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INITIALLY STRUCTURED CATEGORIES AND CARTESIAN CLOSEDNESS

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Introduction. In recent papers Horst Herrlich [**4**; **5**] has demonstrated the usefulness of topological categories for applications to a large variety of special structures. A particularly striking result is his characterization of cartesian closedness for topological categories (see [**5**]). Spaces satisfying a separation axiom usually cannot form a topological category in Herrlich's sense however and some interesting special cases, e.g. Hausdorff k-spaces, remain excluded from his theory despite having many analogous properties. It therefore seems worthwhile to undertake a similar study in a wider setting. To this end we relax one of the axioms for a topological category and show that in the resulting initially structured categories a significant selection of results can still be proved, including the characterization of cartesian closedness. Moreover initially structured categories have useful hereditary properties not shared by topological categories. We apply the categorical results to several special cases. These include categories of convergence spaces, limit spaces and pseudo-topological spaces (with or without separation axioms) and also some "non-topological" examples such as preordered, ordered and bornological spaces. The category theory enables us to establish in a rather effortless way that each of these categories have well-behaved function space structures i.e. is cartesian closed. The same is true of their quotient-reflective or non-trivial finitely productive coreflective subcategories.

1. Initially structured categories. This section is mainly but not exclusively to prepare the way for the study of cartesian closedness in Section 2. Some key facts to be established include the equivalence of the notions extremal epi-sink and final epi-sink, the characterization of coreflective subcategories and the closedness of initially structured categories under formation of certain reflective and coreflective subcategories.

Frequent use will be made of the usual functorial version of Bourbaki's initial structures, which can be formulated as follows. Let $U : \mathfrak{A} \to \mathfrak{X}$ be a functor and $(A \xrightarrow{a_i} A_i)_{i \in I}$ a source in $\mathfrak{A}$. To say that (a_i) is *U-initial* means that for any source $(B \xrightarrow{b_i} A_i)_{i \in I}$ and any morphism f in $\mathfrak{X}$ such that $Ua_i \circ f = Ub_i$ for all i there is precisely one $\mathfrak{A}$ morphism c such that $Uc = f$ and $a_i \circ c = b_i$ for all $i \in I$. The dual concept will be called a *U-final sink*.

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We are now ready to formulate the central concept of this paper.

Definition: A category $\mathfrak{A}$ is *initially structured* with forgetful functor U if there exists a functor $U : \mathfrak{A} \to \mathbf{Set}$ such that the following hold:

IS1. Any source $(X \xrightarrow{f_i} UA_i)_{i \in I}$ in **Set** has an (epi, mono-source)-factorization

$$(X \xrightarrow{e} UB \xrightarrow{Ug_i} UA_i)_{i \in I}$$

such that (g_i) is U-initial.

IS2. U has small fibres, i.e. for every object X in **Set** there is at most a set of pairwise non-isomorphic $\mathfrak{A}$-objects A with $UA = X$.

IS3. There is precisely one object P (up to isomorphism) such that UP is terminal and separating in **Set**.

1.0 *Examples.*

(1) Every topological category in the sense of [**5**] is initially structured. Indeed, if in IS1 we were to replace (epi, mono-source)-factorization by (iso, any source)-factorization then the above definition would reduce to that of a topological category. It will be shown below (see 1.13) that initially structured categories are closed under formation of non-trivial coreflective and epi$_U$-reflective subcategories. Thus if such subcategories are formed iteratively from the basic topological categories **Born** (bornological spaces), **PNear** (prenear spaces, see [**4**]), **Con** (convergence spaces), **PrOrd** (preordered spaces) one obtains a vast collection of examples. These include the usual categories of Hausdorff topological spaces, k-spaces, uniform spaces, ordered spaces and many others.

(2) If "initially structured" is defined by IS1 and IS2 alone, then all results through 1.14 would still hold and moreover in this case **Set** could be replaced by **Vec** (the category of vector spaces over the real or complex numbers, with linear maps as morphisms); more generally, **Set** could be replaced by any category $\mathfrak{X}$ which is balanced, complete, co-complete, well-powered, cowell-powered, with every epimorphism regular and every monomorphism a section. To make clear the possibility of such replacement we retain general terminology and thus speak of epimorphism in **Set** rather than onto function. With **Vec** in the role of **Set** the results have obvious applications to functional analysis, e.g. to topological vector spaces, bornological vector spaces, ordered vector spaces.

For the remainder of the paper $\mathfrak{A}$ will be an initially structured category with forgetful functor U. Initiality and finality in any initially structured category will always be with respect to the forgetful functor in question.

The statement IS1 is an equivalent way of saying that U is (epi, mono-sources)-topological in the sense of [**3**]. Thus several useful properties of $\mathfrak{A}$ and U are known from [**3**]. We summarize some of them in the next theorem to which frequent reference will be made, not always explicitly.

1.1 THEOREM.

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(1) *U is faithful.*

(2) *A source $(A \xrightarrow{f_i} D_i)_{i \in I}$ is the limit of a diagram $D : I \to \mathfrak{A}$ if and only if this source is initial and $(UA \xrightarrow{Uf_i} UD_i)_{i \in I}$ is the limit of UD.*

(3) *For any sink $(UD_i \xrightarrow{f_i} X)_{i \in I}$ there exists a sink $(D_i \xrightarrow{a_i} A)$ and an epimorphism e with $e \circ f_i = Ua_i$ for all i and such that for any $d : X \to UB$ and any sink $(D_i \xrightarrow{b_i} B)_{i \in I}$ with $d \circ f_i = Ub_i$ for all i there exists a (unique) $c : A \to B$ such that $Uc \circ e = d$.*

$$\begin{tikzcd} & & UB \\ UD_i \arrow[urr, "Ub_i"] \arrow[r, "f_i"] \arrow[drr, "Ua_i"'] & X \arrow[ur, "d"] \arrow[dr, "e"] & \\ & & UA \arrow[uu, "Uc"'] \end{tikzcd}$$

(4) *$\mathfrak{A}$ is complete and cocomplete.*

Note that the e in 1.1.3 is a monomorphism hence an isomorphism whenever the available morphisms d form a mono-source.

1.2 LEMMA. *Any sink $(A_i \xrightarrow{f_i} C)_{i \in I}$ in $\mathfrak{A}$ has a factorization*

$$(A_i \xrightarrow{a_i} A \xrightarrow{c} C)_{i \in I}$$

such that (a_i) is a final sink and Uc an isomorphism.

Proof. By 1.1.3 there exists a final sink (a_i) and an epimorphism e such that $e \circ Uf_i = a_i$ and also there exists c such that $Uc \circ e = U1_A$. But then e is a monomorphism and since **Set** is balanced we conclude that e and Uc are isomorphisms.

A morphism f in $\mathfrak{A}$ which forms an initial mono-source will be called an *embedding*. Dually a morphism which forms a final epi-sink will be called a *quotient*.

1.3 PROPOSITION.
(1) *f is a monomorphism in $\mathfrak{A}$ if and only if Uf is a monomorphism in* **Set**.
(2) *f is an epimorphism in $\mathfrak{A}$ whenever Uf is an epimorphism in* **Set**.
(3) *Every extremal monomorphism in $\mathfrak{A}$ is an embedding.*

(4) *Every product in* $\mathfrak{A}$ *of embeddings is an embedding.*
(5) *If* $(g_i \circ f)$ *is an initial mono-source, then* f *is an embedding.*
(6) $(g_i \circ f)$ *is an initial source whenever* (g_i) *is an initial source and* f *is initial.*

1.4 PROPOSITION. *If* $(A_i \xrightarrow{f_i} A)_{i \in I}$ *is a final epi-sink in* $\mathfrak{A}$ *then* (Uf_i) *is an epi-sink in* **Set**.

Proof. Form the (epi-sink, extremal mono)-factorization $(UA_i \xrightarrow{e_i} X \xrightarrow{m} UA)$ of (Uf_i) in **Set**. Then m is a section, so there exists r such that $r \circ m = 1_X$. It follows that $m \circ r \circ Uf_i = Uf_i$ and since (f_i) is final we have $m \circ r = Ud$ for some $d : A \to A$. But $d \circ f_i = f_i = 1_A \circ f_i$ for all i. Hence $d = 1_A$ and $m \circ r = Ud = 1_{UA}$. We conclude that m is an isomorphism.

1.5 PROPOSITION. *A sink in* $\mathfrak{A}$ *is an extremal epi-sink if and only if it is a final epi-sink.*

Proof. Factorize the given sink (f_i) as $(c \circ a_i)$ with (a_i) final and Uc an isomorphism (see 1.2). Then c is a monomorphism (1.3). So if (f_i) is an extremal epi-sink, c is an isomorphism and (f_i) is final. Conversely, suppose (f_i) is a final epi-sink and $(f_i) = (m \circ g_i)$ a factorization with m a monomorphism. Then (Uf_i) is an epi-sink (see 1.4) and hence Um is an epimorphism in **Set** therefore an isomorphism. By finality $(Um)^{-1} = Uk$ for some k which is clearly the required inverse for m.

1.6 PROPOSITION.
(1) *If* $(f \circ g_i)$ *is a final epi-sink, then* f *is a quotient.*
(2) $(f \circ g_i)$ *is final whenever* f *is final and* (g_i) *is final.*
(3) *For any* $\mathfrak{A}$*-morphism* f *the following are equivalent:*
  (a) f *is a quotient,*
  (b) f *is a regular epimorphism,*
  (c) f *is an extremal epimorphism.*

Proof. 3 (a) implies (b): If f is a quotient, then Uf is an epimorphism hence a coequalizer in **Set** of (say) $c, c' : X \to UA$. By IS1 we have factorizations $c = Ud \circ e$, $c' = Ud' \circ e$ where e is an epimorphism. It is easy to check that Uf is then also the coequalizer of Ud, Ud'. Hence (see 1.1(3)) f is the coequalizer of d, d'.

1.7 PROPOSITION. $\mathfrak{A}$ *is well-powered,* epi_U*-cowell-powered, and quotient-cowell-powered.*

Proof. This follows from the well-poweredness and co-wellpoweredness of **Set** by IS2, 1.3 and 1.6. Note: f is epi_U (respectively iso_U) means Uf is epi (respectively iso).

1.8 PROPOSITION. $\mathfrak{A}$ *is an* $(\mathfrak{E}, \mathfrak{M})$*-category where* $(\mathfrak{E}, \mathfrak{M})$ *is any of the following pairs:* (a) $(epi_U$, *embedding*$)$, (b) $(epi_U$, *initial mono-sources*$)$, (c) (*quotient*, *mono*), (d) (*quotient*, *mono-sources*), (e) (*final epi-sink*, *mono*).

Proof. For (a) the result is immediate from IS1. Let $(A \xrightarrow{f_i} A_i)_{i \in I}$ be any source in $\mathfrak{A}$. By IS1 we have an (epi, mono-sources)-factorization $Uf_i = Um_i \circ e$ in **Set**, with (m_i) initial. Hence $e = Ud$ for some d with $f_i = m_i \circ d$ an (epi_U, initial mono-source)-factorization in $\mathfrak{A}$. The (epi_U, initial mono-sources)-diagonalization property follows by initiality from the (epi, mono-sources)-diagonalization property in **Set**. Thus (b) holds. By virtue of its properties $\mathfrak{A}$ is clearly an (extremal epi, mono)-category (see [**2**]) hence (c) follows by 1.6. From the factorization $f_i = m_i \circ d$ above we obtain a (quotient, mono-sources)-factorization $f_i = (m_i \circ m) \circ q$ by forming the (quotient, mono)-factorization $d = q \circ m$ and the (quotient, mono-sources)-diagonalization property follows by finality from the (epi, mono-sources)-diagonalization property in **Set**. Thus (d) holds. Again $\mathfrak{A}$ has the (extremal epi-sink, mono)-factorization property (see 35.6 in [**2**]) hence by 1.5 the (final epi-sink, mono)-factorization property and the corresponding diagonalization property follows as above by finality.

1.9 Lemma. *For any final epi-sink $(f_i)_{i \in I}$ in $\mathfrak{A}$ there is a set $J \subset I$ such that $(f_j)_{j \in I}$ is likewise a final epi-sink.*

Proof. Form the (quotient, mono)-factorization of each f_i and use the quotient cowell-poweredness of $\mathfrak{A}$ to get the required set-indexed sink.

1.10 Lemma. *A functor $F : \mathfrak{A} \to \mathfrak{A}'$ between initially structured categories which preserves colimits preserves final epi-sinks.*

Proof. Let $(A_i \xrightarrow{f_i} A)_{i \in I}$ be a final epi-sink in $\mathfrak{A}$. Form a set-indexed sink $(f_j)_{j \in J}$ as in 1.9 and let

$$A_j \xrightarrow{s_j} \coprod_{k \in J} A_k \xrightarrow{q} A$$

be its obvious factorization through the coproduct. Then q is a quotient hence a regular epimorphism. By assumption $(Fs)_j)_{j \in J}$ is a coproduct and Fq a regular epimorphism. Since every colimit is an extremal epi-sink and thus a final epi-sink we conclude that $(Ff_j)_{j \in J}$ is a final epi-sink. Then so is $(Ff_i)_{i \in I}$.

A class $\mathfrak{C}$ of $\mathfrak{A}$-objects will be called *non-trivial* if for any object A in $\mathfrak{A}$ there exists a sink $(C_i \xrightarrow{f_i} A)_{i \in I}$ with all C_i in $\mathfrak{C}$ such that (Uf_i) is an epi-sink in **Set**. Note that $\mathfrak{C}$ is non-trivial whenever $P \in \mathfrak{C}$ but this criterion does not apply when **Set** is replaced by **Vec** (see 1.0.2). We call $\mathfrak{C}$ *closed under formation of final epi-sinks* if A is in $\mathfrak{C}$ whenever there exists a final epi-sink $(C_i \xrightarrow{f_i} A)_{i \in I}$ with all C_i in $\mathfrak{C}$.

1.11 Theorem. *For a non-trivial subcategory $\mathfrak{C}$ of $\mathfrak{A}$ the following statements are equivalent.*

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(a) *$\mathfrak{C}$ is coreflective in $\mathfrak{A}$.*
(b) *$\mathfrak{C}$ is iso$_U$-coreflective in $\mathfrak{A}$.*
(c) *$\mathfrak{C}$ is closed under formation of colimits.*
(d) *$\mathfrak{C}$ is closed under formation of coproducts and quotients.*
(e) *$\mathfrak{C}$ is closed under formation of final epi-sinks.*

Proof. (c) $\Rightarrow$ (d) $\Rightarrow$ (e) is patterned after 1.10 and its proof. To prove (e) $\Rightarrow$ (b) we consider the sink $(C_i \xrightarrow{f_i} A)_{i \in I}$ of all possible f_i with the fixed codomain A and with domain C_i in $\mathfrak{C}$. By 1.2 there exists a factorization $(C_i \xrightarrow{g_i} B \xrightarrow{e_A} A)_{i \in I}$ of (f_i) such that (g_i) is final and Ue_A an isomorphism. Since $\mathfrak{C}$ is non-trivial, (f_i) and (g_i) are epi-sinks. Hence B is in $\mathfrak{C}$ and e_A is the required coreflection (counit). (a) $\Rightarrow$ (b) trivially and (b) $\Rightarrow$ (c) holds for any coreflective subcategory.

1.12 COROLLARY. *Let $\mathfrak{B}$ be a non-trivial class of $\mathfrak{A}$-objects. The following statements are equivalent for an $\mathfrak{A}$-object A.*
(a) *A is in the coreflective hull of $\mathfrak{B}$ in $\mathfrak{A}$.*
(b) *A is a quotient of a coproduct of $\mathfrak{B}$-objects.*
(c) *There exists a final epi-sink from $\mathfrak{B}$-objects to A.*
(d) *The sink of all possible morphisms from $\mathfrak{B}$-objects to A is a final epi-sink.*

Of course, several additional statements are known to be equivalent with these in 1.11 and 1.12 respectively; this is by virtue of $\mathfrak{A}$ being cocomplete, well-powered and epi$_U$-cowell-powered (see [**2**, section 37]).

1.13 THEOREM. *Let $\mathfrak{B}$ be a subcategory of $\mathfrak{A}$ with embedding functor $E : \mathfrak{B} \to \mathfrak{A}$. Then $\mathfrak{B}$ is initially structured with forgetful functor UE whenever one of the following holds:*
(a) *$\mathfrak{B}$ is a non-trivial coreflective subcategory of $\mathfrak{A}$.*
(b) *$\mathfrak{B}$ is an epi$_U$-reflective subcategory of $\mathfrak{A}$.*

Proof. Given the source $(X \xrightarrow{f_i} UEB_i)_{i \in I}$ we apply IS1 to get the factorization

$$(X \xrightarrow{e} UA \xrightarrow{Ug_i} UEB_i)_{i \in I}$$

with e an epimorphism and (g_i) a U-initial source. To prove IS1 in case (a) consider the right adjoint R of E and coreflection $e_A : ERA \to A$ and observe that

$$(X \xrightarrow{(Ue_A)^{-1} \circ e} UERA \xrightarrow{U(g_i \circ e_A)} UEB_i)_{i \in I}$$

is an (epi, mono-source)-factorization of (f_i). By fullness of E we have $g_i \circ e_A =$ some Eh_i and the U-initiality of (g_i) together with the universal property of e_A forces (h_i) to be UE-initial. To prove IS1 in case (b) we consider the left adjoint R of E and unit of adjunction $\eta : I \to ER$ and observe that they give rise to a factorization $(Eh_\iota \circ \eta_A)$ of the mono-source (g_ι). Hence $U\eta_A$ is a monomorphism and thus an isomorphism. It is easy to check that (h_i) is a UE-initial mono-source and thus that $UEh_\iota \circ (U\eta_A \circ e)$ will be a factorization of (f_ι) as required by IS1. Verification of IS2 and IS3 is straightforward in both cases.

It should be noted that the above proof in conjunction with 1.11 also shows that if $\mathfrak{A}$ is topological then $\mathfrak{B}$ is topological whenever (a) holds. The corresponding conclusion cannot be made when (b) holds.

We note for convenient reference the following fact which is a special case of [**3**, 7.4].

1.14 Lemma. *A subcategory $\mathfrak{B}$ is quotient-reflective in $\mathfrak{A}$ if and only if an object A is in $\mathfrak{B}$ whenever there exists a mono-source $(A \xrightarrow{f_\iota} B_\iota)_{\iota \in I}$ with all B_i in $\mathfrak{B}$.*

We now begin to exploit axiom IS3. It gives us an object P in $\mathfrak{A}$ such that UP is a set with precisely one member, $UP = \{p\}$ (say).

1.15 Proposition.

(1) *Every $f \in \mathfrak{A}(P, A)$ is an embedding.*

(2) *Every $f \in \mathfrak{A}(A, P)$ is a quotient.*

(3) *The correspondence $a \mapsto Ua$ is a natural bijection of $\mathfrak{A}(P, A)$ onto* **Set** *(UP, UA). Hence P is a separator in $\mathfrak{A}$.*

(4) *The correspondence $t \mapsto Ut$ is a natural bijection of $\mathfrak{A}(A, P)$ onto* **Set** *(UA, UP). Hence P is terminal in $\mathfrak{A}$.*

(5) *There exists a natural isomorphism $\varphi : U \to \mathfrak{A}(P, -)$. In fact φ can be defined so that for each $a \in UA$ we have $U\varphi_A(a)(p) = a$.*

(6) *Every projection $p_A : A \times P \to A$ has $(1_A, t_A)$ as inverse, where t_A is the unique morphism in $\mathfrak{A}(A, P)$.*

(7) *U preserves coproducts.*

1.16 Proposition. *Every projection $p_A : A \times B \to A$ is a quotient $(UB \neq \emptyset)$.*

Proof. Take any $b : P \to B$ and let q_A denote the projection $A \times P \to A$. By 1.15.6 q_A is a quotient and by 1.1.2 we have

$$Up_A \circ U(1_A \times b) = Up_A \circ (U1_A \times Ub) = Uq_A.$$

Thus p_A is a quotient by 1.6.1.

1.17 Proposition. *For an $\mathfrak{A}$-object D the following statements are equivalent.*

(a) *$U_A : \mathfrak{A}(D, A) \to$* **Set** *$(UD, UA)$ is a bijection, where $U_A(f) = Uf$.*

(b) *$(P \xrightarrow{d} D)_{d \in \mathfrak{A}(P,D)}$ is a coproduct.*

Objects D which enjoy these equivalent properties will be called *discrete*.

2. Cartesian closedness. We are now going to characterize cartesian closedness for initially structured categories $\mathfrak{A}$ along the lines of Herrlich's characterization for topological categories.

Cartesian closedness gives a **Set**-like quality to $\mathfrak{A}$. To emphasize this aspect and also because it seems technically more natural we will until further notice replace the functor U by $\mathfrak{A}(P, -)$. Since $\mathfrak{A}(P, -)$ is naturally isomorphic to U (see 1.15) all of Section 1 remains applicable. The use of $\mathfrak{A}(P, -)$ emphasizes the role of morphisms $a : P \rightarrow A$ as "points" of the object A.

2.1 THEOREM. *The following statements are equivalent.*

(a) *$\mathfrak{A}$ is cartesian closed i.e. it has finite products and for every $\mathfrak{A}$-object A the functor $A \times -$ has a right adjoint $(-)^A$.*

(b) *$A \times -$ preserves colimits for all A in $\mathfrak{A}$.*

(c) *$A \times -$ preserves coproducts and quotients for all A in $\mathfrak{A}$.*

(d) *$A \times -$ preserves final epi-sinks for all A in $\mathfrak{A}$.*

Proof. As in [**5**] the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear and (c) $\Rightarrow$ (d) follows by 1.10 and 1.6.

(d) $\Rightarrow$ (a): The diagram

$$\text{(1)}\qquad \begin{tikzcd} P \arrow[r, "a"] \arrow[d, "c"'] & A \arrow[r, "{(1, t)}"] & A \times P \arrow[d, "1 \times c"] & \\ C \arrow[r, "{(t, 1)}"'] & P \times C \arrow[r, "a \times 1"'] & A \times C \arrow[r, "f"] & B \end{tikzcd}$$

in $\mathfrak{A}$ is always commutative. The definition

$$g_{fC}(c) = f \circ (1 \times c) \circ (1, t)$$

gives us an epi-sink $\mathfrak{A}(P, C) \xrightarrow{g_{fC}} \mathfrak{A}(A, B)$ (indexed by f, C). By 1.1.3 there is a final epi-sink $(C \xrightarrow{f^*} B^A)$ and an epimorphism h_{AB} such that the diagram

$$\text{(2)}\qquad \begin{tikzcd} \mathfrak{A}(P, C) \arrow[rr, "g_{fC}"] \arrow[dr, "{\mathfrak{A}(P, f^*)}"'] & & \mathfrak{A}(A, B) \arrow[dl, "h_{AB}"] \\ & \mathfrak{A}(P, B^A) & \end{tikzcd}$$

commutes for all f and C. But h_{AB} is monic hence invertible since there exists a mono-source $\mathfrak{A}(A, B) \xrightarrow{m_a} \mathfrak{A}(P, B)$ indexed by $a \in \mathfrak{A}(P, A)$ such that $m_a \circ g_{fC} = \mathfrak{A}(P, d_{a,f})$ for $\mathfrak{A}$-morphisms $d_{a,f}$ (see 1.1.3); indeed we can take $m_a = \mathfrak{A}(a, B)$ and $d_{af} = f \circ (a \times 1) \circ (t, 1)$ (see Diagram 1). We can now form the commu-

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tative diagram

(3)
$$\begin{tikzcd}
 & \mathfrak{A}(P, B) \\
\mathfrak{A}(P, A) \times \mathfrak{A}(P, C) \arrow[ur, "\mathfrak{A}(P, f)"] \arrow[r, "1 \times g_{fC}"] \arrow[dr, "\mathfrak{A}(P, 1 \times f^*)"'] & \mathfrak{A}(P, A) \times \mathfrak{A}(A, B) \arrow[u, "k_{AB}"'] \\
 & \mathfrak{A}(P, A) \times \mathfrak{A}(P, B^A) \arrow[u, "1 \times h_{AB}^{-1}"']
\end{tikzcd}$$

where we define $k_{AB}(a, r) = r \circ a$. Since $(f^*)_{f,C}$ is a final epi-sink, so by assumption is $(1 \times f^*)_{f,C}$. Hence there is a unique $\mathfrak{A}$-morphism e_{AB} with $\mathfrak{A}(P, e_{AB}) = k_{AB} \circ (1 \times h_{AB}^{-1})$ and $e_{AB} \circ (1 \times f^*) = f$ for all f. If $e_{AB} \circ (1 \times f^*) = e_{AB} \circ (1 \times g)$ then (see Diagram (3)) $h_{AB}^{-1}(f^* \circ c) \circ a = h_{AB}^{-1}(g \circ c) \circ a$ for all $a : P \to A$, $c : P \to C$ whence $f^* \circ c = g \circ c$ for all c and $f^* = g$. We conclude that e_{AB} is co-universal for B with respect to $A \times -$.

2.2 COROLLARY. *Suppose $\mathfrak{A}$ is the coreflective hull of a class of objects $\mathfrak{K}$. Then $\mathfrak{A}$ is cartesian closed whenever one of the following holds for all A in $\mathfrak{A}$:*

(a) *$A \times -$ preserves colimits of $\mathfrak{K}$-valued diagrams in $\mathfrak{A}$.*

(b) *$A \times -$ preserves coproducts of $\mathfrak{K}$-objects and quotients.*

(c) *$A \times -$ preserves final epi-sinks $(K_i \xrightarrow{f_i} C)_{i \in I}$ with all K_i in $\mathfrak{K}$.*

When $A \times -$ has a right adjoint the associated natural bi-jection of $\mathfrak{A}(A \times C, B)$ onto $\mathfrak{A}(C, B^A)$ will be denoted by $\psi_{CB}(f) = f^*$ where f^* is characterized by the commutative diagram

$$\begin{tikzcd}
 & A \times B^A \arrow[dr, "e_{AB}"] & \\
A \times C \arrow[ur, "1 \times f^*"] \arrow[rr, "f"'] & & B
\end{tikzcd}$$

Henceforth r_P will denote the image of r under the isomorphism $h_{AB} : \mathfrak{A}(A, B) \to \mathfrak{A}(P, B^A)$.

The next theorem lists a number of nice properties of cartesian closed initially structured categories. Most of them have been noted for topological categories in [**5**]. We will sometimes denote B^A by $[A, B]$.

2.3 THEOREM. *When $\mathfrak{A}$ is cartesian closed the following hold ($\simeq$ denotes isomorphism).*

(1) *The counit* e_A *gives an internal evaluation for* $\mathfrak{A}$ *in the sense that the diagram*

$$\begin{tikzcd} P \arrow[r, "{(a, r_P)}"] \arrow[dr, "r\ a"'] & A \times B^A \arrow[d, "e_{AB}"] \\ & B \end{tikzcd}$$

commutes for all $a : P \to A$, $r : A \to B$.

(2) $\mathfrak{A}$ *has internal compositions i.e. there exist morphisms* $\circ$, $s \circ -$, $- \circ r$ $(r \in \mathfrak{A}(A, B),\ s \in \mathfrak{A}(B, C))$ *such that the following diagrams always commute*

$$\begin{tikzcd} P \arrow[r, "{(r_P, s_P)}"] \arrow[dr, "(s \circ r)_P"'] & B^A \times C^B \arrow[d, "\circ"] \\ & C^A \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "r_P"] \arrow[dr, "(s \circ r)_P"'] & B^A \arrow[d, "s \circ -"] \\ & C^A \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "s_P"] \arrow[dr, "(s \circ r)_P"'] & C^B \arrow[d, "- \circ r"] \\ & C^A \end{tikzcd}$$

(3) $\mathfrak{A}$ *has an internal hom-functor i.e. for each pair of morphisms* $(B \xleftarrow{r} A,\ E \xrightarrow{w} F)$ *there exists a morphism* $\circ_{rw}$ *such that the following diagram commutes for all* $u : B \to E$

$$\begin{tikzcd} P \arrow[r, "u_P"] \arrow[dr, "(w \circ u \circ r)_P"'] & E^B \arrow[d, "\circ_{rw}"] \\ & F^A \end{tikzcd}$$

Moreover the assignment $(r, w) \to \circ_{rw}$ *defines a functor* $(-)^{(-)}: \mathfrak{A}^{op} \times \mathfrak{A} \to \mathfrak{A}$ *such that* $\mathfrak{A}(P, -) \circ (-)^{(-)}$ *is naturally isomorphic to* $\mathfrak{A}(-, -)$.

(4) *The object* B^A *is characterized by the properties* $\mathfrak{A}(P, B^A) \simeq \mathfrak{A}(A, B)$ *and for any class* $\mathfrak{K}$ *of objects whose coreflective hull is all of* $\mathfrak{A}$ *the epi-sink*

$$(K \xrightarrow{\psi_{KB}(f)} B^A)_{f : A \times K \to B,\, K \in \mathfrak{K}}$$

is final.

(5) *Finite products of quotients are quotients.*

(6) $A^{B \times C} \simeq (A^B)^C$

(7) $[A, \prod_{i \in I} B_i] \simeq \prod_{i \in I} [A, B_i]$

(8) $A \times \coprod_{i \in I} B_i \simeq \coprod_{i \in I} A \times B_i$; *in particular* $A \times D = \coprod_{d \in \mathfrak{A}(P, D)} A$ *when* D *is discrete.*

(9) $[\coprod_{i \in I} A_i, B] \simeq \prod_{i \in I} [A_i, B]$; *in particular* $B^D = \prod_{d \in \mathfrak{A}(P, D)} B$ *when* D *is discrete.*

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Proof. (1) Implicit in the proof of 2.1. (2) Put $D = B^A \times C^B$, $E = B^A$, $F = C^B$, $\circ = \psi_{DC}(g)$ where g is the composition

$$A \times B^A \times C^B \xrightarrow{e_{AB} \times 1_F} B \times C^B \xrightarrow{e_{BC}} C;$$

let $s \circ -$, $- \circ r$ respectively be the compositions

$$B^A \xrightarrow{(1_E, t_E)} B^A \times P \xrightarrow{1_E \times s_P} B^A \times C^B \xrightarrow{\circ} C^A$$

$$C^B \xrightarrow{(t_F, 1_F)} P \times C^B \xrightarrow{r_P \times 1_F} B^A \times C^B \xrightarrow{\circ} C^A.$$

(3) Apply 2 and calculate. (4) We note that for each C in $\mathfrak{A}$ there is a final epi-sink $(K_i \xrightarrow{g_i} C)_{i \in I}$ with all K_i in $\mathfrak{K}$; hence if $C \xrightarrow{\psi_{CB}(f)} B^A$ is a final epi-sink, then so is $\psi_{CB}(f) \circ g_i$ and any larger sink. (5) With $f : A \to B$ and $g : C \to D$ we have

$$f \times g = (1_B \times g) \circ (f \times 1_C).$$

(6) The right adjoint of $B \times C \times -$ is the composition of the right adjoints of $C \times -$ and $B \times -$. (7) and (8) Right and left adjoints preserve products and coproducts respectively. (9) Given the coproduct $(A_j \xrightarrow{e_j} \coprod_{i \in I} A_i)_{j \in I}$ we have the $1 - 1$ correspondence $(r_k)_{k \in I} \leftrightarrow [r_i]_{i \in I}$ between $\prod_k \mathfrak{A}(A_k, B)$ and $\mathfrak{A}(\coprod_i A_i, B)$ where $[r_i] \circ e_j = r_j$ for all j. We lift this correspondence to $\mathfrak{A}$-morphisms. Define $m : [\coprod_i A_i, B] \to \prod_k [A_k, B]$ by $m = (- \circ e_k)$. Then $m \circ [r_i]_P = (r_{kP})$. To get an inverse for m we need n such that $n \circ (r_{kP}) = [r_i]_P$. It is enough to put $n = g^*$ where $g : \coprod_i A_i \times \prod_k [A_k, B] \to B$ is so chosen that $g \circ (a, (r_{kP}))$ always has a factorization $P \xrightarrow{a_j} A_j \xrightarrow{r_j} B$ for some a_j, since in that case we can conclude $g^* \circ (r_{kP}) = r_P$ where $r \circ e_j = r_j$ for all j. Now the composition

$$\coprod_i A_i \times \prod_k [A_k, B] \xrightarrow{h} \coprod_i (A_i \times \prod_k [A_k, B])$$
$$\xrightarrow{[1 \times p_i]} \coprod_i (A_i \times [A_i, B]) \xrightarrow{[e_{A_i B}]} B$$

will furnish such a morphism g if we can find an isomorphism h such that $h \circ (a, (r_{kP}))$ always factors through some $(a_j, (r_{kP}))$. The existence of such h follows by finality from the fact that $\mathfrak{A}(P, -)$ preserves coproducts (see 1.15).

Indeed we can sharpen (8) above by saying there exists an isomorphism

$$h : C \times \coprod_i A_i \to \coprod_i (C \times A_i)$$

such that for every $(c, a) : P \to C \times \coprod_i A_i$ we have $h \circ (c, a) = (1, e_j) \circ (c, a_j)$ for some j.

We proceed to show that cartesian closedness is inherited by certain kinds of reflective and coreflective subcategories. These inheritance properties will be used frequently in section 3. We resume the use of U as forgetful functor in preference to $\mathfrak{A}(P, -)$.

2.4 Proposition. *If $\mathfrak{A}$ is cartesian closed and $\mathfrak{B}$ is a non-trivial coreflective subcategory of $\mathfrak{A}$ such that the embedding functor $E : \mathfrak{B} \to \mathfrak{A}$ preserves finite products, then $\mathfrak{B}$ is cartesian closed.*

Proof. Note first that $\mathfrak{B}$ is initially structured by 1.13. Let $D : I \to B$ be a diagram and $(D_i \xrightarrow{f_i} C)_{i\in I}$ its colimit in $\mathfrak{B}$. Then $(ED_i \xrightarrow{Ef_i} EC)_{i\in I}$ is a colimit in $\mathfrak{A}$. Since $\mathfrak{A}$ is cartesian closed,

$$(EB \times ED_i \xrightarrow{1 \times Ef_i} EB \times EC)_{i\in I}$$

is a colimit in $\mathfrak{A}$ for each object $B \in \mathfrak{B}$. But $EB \times ED = E(B \times D)$ by assumption and since $\mathfrak{B}$ is closed under colimits in $\mathfrak{A}$ we conclude that

$$(B \times D_i \xrightarrow{1 \times f_i} B \times C)_{i\in I}$$

is a colimit in $\mathfrak{B}$.

2.5 Proposition. *If $\mathfrak{A}$ is cartesian closed and $\mathfrak{B}$ a quotient-reflective subcategory of $\mathfrak{A}$, then $\mathfrak{B}$ is cartesian closed.*

Proof. Note first that $\mathfrak{B}$ is initially structured by 1.13 and 1.6. Note also that since $\mathfrak{A}$ is a (quotient, mono-sources)-category (see 1.8), $\mathfrak{B}$ has the characteristic property of being closed under mono-sources (see 1.14), in particular, $C \in \mathfrak{B}$ whenever there is a monomorphism $m : C \to B$ and $B \in \mathfrak{B}$. Suppose now that $(B_i \xrightarrow{f_i} C)_{i\in I}$ is a final epi-sink in $\mathfrak{B}$ and let $(EB_i \xrightarrow{g_i} A \xrightarrow{m} EC)_{i\in I}$ be the (final epi-sink, mono)-factorization of (Ef_i) in $\mathfrak{A}$, where $E : \mathfrak{B} \to \mathfrak{A}$ is the embedding functor. Then $A = ED$ and by fullness $g_i = Eg_i'$, $m = Em'$ for some D, g_i', m' in $\mathfrak{B}$. Since $f_i = m' \circ g_i'$ is an extremal epi-sink (see 1.5) the monomorphism m' is an isomorphism and we conclude that $(EB_i \xrightarrow{Ef_i} EC)_{i\in I}$ is a final epi-sink in $\mathfrak{A}$. Since $\mathfrak{A}$ is cartesian closed and E preserves products,

$$(E(B \times B_i \xrightarrow{E(1 \times f_i)} E(B \times C))_{i\in I}$$

is a final epi-sink in $\mathfrak{A}$ for any object B in $\mathfrak{B}$. The left adjoint R of E preserves colimits, hence final epi-sinks (see 1.10). Hence $(1_B \times f_i)$ is a final epi-sink in $\mathfrak{B}$ as required.

The following result is useful in applications, particularly where the class $\mathfrak{K}$ of compact Hausdorff spaces are involved.

2.6 Proposition. *Suppose $\mathfrak{A}$ is cartesian closed and $\mathfrak{B}$ is the coreflective hull in $\mathfrak{A}$ of a nontrivial class $\mathfrak{K}$ such that $\mathfrak{B}$ contains all finite $\mathfrak{A}$-products of $\mathfrak{K}$-objects. Then $\mathfrak{B}$ is cartesian closed.*

Proof. In view of 2.4 it is enough to show that $\mathfrak{B}$ is finitely productive. If B and B' are $\mathfrak{B}$-objects, we have final epi-sinks $(K_i \xrightarrow{f_i} B)_{i \in I}$ and $(K_j \xrightarrow{g_j} B')_{j \in I'}$ with all K_i, K_j in $\mathfrak{K}$. Then $(f_i \times g_j)_{i,j}$ is an epi-sink with the factorization

$$\begin{tikzcd} K_i \times K_j' \arrow[rr, "f_i \times g_j"] \arrow[dr, "f_i \times 1"'] & & B \times B' \\ & B \times K_j' \arrow[ur, "1 \times g_j"'] & \end{tikzcd}$$

Since $\mathfrak{A}$ is cartesian closed $(f_i \times 1)$ is a final epi-sink (with j fixed) and $(1 \times g_j)_{j \in I'}$ likewise. We conclude that $(f_i \times g_j)_{i,j}$ is a final epi-sink and hence that $B \times B'$ is in $\mathfrak{B}$ (see 1.12).

3. Applications. In this section we apply the preceding theory to special cases. The categories of Hausdorff k-spaces and sequential spaces are perhaps the best known examples of cartesian closed initially structured categories. We prefer to consider some less known examples where more new results will be generated by our applications.

All special categories will have objects which are pairs (X, α) where X is a set and α some structure on X. In all cases the forgetful functor U in question will be the obvious one such that $U(X, \alpha) = X$.

3.1 *Categories of ordered spaces.* A preordered space is a pair $(X, \leqq)$ where X is a set and $\leqq$ a reflexive transitive relation on X. **PrOrd** will denote the category of preordered spaces and order preserving functions.

(3.1.1) **PrOrd** *is a topological category.*

(3.1.2) *A source* $(A \xrightarrow{f_i} A_i)_{i \in I}$ *in* **PrOrd** *is initial if and only if* $x \leqq y$ *holds in* A *precisely when* $f_i(x) \leqq f_i(y)$ *holds in* A_i *for all* $i \in I$. *In particular,* $(a, b) \leqq (a', b')$ *holds in the product space* $A \times B$ *if and only if* $a \leqq a'$ *and* $b \leqq b'$ *holds in* A *and* B *respectively.*

(3.1.3) *An epi-sink* $(B_i \xrightarrow{f_i} C)_{i \in I}$ *in* **PrOrd** *is final if and only if* $c \leqq c'$ *holds in* C *precisely when there exists an* (f_i)*-chain from* c *to* c', *i.e. a finite chain* $c = w_0 \leqq w_1 \leqq \ldots \leqq w_n = c'$ *such that for each* $k = 0, 1, \ldots (n-1)$ *there is a pair* $b_k \leqq b_k'$ *in some* $B_{i(k)}$ *such that* $f_{i(k)}(b_k) = w_k$ *and* $f_{i(k)}(b_k') = w_{k+1}$.

(3.1.4) **PrOrd** *is cartesian closed.*

Proof. Let A be any object and $(B_i \xrightarrow{f_i} C)_{i \in I}$ any final epi-sink in **PrOrd**. Our aim is to show that

$$(A \times B_i \xrightarrow{1 \times f_i} A \times C)_{i \in I}$$

is a final epi-sink. Suppose $(a, c) \leqq (a', c')$ holds in $A \times C$. Then $a \leqq a'$, $c \leqq c'$ holds in A, C respectively. Hence there is an (f_i)-chain from c to c', say $w_0 = c, w_1, \ldots, w_n = c'$. But then $(a, w_0), (a, w_1), \ldots (a, w_{n-1}), (a', w_n)$ is a $(1 \times f_i)$-chain from (a, c) to (a', c'). On the other hand, if $z_0, \ldots, z_n$ is a $(1 \times f_i)$-chain from (a, c) to (a', c') then clearly $z_0 \leqq z_1 \leqq \ldots \leqq z_n$ and $(a, c) \leqq (a', c')$. Applying 3.1.3 we conclude that $(1 \times f_i)$, which is clearly an epi-sink, is final.

Ord will denote the subcategory of **PrOrd** determined by those objects $(X, \leqq)$ for which $x \leqq y$ and $y \leqq x$ imply $x = y$. If $(A \overset{f_i}{\rightarrow} A_i)_{i \in I}$ is a mono-source with all A_i in **Ord**, then clearly A is in **Ord**. By applying 1.14, 1.13 and 2.5 we have the following result.

(3.1.5) **Ord** *is quotient reflective in* **PrOrd**, *hence* **Ord** *is an initially structured cartesian closed category.*

3.2 *Categories of convergence structures.* Let X be a set and q a function whose value at each x in X is a set qx of filters on X "convergent to x" such that

(1) $\dot{x} \in qx$ where $\dot{x}$ is the ultrafilter with base $\{x\}$, and

(2) $\mathscr{F} \in qx$ and $\mathscr{G} \supset \mathscr{F}$ implies $\mathscr{G} \in qx$.

Then (X, q) is called a:

convergence space if and only if $\mathscr{F} \cap \dot{x} \in qx$ whenever $\mathscr{F} \in qx$;

limit space if and only if $\mathscr{F} \cap \mathscr{G} \in qx$ whenever $\mathscr{F}, \mathscr{G} \in qx$;

pseudotopological if and only if $\mathscr{F} \in qx$ whenever $\mathscr{G} \in qx$ holds for every ultrafilter $\mathscr{G} \supset \mathscr{F}$;

pretopological space if and only if $\mathscr{N}_x \in qx$ where $\mathscr{N}_x$ is the intersection of all filters in qx.

A pretopological space is topological when $\mathscr{N}_x$ always has a member V such that $V \in \mathscr{N}_y$ for each $y \in V$. For background we refer to [**1**] and [**7**] where references to further basic papers will be found. The above spaces are the objects of the categories **Con**, **Lim**, **PsT**, **PrT** and **Top** to be discussed; the morphisms in each case are all continuous functions f, i.e. those carrying filters convergent to x to filters convergent to $f(x)$. The following are easy to check and essentially well-known.

(3.2.1) **Con**, **Lim**, **PsT**, **PrT** *and* **Top** *are topological categories. Each is a bireflective subcategory of every category preceding it in this list.*

We proceed to examine these categories for cartesian closedness.

(3.2.2) *A source* $(A \overset{f_i}{\rightarrow} A_i)_{i \in I}$ *is initial in* **Con**, **Lim**, **PsT**, **PrT** *and* **Top** *if and only if* $\mathscr{F} \in qx$ *holds in* $A = (X, q)$ *precisely when* $f_i(\mathscr{F}) \in q_i f_i(x)$ *holds in* $A_i = (X_i, q_i)$ *for all* $i \in I$. *In particular,* $\mathscr{F}$ *converges to* (x, y) *in a product space* $A \times B$ *if and only if the projections* $p_A\mathscr{F}$, $p_B\mathscr{F}$ *converge to* x, y *respectively.*

The characterization of final epi-sinks for various categories of convergence structures in this section generalize the characterizations for quotient maps given in [**7**].

(3.2.3) *An epi-sink* $(X_i, q_i) \xrightarrow{f_i} (X, q)(i \in I)$ *in* **Con** *is final if and only if for each* $y \in X$ $\mathscr{F} \in qy$ *implies that there exists* f_i *and* $\mathscr{E}$ *such that* $F \supset f_i(\mathscr{E})$ $\mathscr{E} \in q_i x$ *and* $f_i(x) = y$.

Proof. It is straightforward to verify that the refinements of filters $f_i(\mathscr{E})$ described above form a convergence structure and moreover that this is the finest such structure for which all functions f_i will be continuous.

(3.2.4) **Con** *is cartesian closed.*

Proof. Let $((X_i, q_i) \xrightarrow{f_i} (Y, q))_{i \in I}$ be any final epi-sink and (Z, r) any object in **Con**. Suppose $\mathscr{F}$ converges to (z, y) in $(Z \times Y, r \times q) \equiv (Z, r) \times (Y, q)$. By 3.2.2 $p_Y(\mathscr{F}) \in qy$ and by 3.2.3 $p_Y(\mathscr{F}) \supset f_i(\mathscr{E})$ where $\mathscr{E} \in q_i x$ and $f_i(x) = y$. But then $p_Z(\mathscr{F}) \times \mathscr{E} \in (r \times q_i)(z, x)$, $(1 \times f_i)(z, x) = (z, y)$ and $\mathscr{F} \supset (1 \times f_i)(p_Z(\mathscr{F}) \times \mathscr{E})$. By 3.2.3 $(1 \times f_i)$ is a final epi-sink and thus the result follows by 2.1.

Next we consider the subcategory **HCon** of **Con** as prototype of many others that may be defined by a separation axiom. Objects of **HCon** are the Hausdorff convergence spaces (i.e. those in which limits are unique). If $(A \xrightarrow{f_i} H_i)_{i \in I}$ is a mono-source in **Con** with all H_i in **HCon**, then clearly A is in **HCon**. By 1.14, 1.13 and 2.5 we have the following.

(3.2.5) **HCon** *is quotient reflective in* **Con**. *Hence* **HCon** *is a cartesian closed initially structured category.*

(3.2.6) *An epi-sink* $(X_i, q_i) \xrightarrow{f_i} (X, q)(i \in I)$ *in* **Lim** *is final if and only if for each* $y \in X$ $\mathscr{E} \in qy$ *implies that*

$$\mathscr{F} \supset f_{i(1)}(\mathscr{E}^1) \cap f_{i(2)}(\mathscr{E}^2) \ldots \cap f_{i(n)}(\mathscr{E}^n)$$

for some finite set of filters $\mathscr{E}^1, \ldots \mathscr{E}^n$ *such that* $\mathscr{E}^k \in q_{i(k)} x^k$ and $f_{i(k)}(x^k) = y$ *for* $k = 1, 2, \ldots n$.

The proof of the next result may be patterned after that of 3.2.4.

(3.2.7) **Lim** *is cartesian closed.*

This was already established in [**1**] by a different method. As in the case of 3.2.5 we have also for **HLim** (Hausdorff limit spaces) the following.

(3.2.8) **HLim** *is quotient-reflective in* **Lim**. *Hence* **HLim** *is a cartesian closed initially structured category.*

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(3.2.9) *An epi-sink* $(X_i, q_i) \xrightarrow{f_i} (X, q)(i \in I)$ *in* **PsT** *is final if and only if for each* $y \in X$ *and each ultrafilter* $\mathscr{F}$ *on* X, $\mathscr{F} \in qy$ *implies* $\mathscr{F} = f_i(\mathscr{E})$ *for some ultrafilter* $\mathscr{E}$ *such that* $\mathscr{E} \in q_i x$ *and* $f_i(x) = y$.

The proof of the next result may again be patterned after that of 3.2.4 ($p_Z(\mathscr{F}) \times \mathscr{E}$ has to be replaced by an ultrafilter containing it).

(3.2.10) **PsT** *is cartesian closed.*

For **HPsT** (Hausdorff pseudotopological spaces) we have the following analogue of 3.2.5.

(3.2.11) **HPsT** *is quotient reflective in* **PsT**. *Thus* **HPsT** *is a cartesian closed initially structured category.*

As a consequence of the preceding results we also have at once the following fact.

(3.2.12) *Every finitely productive non-trivial coreflective subcategory of* **Con**, **HCon**, **Lim**, **HLim**, **PsT**, **HPsT**, *is an initially structured cartesian closed category. So is in particular the coreflective hull of all compact Hausdorff spaces in each category or (if sequential spaces are desired) the coreflective hull of the space* N^* *(= one-point compactification of the discrete space of natural numbers).*

Thus in particular the category of locally compact Hausdorff convergence spaces studied in [**8**] is cartesian closed. It is well known that **Top** is not cartesian closed.

3.3 *The category of bornological spaces.* A bornological space (see [**6**; **9**]) is a pair $(X, \mathscr{B})$ where X is a set and $\mathscr{B}$ a family of "bounded" subsets of X such that

(1) $A, B \in \mathscr{B}$ implies $A \cup B \in \mathscr{B}$,
(2) $B \in \mathscr{B}$ and $A \subset B$ imply $A \in \mathscr{B}$, and
(3) B finite implies $B \in \mathscr{B}$.

A function between bornological spaces is called bounded if and only if it carries bounded sets onto bounded sets. **Born** will denote the category of bornological spaces with bounded functions. The utility of this category for functional analysis has become apparent through [**6**] and [**9**].

(3.3.1) **Born** *is a topological category.*

(3.3.2) *A source* $(X, \mathscr{B}) \xrightarrow{f_i} (X_i, \mathscr{B}_i)(i \in I)$ *in* **Born** *is initial if and only if* $B \in \mathscr{B}$ *holds whenever* $f_i(B) \in \mathscr{B}_i$ *for all* $i \in I$.

(3.3.3) *An epi-sink* $(X_i, B_i) \xrightarrow{f_i} (X, B)(i \in I)$ *in* **Born** *is final if and only if every* $B \in \mathscr{B}$ *is contained in a finite union of sets* $f_i(M)$ *with* $M \in \mathscr{B}_i$.

(3.3.4) **Born** *is cartesian closed.*

Proof. Let $(X_i, \mathscr{B}_i) \xrightarrow{f_i} (X, \mathscr{B})(i \in I)$ be an epi-sink and $(Z, \mathscr{C})$ an object in **Born**. Suppose $A \subset Z \times X$ is bounded in the product structure $\mathscr{B} \times \mathscr{C}$ (see 3.3.2). Then by 3.3.2 and 3.3.3

$$p_X(A) \subset f_{i(1)}(M^1) \cup \ldots \cup f_{i(n)}(M^n)$$

where $M^k \in \mathscr{B}_{i(k)}$ for each k. Then $p_Z(A) \times M^k \in \mathscr{C} \times \mathscr{B}_{i(k)}$ and

$$A \subset \cup_k (1 \times f_{i(k)})(p_Z(A) \times M^k).$$

Thus $(1 \times f_i)$ is a final epi-sink and we can apply 2.1.

REFERENCES

1. E. Binz and H. H. Keller, *Funktionräume in der Kategorie der Limesräume*, Ann. Acad. Sci. Fenn. *383* (1966), 1–21.

2. H. Herrlich and G. E. Strecker, *Category theory* (Allyn and Bacon, 1973).

3. H. Herrlich, *Topological functors*, Gen. Top. Appl. *4* (1974), 125–142.

4. ——— *Topological structures*, Math. Centre Tract *52* (1974), 59–122.

5. ——— *Cartesian closed topological categories* Math. Coll. Univ. Cape Town *9* (1974). 1–16.

6. H. Hogbe-Nlend, *Theorie des bornologies et applications*, Lecture Notes in Mathematics 213 (Springer 1971).

7. D. C. Kent, *Convergence quotient maps*, Fund. Math. *65* (1969), 197–205.

8. D. C. Kent and G. D. Richardson, *Locally compact convergence spaces* (preprint).

9. L. Waelbroeck, *Etude spectrale des algebres completes*, Mem. Acad. Royale Belgique, 1960.

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Ottawa, Ontario