial_2$ in these coordinates, then

$$g_1\partial_1 + g_2\partial_2 + f = \partial_1 g_1 + \partial_2 g_2 - \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2}\right) + f.$$

Thus, in $\mathbb{C}[\partial_1, \partial_2]$ we have that

$$(d + f) \cdot 1 = -\operatorname{tr}(j_p(d)) + f(0,0) = 0$$

by (2). Hence, $M_X(d, f)$ has a holonomic quotient supported at the origin, and the proof is complete. □

If we drop the hypotheses on the derivation, we can still prove a lemma like 3.3 by strengthening the hypothesis on f. Since this is a simple generalization of [**7**, Lemma 2.1, p. 263], we omit the proof.

Lemma 3.4. *Let d be a derivation of $\mathcal{O}(X)$ and let $f \in \mathcal{O}(X)$. If the module $M_X(d, f)$ has a factor Q supported at a point $p \in X$, then*

(1) $p \in \operatorname{Sing}(d)$;

(2) $f(p) \in \Lambda_p(d)$ *for all* $p \in S$.

Theorem 3.5. *Let d be a derivation of $\mathcal{O}(X)$, and let $f \in \mathcal{O}(X)$. Assume that*

(1) *d has no invariant elements in $\mathcal{O}(X)$;*

(2) *S is a subset of* $\operatorname{Sing}_0(d)$;

(3) $\operatorname{Sing}(d) \setminus S \neq \emptyset$;

(4) $f(p) \notin \Lambda_p(d)$ *for all* $p \in \operatorname{Sing}(d) \setminus S$.

The module $M_X(d, f)$ is simple if and only if $f(p) \neq \operatorname{tr}(j_p(d))$ for all $p \in S$.

Proof. If $f(p) = \operatorname{tr}(j_p(d))$ for some $p \in S \subseteq \operatorname{Sing}_0(d)$, then $M_X(d, f)$ has a non-zero factor supported at p by Lemma 3.3. In order to prove the converse, suppose that $f(p) \neq \operatorname{tr}(j_p(d))$ for all $p \in S$.

Assume, by contradiction, that $\mathcal{D}(X)/\mathcal{D}(X)(d + f)$ admits a simple proper quotient module H. By Proposition 3.1, H must be a holonomic module whose support is either the whole of X or a finite set of points of X. We analyse these two cases separately.

First case: the support of H is X

By Lemma 3.2, $\mathrm{Ch}(H)$ must be equal to the zero section. Hence, H is finitely generated and locally free over $\mathcal{O}(X)$ by [**4**, Chapter VI, Proposition 1.7]. Now choose a singular point $p \in \mathrm{Sing}(d) \setminus S$ of d and let x_1 and x_2 be local coordinates at p. Without loss of generality we may assume that p is the origin and that the 1-jet of d at p is $(\lambda_1 x_1)\partial_1 + (\lambda_2 x_2 + c x_1)\partial_2$ for some complex number c. Let $\mathcal{D}_2$ be the ring of germs of analytic differential operators in a neighbourhood of the origin. Thus, $\mathcal{D}_2 \otimes H$ is a quotient of $\mathcal{D}_2/\mathcal{D}_2(d+f)$. Since H is finitely generated over $\mathcal{O}(X)$, it follows that $\mathcal{D}_2 \otimes H$ is finitely generated over the ring $\mathcal{O}_2$ of germs of holomorphic functions at the origin. Thus, by [**3**, Theorem 7.1, p. 207] there exists a map of $\mathcal{D}_2$-modules

$$\frac{\mathcal{D}_2}{\mathcal{D}_2(d+f)} \to \mathcal{O}_2.$$

In other words, there exists a germ of holomorphic function $h \in \mathcal{O}_2$ such that $(d+f)h = 0$. Let $\sum_{i+j=k} a_{ij} x_1^i x_2^j$ be the non-zero homogeneous component of smallest degree of the Taylor series of h at the origin. Comparing coefficients in $x_1^i x_2^j$ on both sides of $d(h) = -fh$, we have that

$$i\lambda_1 a_{i,j} + (j+1)c a_{i-1,j+1} + j\lambda_2 a_{i,j} = -f(0) a_{i,j}.$$

Choosing the smallest i for which $a_{i,j} \neq 0$ we obtain

$$i\lambda_1 + j\lambda_2 = -f(0) \notin \Lambda_p(d),$$

a contradiction. We conclude that H cannot have the zero section as its characteristic variety.

Second case: $\mathrm{Ch}(H)$ *is not supported at X*

Since H is simple, it follows from the first case that it must be supported at a point $p \in X$. Furthermore, this point must be stable under d; so it has to be a singular point of d. Thus $\mathrm{Ch}(H) = T_p^*\mathbb{C}^2$. There are two cases to consider. If $p \in \mathrm{Sing}(d) \setminus S$, then we have a contradiction by Lemma 3.4; while if $p \in S$, the contradiction follows from Lemma 3.3. In any case we conclude that $\mathcal{D}(X)/\mathcal{D}(X)(d+f)$ has no proper factor modules. □

Since the characteristic variety is an invariant of the module, it follows that if $M_X(d,f) \cong M_X(d',f')$, then $d = cd'$, for some non-zero constant c. However, even for a fixed characteristic variety there exist infinitely many non-isomorphic modules of the form $M_X(d,f)$, as the next proposition shows.

Proposition 3.6. *Let d be a derivation of $\mathcal{O}(X)$ and let c_1 and c_2 be complex numbers. Suppose that*

(1) *the eigenvalues of d at some point $p \in \mathrm{Sing}_0(d)$ are linearly independent over $\mathbb{Q}$;*

(2) *c_1, c_2 and $c_1 - c_2$ do not belong to $\Lambda_p(d)$.*

Then $M_X(d,c_1) \ncong M_X(d,c_2)$.

Proof. Without loss of generality we may assume that p is the origin. Localizing at p, and tensoring with $\mathcal{D}_2$ we conclude that

$$\frac{\mathcal{D}_2}{\mathcal{D}_2(d+c_1)} \cong \frac{\mathcal{D}_2}{\mathcal{D}_2(d+c_2)}. \tag{3.2}$$

By Corollary 2.4 there exists an automorphism ϕ of $\mathcal{D}_2$ such that

$$d' = \phi(d) = \lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2.$$

Twisting both sides of (3.2) with ϕ^{-1} we conclude that

$$\frac{\mathcal{D}_2}{\mathcal{D}_2(d'+c'_1)} \cong \frac{\mathcal{D}_2}{\mathcal{D}_2(d'+c'_2)}.$$

Now let i be the embedding $i(x_1) = (x_1, 0)$. Taking the inverse image of $\mathcal{D}_2/\mathcal{D}_2(d'+c_i)$ under i we obtain

$$\frac{\mathcal{D}_2}{(x_2\mathcal{D}_2 + \mathcal{D}_2(d'+c_i))} \cong \sum_{k\geqslant 0} \frac{\mathcal{D}_1}{\mathcal{D}_1(x_1\partial_1 + (c_i + k\lambda_2)/\lambda_1)} \partial_2^k,$$

which is isomorphic to an infinite direct sum of the modules

$$U_k^i = \frac{\mathcal{D}_1}{\mathcal{D}_1(x_1\partial_1 + (c_i + k\lambda_2)/\lambda_1)},$$

for $k \geqslant 0$. But the hypotheses on c_1 and c_2 imply that each U_k^i is a simple $\mathcal{D}_1$-module by Proposition 2.8. Thus

$$i^*\left(\frac{\mathcal{D}_2}{\mathcal{D}_2(d'+c'_1)}\right) \cong i^*\left(\frac{\mathcal{D}_2}{\mathcal{D}_2(d'+c'_2)}\right),$$

are semisimple modules. Therefore, there exists an integer $r \geqslant 0$ such that $U_0^1 \cong U_r^2$. Hence, by Proposition 2.8,

$$c_1 - c_2 - r\lambda_2 \in \mathbb{Z}\lambda_1.$$

It follows that $c_1 - c_2 \in \Lambda_p(d)$, which contradicts (2). □

We may now turn to GK-critical modules.

Theorem 3.7. *Let d be a derivation of $\mathcal{O}(X)$ and let $f \in \mathcal{O}(X)$. Assume that*

(1) *d has no invariant curves in X;*

(2) *S is a subset of $\mathrm{Sing}_0(d)$ with k points;*

(3) *$\mathrm{Sing}(d) \setminus S \neq \emptyset$; and*

(4) *$f(p) \notin \Lambda_p(d)$ for all $p \in \mathrm{Sing}(d) \setminus S$.*

If $f(p) = \mathrm{tr}(j_p(d))$ for all $p \in S$, then $M_X(d, f)$ is GK-critical of length $k+1$.

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Proof. Let $g=0$ be a hypersurface of $\mathbb{C}^n$ which contains S but not $\mathrm{Sing}(d)\setminus S$. If Q is a quotient of $M_X(d,f)$, then Q_g is a quotient of $M_Y(d,f)$, where $Y=X\setminus\mathcal{Z}(g)$. Since $\mathrm{Pic}(X)=0$, we have by [**12**, Proposition 6.5, p. 133] that $\mathrm{Pic}(Y)=0$. Applying Theorem 3.5 to the $\mathcal{D}(Y)$-module $M_Y(d,f)$ we conclude that $Q_g=0$. Hence, the support of Q is contained in $g=0$. By Proposition 3.1 this implies that Q is supported at a finite number of points. It follows from Lemma 3.3 that Q has length at most k.

Now let N be a submodule of $M_X(d,f)$ maximal, subject to the condition that $M_X(d,f)/N$ has length k. If N' is a non-zero submodule of N, then $M_X(d,f)/N$ is a quotient of $M_X(d,f)/N'$. However, $M_X(d,f)/N$ has maximum possible length. Hence, $N'=N$. Therefore, N is simple and $M_X(d,f)$ has length $k+1$. □

Theorem 1.1 (1) is an immediate consequence of the following corollary, which is itself a combination of Proposition 2.6 and Theorem 3.7.

Corollary 3.8. *Let k be a positive integer, and assume that X is an algebraic irreducible complex affine surface with $\mathrm{Pic}(X)=0$ whose projective closure $\bar{X}$ is smooth. There exist infinitely many derivations of X and coordinate functions $f\in\mathcal{O}(X)$ such that $M_X(d,f)$ is GK-critical of length k.*

We have shown that there exist indecomposable GK-critical modules of finite length whose characteristic variety is the hypersurface $\sigma(d)=0$, where d is a derivation of $\mathcal{O}(X)$ without invariant algebraic curves. Thus, it is natural to ask if there exists an indecomposable module with this same characteristic variety that is not critical. The answer is yes, as the next theorem shows.

Theorem 3.9. *Let d be a derivation of $\mathcal{O}(X)$ and let c be a complex number. Assume that*

(1) *d has no invariant curves in X;*

(2) *$c\notin\Lambda_p(d)$ for some $p\in\mathrm{Sing}(d)$.*

The module $\mathcal{D}(X)/\mathcal{D}(X)(d+c)^2$ is indecomposable of length 2 but not GK-critical.

Proof. The module $N=\mathcal{D}(X)/\mathcal{D}(X)(d+c)^2$ has a composition series of the form

$$0\subsetneq\frac{\mathcal{D}(X)(d+c)}{\mathcal{D}(X)(d+c)^2}\subsetneq\frac{\mathcal{D}(X)}{\mathcal{D}(X)(d+c)^2}$$

whose subfactors are isomorphic to $M_X(d,c)$. Hence, this is a module of length 2 that is neither simple nor GK-critical; and we must prove that it is indecomposable. Moreover, all simple factor modules of N are isomorphic to $M_X(d,c)$.

Suppose, by contradiction, that $N=Q_1\oplus Q_2$, where Q_1 and Q_2 are simple $\mathcal{D}(X)$-modules. Since Q_1 and Q_2 are quotients of N, we know that

$$Q_1\cong Q_2\cong M_X(d,c).$$

Let p be the singularity singled out in (2). Tensoring by $\mathcal{D}_2$ on a neighbourhood of p, we have that

$$\mathcal{D}_2 \otimes N \cong (\mathcal{D}_2 \otimes Q_1) \oplus (\mathcal{D}_2 \otimes Q_2).$$

Note that for $j = 1, 2$,

$$\mathcal{D}_2 \otimes Q_j \cong \mathcal{D}_2 \otimes M_X(d, c) \neq 0,$$

so that $\mathcal{D}_2 \otimes N$ cannot be indecomposable. However, by Corollary 2.4 there exists an automorphism ϕ of $\mathcal{D}_2$ such that the corresponding twisted module $(\mathcal{D}_2 \otimes N)_\phi$ is isomorphic to

$$\mathcal{D}_2/(\mathcal{D}_2(d' + c))^2,$$

where $d' = \lambda_1 x_1 \partial_1 + \lambda_2 x_2 \partial_2$.

Now let i be the embedding $i(x_1) = (x_1, 0)$. The inverse image of $(\mathcal{D}_2 \otimes N)_\phi$ under i is isomorphic to

$$\frac{\mathcal{D}_2}{x_2\mathcal{D}_2 + \mathcal{D}_2(d' + c)^2} \cong \sum_{k \geqslant 0} \left(\frac{\mathcal{D}_1}{\mathcal{D}_1(x_1\partial_1 + (c + k\lambda_2)/\lambda_1)^2} \right) \partial_2^k.$$

Similarly, the inverse image of $(\mathcal{D}_2 \otimes Q_j)_\phi$ under i is isomorphic to

$$\sum_{k \geqslant 0} \left(\frac{\mathcal{D}_1}{\mathcal{D}_1(x_1\partial_1 + (c + k\lambda_2)/\lambda_1)} \right) \partial_2^k.$$

Hence, $i^*((\mathcal{D}_2 \otimes (Q_1 \oplus Q_2))_\phi)$ is semisimple. In particular,

$$\frac{\mathcal{D}_1}{\mathcal{D}_1(x_1\partial_1 + (c + k\lambda_2)/\lambda_1)^2}$$

cannot be indecomposable. But this contradicts Proposition 2.8, since $c \notin \Lambda_p(d)$. It follows that $\mathcal{D}(X)/\mathcal{D}(X)(d + c)^2$ is indecomposable, as required. □

This completes the proof of Theorem 1.1. Finally, passing to an affine open set of X, we can prove all the above results for the local ring $\mathcal{O}_p(X)$, and a derivation d with an isolated singularity at p.

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