On the Foundations of Dynamics.

By Dr Peddie.

Note on a Theorem in connection with the Hessian of a Binary Quantic.

By Charles Tweedie, M.A., B.Sc.

Extension of the "Medial Section" problem (Euclid II:11, VI:30, etc.) and derivation of a Hyperbolic Graph.

By R. E. Anderson, M.A.

To divide the straight line AB (containing a units) at C so that

$$AB \, . \, BC = p \, . \, AC^2.$$

§ I.

By algebra, taking the positive root,

$$AC = \frac{AB}{2p}(\sqrt{4p+1} - 1), \quad . \quad . \quad . \quad (1.)$$

The number p may therefore have any positive value, integral or fractional, and when negative cannot exceed $\frac{1}{4}$. Secondly, AC and AB are incommensurable except when $4p+1$ is a square:— *e.g.*, if $4p = (q-1)(q+1)$ or if $p = q(q+1)$, q being any positive integer or fraction.

To find the surd-line $\sqrt{4p+1}$ geometrically is the heart of the problem. *Euclid* solves it (II:11) when $p = 1$ by I:47, which is also used in Ex. i., ii., iii. following; but II:14 will sometimes be easier. Since equation (1.) becomes $AC = \sqrt{4p+1} - 1$, if $AB = 2p$, *i.e.*, if the unit line is $\frac{AB}{2p}$ or $\frac{AM}{p}$ *, we construct thus:—

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(Figure 1.)

Ex. i. Solve $AB.BC = 3AC^2$. Here $p = 3$, surd $= \sqrt{13}$.
Take $AK = \frac{1}{3}AM$, $AR = 2AK$, and $KS = RM$.
C the point required is determined by a square described on AS.

(Figure 2.)

Ex. ii. Solve $AB.BC = 4AC^2$. Here $p = 4$, surd $= \sqrt{17}$.
Take $AK = \frac{1}{4}AM$ and $KS = KM$.
C the point required is determined by square on AS.

(Figure 3.)

Ex. iii. Solve $AB.BC = 7AC^2$. Here $p = 7$, surd $= \sqrt{29}$.
Take $AK = \frac{1}{7}AM$, $AR = 2AK$, $AT = 5AK$, and $KS = RT$.
C is determined by square on AS.

(Figure 4.)

Ex. iv. Solve $AB.BC = AC^2$ by II : 14. Here $p = 1$, surd $= \sqrt{5}$. Produce AB both ways till $AK = AM$ the unit, and $AH = 5AM$ $\therefore$ $AR = \sqrt{5}$ if $\perp$ AB and limited by $\frac{1}{2}$ $\odot$ on KH : $\therefore$ if $KC = AR$, C gives the medial section of AB; and if $KC' = KC$, C′ is point of external section, corresponding to the negative root of equation (1.).

(Figure 5.)

Ex. v. Solve $AB.BC = \frac{2}{3}AC^2$. Here $p = \frac{2}{3}$, surd $= \sqrt{\frac{11}{3}} = \frac{1}{3}\sqrt{33}$. Produce AB both ways. In AB produced take $AK = \frac{3}{2}AM$, $AW = 3AK$, and $AY = 11AK$. $\therefore$ $AR = \sqrt{3 \times 11}$ if $\perp$ AB and limited by $\frac{1}{2}$ $\odot$ on WY. Hence if $KC = \frac{1}{3}AR$, C is the point required; and if $KC' = KC$, C′ is the point of external section, as in Ex. iv. above.

Thus for a given value of p the surd number $\sqrt{4p+1}$ though intractable to Arithmetic can always be found by Geometry. In certain cases lower surds should be subsidised: thus for $\sqrt{33}$ (just found by II : 14) we may say $33 = 5^2 + (2\sqrt{2})^2$.
Similarly $21 = 4^2 + (\sqrt{5})^2$; $181 = 13^2 + (2\sqrt{3})^2$.

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* In the diagrams M is the mid-point of AB and in Figs. 1, 2, 3, SAK is $\perp$AB.

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§ II.

(Figure 6.)

For the general case of equation (1.), the surd line $\sqrt{4p+1}$ can be expressed either by I : 47, since $4p+1=(2\sqrt{p})^2+1^2$, or by II : 14. Thus, by the latter, produce AB both ways till

AK = the unit $=\dfrac{AB}{2p}$ and BH = BK, so that AH = $4p+1$ units. $\therefore$ AR = $\sqrt{4p+1}$ if $\perp$ AB and limited by $\frac{1}{2}$ ⊙ on KH. Hence if KC = KC′ = AR, C and C′ are the required points of section.

(Figure 7.)

Mr G. Duthie suggests a third general construction for eq. (1.). "Produce AB till BK = MB = MA, and take MH $=\dfrac{AM}{p}$: with centre K and radius KM describe ⊙ MQW. Finally take HR = HQ the tangent from H. Then if AC = MR, C is the point required."

As shown above, C and C′ are two points in AB or its production which determine the roots of the original quadratic. Thus, if AR $=\dfrac{AB}{2p}$, then

(α.), when p is positive, with limits ∞ and 0,

$$AC = AR(\sqrt{1+4p}-1),\ \text{and}\ AC' = -AR(\sqrt{1+4p}+1),$$

$\therefore$ as AR grows continuously from 0 to ∞, so C and C′ move further apart from A in opposite directions;

(β.), when p is negative, with limits $-\frac{1}{4}$ and $-\frac{1}{\infty}$,

$$AC = AR(1-\sqrt{1+4p}),\ \text{and}\ AC' = AR(1+\sqrt{1+4p}),$$

$\therefore$ as AR grows continuously from $-2a$ to $-\infty$, so C and C′ move in opposite directions further and further apart from Z which is a point in AB produced positively so that AZ = $2a$.

(γ.). At any instant both for (α.) and (β.), the distance CC′ = twice the surd-line; and, numerically, AC′ − AC = 2AR.

We may note also from equation (1.) that generally $\frac{AC}{AM} = \frac{\sqrt{4p+1}-1}{p}$; $\therefore$ AC$\lesseqgtr$AM according as $\sqrt{4p+1} \lesseqgtr 1+p$, or as $2 \lesseqgtr p$: [Thus AC$=\frac{1}{2}$AB when $p=2$; AC$>\frac{1}{2}$AB when $p<2$ as in *Euclid* II:11 and Ex. v. above; AC$<\frac{1}{2}$AB when $p>2$ as Ex. i., ii., iii.], ... (2.)

§ III.

(Figure 8.)

To show graphically the variation of the segment AC, as obtained by equation (1.), I have placed P_1R_1 P_2R_2 P_3R_3 . . . . perpendicular to the fixed line AB, so that

$P_1R_1 = AR_1(\sqrt{5}-1)$, $P_2R_2 = AR_2(\sqrt{9}-1)$, $P_3R_3 =$ etc., etc.,

and generally $PR = AR(\sqrt{4p+1}-1)$, where PR = AC, $AR = \frac{AB}{2p}$

$$\therefore \quad PR + AR = PS = AR\sqrt{4p+1} = \frac{AS}{\sqrt{2}}\sqrt{4p+1}$$

$$\therefore \quad \left.\begin{aligned} 2PS^2 &= AS^2(4p+1) \\ &= AS\,(2a\sqrt{2}+AS) \end{aligned}\right\} \text{ since } 4p = \frac{2a}{AR} = \frac{2a\sqrt{2}}{AS},$$

$$= AS\,.\,A'S, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Hence for P, any point in the locus, the square of PS has a constant ratio to the rectangle AS . A′S, and that is the geometrical property of a Hyperbola having A′OASE as a diameter and PS an ordinate to it. Thus PS = SQ, FAD is a tangent at A, F′A′ another at A′, and O is the centre of the curve.

With reference to the original problem, PS (or SQ) is the surd-line $\sqrt{4p+1}$, AR (or RS) the unit-line, and the two roots of equation (1.) are PR and RQ. If, for example,

$$\left.\begin{aligned} p = 1,\ SP &= SQ = \sqrt{5} \\ PR &= \quad \sqrt{5}-1 = AC, \text{ internal segment} \\ RQ &= -\sqrt{5}-1 = AC', \text{ external segment} \end{aligned}\right\} \begin{aligned} &\textit{cf. } \text{Ex. iv. of § I.} \\ &\text{and } \textit{Euclid} \text{ II : 11.} \end{aligned}$$

The ordinate at P′ if produced upward to meet the branch P′V′A′ in Q′, and downward to meet AB in R′, gives P′R′ = AC and Q′R′ = AC′, for p negative. The Euclidian solution therefore of this case must place C and C′ in AB produced through B, as already shown, § II.

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§ IV.

The Cartesian equation to this Hyperbola is at once derived from equation (3.) thus, choosing OFG as the axis of x,

$$2PS^2 = AS \,.\, A'S = (OS - a\sqrt{2})(OS + a\sqrt{2})$$

$$\left.\begin{aligned} &= OS^2 - 2a^2 \\ \therefore \quad 2(x+y)^2 &= 2x^2 - 2a^2 \end{aligned}\right\} \text{ putting } x+y \text{ for PS and } x\sqrt{2} \text{ for OS.}$$

$$\therefore \qquad y^2 + 2xy + a^2 = 0. \qquad \text{(4.)}$$

The form of this equation shows that one asymptote is parallel to the axis of x $\therefore$ OFG is that asymptote. But $y^2 + 2xy = y(y+2x)$, $\therefore$ $y + 2x = 0$ is the other asymptote, viz., the line OD. Thus V′OV bisecting the angle FOD is the transverse axis of the Hyperbola, V and V′ are the vertices.

Finally, referring the curve to its own axes, equation (4.) becomes

$$\frac{x^2}{\sqrt{5}+1} - \frac{y^2}{\sqrt{5}-1} = \frac{a^2}{2}, \quad \text{or} \quad \left(\frac{x}{h}\right)^2 - \left(\frac{y}{k}\right)^2 = 1, \qquad \text{(5.)}$$

where

$$h^2 = OV^2 = \frac{a^2}{2}(\sqrt{5}+1) = 2a^2\cos\frac{\pi}{5}$$

$$k^2 = OW^2 = \frac{a^2}{2}(\sqrt{5}-1) = 2a^2\cos\frac{2\pi}{5}$$

Thus $hk = a^2 = \frac{1}{4}LL'$, where L is the lat. rectum of the curve, and L′ that of its conjugate. Hence

$$L = 4k\cos 72°.$$

A curious property therefore of our Hyperbola is that

$$\frac{L}{WW'} = \frac{WW'}{VV'} = \frac{VV'}{L'} = 2\cos 72°, \qquad \text{(6.)}$$

In other words, an isosceles Δ satisfying *Euclid* IV : 10 is found by taking the two terms of any one of those three fractions, and making the numerator the base.

Another singular property, easily deduced, is that

$$r^{\frac{2}{3}} + a^{\frac{2}{3}} = (4L)^{\frac{2}{3}}, \qquad \text{(7)}$$

where r = radius of curvature at the extremity of L the latus rectum.

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