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BLOCKING SETS AND SKEW SUBSPACES OF PROJECTIVE SPACE

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In what follows, a theorem on blocking sets is generalized to higher dimensions. The result is then used to study maximal partial spreads of odd-dimensional projective spaces.

Notation. The number of elements in a set X is denoted by $|X|$. Those elements in a set A which are not in the set B are denoted by $A - B$. In a projective space $\Sigma = PG(n, q)$ of dimension n over the field $GF(q)$ of order q, $\Gamma_d(\Omega_d, \Lambda_d$, etc.) will mean a subspace of dimension d. A *hyperplane* of Σ is a subspace of dimension $n - 1$, that is, of co-dimension one.

A *blocking set* in a projective plane π is a subset S of the points of π such that each line of π contains at least one point in S and at least one point not in S. The following result is shown in [**1**], [**2**].

THEOREM 1. *Let S be a blocking set in the plane π of order n. Then $|S| \geqq n + \sqrt{n} + 1$. If equality holds, then S is the set of points of a Baer subplane of π.*

We proceed to generalize this to higher dimensions.

THEOREM 2. *Let S be a set of points in $\Sigma = PG(n, q)$, $n \geqq 2$. Suppose that*

(1) *Every hyperplane of Σ contains at least one point of S.*

(2) *S does not contain any line.*

Then $|S| \geqq q + \sqrt{q} + 1$. If $|S| = q + \sqrt{q} + 1$, the points of S are the points of a Baer subplane of some plane in Σ.

Proof. The case $n = 2$ follows from Theorem 1, and we proceed by induction on n. Let us assume that $|S| \leqq q + \sqrt{q} + 1$. Now let u, v be any two points of S. By hypothesis there exists a point x on the line joining u to v such that x is not in S. The lines of Σ through x form the Points of the Quotient Geometry Σ_x. By joining each to x, the points S of Σ then yield a set of Points S_x in Σ_x. The dimension of Σ_x is $n - 1$. Each hyperplane σ of Σ through x yields a Hyperplane σ_x in Σ_x. By hypothesis, σ_x contains at least one Point of S_x. Since x, u, v are collinear, we have

$$|S_x| < |S| \leqq q + \sqrt{q} + 1.$$

Thus $|S_x| < q + \sqrt{q} + 1$. Then, by induction, some Line in Σ_x consists

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entirely of Points of S_x. Translating back into Σ, this means that some plane π of Σ containing x also contains at least $q + 1$ points of S. Let $\Lambda = \Lambda_{n-1}$ be any hyperplane of Σ containing π. Suppose that some subspace $\Gamma = \Gamma_{n-2}$ of Λ of dimension $n - 2$ contains no point of S. Each member of the pencil of $q + 1$ hyperplanes of Σ that contain Γ contains at least one point of S. Since Λ contains at least $q + 1$ points of S, we get

$$|S| \geqq 1 \cdot (q + 1) + q \cdot 1 = 2q + 1.$$

This contradicts the assumption that $|S| \leqq q + \sqrt{q} + 1$. Thus each hyperplane Γ of Λ contains at least one point of S. Since $S \cap \Lambda$ contains no line we obtain by induction that

$$|S \cap \Lambda| \geqq q + \sqrt{q} + 1,$$

with equality if and only if the points of $S \cap \Lambda$ are the points of a Baer subplane of some plane π of Λ. Now $|S \cap \Lambda| \leqq |S|$, and $|S| \leqq q + \sqrt{q} + 1$, by assumption. Since $|S \cap \Lambda| \geqq q + \sqrt{q} + 1$ it follows that $S \cap \Lambda = S$, and we are done.

We turn our attention to maximal partial spreads of $\Sigma = PG(2t + 1, q)$, $t \geqq 1$. A partial t-spread or, simply, a *partial spread* of Σ is a collection W of t-dimensional subspaces of Σ such that no two members of W have a point of Σ in common (i.e., any two members of W are skew). If each point of Σ lies on a (unique) member of W, then W is called a *spread* of Σ. In that case $|W| = q^{t+1} + 1$. A partial spread W is *maximal* provided that (1) and (2) below are both satisfied.

(1) W is not a spread

(2) W is not contained in any larger partial spread of Σ.

The integer $d = q^{t+1} + 1 - |W|$ is then called the *deficiency* of W.

THEOREM 3. *Let W be a maximal partial t-spread of $\Sigma = PG(2t + 1, q)$. Assume that $q \geqq 4$. Then $|W| \geqq q + \sqrt{q} + 1$.*

Proof. Put $W = \{w_1, w_2, \ldots, w_k\}$. By way of contradiction assume that

$$|W| = k < q + \sqrt{q} + 1.$$

Using this assumption on $|W|$, and counting incidences, it follows that there exists a hyperplane $\Omega = \Omega_{2t}$ of Σ containing none of the w_i. In Ω we now have k skew subspaces of the type $w_i \cap \Omega$. Repeating the above argument we can find a hyperplane of Ω containing none of the $w_i \cap \Omega_{2t}$. Proceeding like this we obtain a subspace $\Lambda = \Lambda_{t+2}$ such that $w_i \cap \Lambda = l_i$, where l_i is a line of Λ. Now put $R = \{l_1, l_2, \ldots, l_k\}$. No two lines l_i, l_j meet if $i \neq j$. Let P be any point on any line l_1 of R. Suppose that u was a line on P, $u \notin R$, such that each point of u is on a line of R. Similarly, let v be any other transversal of R through P. Now $|R - \{l_1\}| < q + \sqrt{q}$. Also $q + (q - 1) > q + \sqrt{q}$ if $q \geqq 3$. It follows

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that some two lines l_α, l_β of R would have as transversals the two co-planar lines u and v. Then l_α and l_β would intersect, a contradiction. Thus, for any point P on any line l_i of R, there is at most one transversal of R through P. So the total number of transversals to R is at most $k = |W|$. Let X denote those lines of Λ which are either lines of R or transversals to R. Then

$$|X| \leqq 2k < 2(q + \sqrt{q} + 1).$$

The number of hyperplanes of Λ that contain a given line is equal to $q^t + q^{t-1} + \ldots + 1$. For $q \geqq 4$ we have

$$2(q + \sqrt{q} + 1)(q^t + q^{t-1} + \ldots + 1) < q^{t+2} + q^{t+1} + \ldots + 1.$$

The total number of hyperplanes of Λ is $q^{t+2} + q^{t+2} + \ldots + 1$. From the above inequality we can therefore find a hyperplane $\Gamma = \Gamma_{t+1}$ of $\Lambda = \Lambda_{t+2}$ such that Γ contains no line of X. Then $w_i \cap \Gamma = x_i$, with x_i being a point of Γ. By our choice of Γ the set $S = \{x_1, x_2, \ldots, x_k\}$ contains no line of Γ. Since W is a maximal partial t-spread, each hyperplane of Γ contains at least one point of S. An appeal to Theorem 2 shows that the assumption $k < q + \sqrt{q} + 1$ leads to a contradiction. Thus $k \geqq q + \sqrt{q} + 1$, and the proof is complete.

Notation. Let W be a maximal partial t-spread of $\Sigma = PG(2t + 1, q)$ having deficiency d. Then we set $f(d) = \frac{1}{2}(d - 1)(d^3 - d^2 + d + 2)$.

THEOREM 4. *The following bounds hold*:

(i) $q + \sqrt{q} + 1 \leqq |W|$ *for* $q \geqq 4$.

(ii) $|W| \leqq q^{t+1} - \sqrt{q}$.

(iii) *If q is not a square, then* $f(d) \geqq q^{t+1}$.

(iv) *If* $t = 1$ *then* $q + \sqrt{q} + 1 < |W|$.

Proof. Part (i) has been shown in Theorem 3. Parts (ii) and (iii) follow exactly as in the proof of Theorem 5 in [**3**] which makes use of Bruck's embedding theorem. Part (iv) is shown in [**4**].

Remark. In Theorem 3.1 of his paper in Math. Zeit. (211–229, 1975) A. Beutelspacher obtained bounds which were stronger than those in Theorem 4 above. However, his proof is in error, as he points in a subsequent paper in Math. Zeit., and his results have been retracted.

REFERENCES

1. A. Bruen, *Baer subplanes and blocking sets*, Bull. Amer. Math. Soc. *76* (1970), 342–344.

2. ——— *Blocking sets in finite projective planes*, SIAM. J. Appl. Math. *21* (1971), 380–392.

3. ——— *Collineations and extensions of translation nets*, Math. Z. *145* (1975), 243–249.

4. A. Bruen and J. A. Thas, *Blocking sets*, Geom. Ded. *6* (1977), 193–203.

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