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ON UNSYMMETRIC DIRICHLET FORMS

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1. Introduction. Let F be a linear, but not necessarily closed, subspace of $L^2[X, dm]$, where $(X, \mathscr{B}, m)$ is a σ-finite measure space with $\mathscr{B}$ the Borel subsets of the locally compact space X. If u and v are measureable functions, then v is called a normalized contraction of u if $|v(x)| \leqq |u(x)|$ and $|v(x) - v(y)| \leqq |u(x) - u(y)|$. Assume that F is stable under normalized contractions, that is, if $u \in F$ and v is a normalized contraction of u, then $v \in F$. Then a symmetric, positive semidefinite quadratic form $\mathscr{E}$ on $F \times F$ is called a symmetric Dirichlet form if and only if

(i) $$\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$$

whenever v is a normalized contraction of u;

(ii) for some (and therefore all) $\lambda > 0$, F is a Hilbert space under the inner product

$$\mathscr{E}_\lambda(f, g) = \mathscr{E}(f, g) + \lambda(f, g)$$

where $(\cdot, \cdot)$ denotes the L^2 inner product.

Each such Dirichlet form gives rise to a symmetric submarkovian pseudo-resolvent $\{G_\lambda; \lambda > 0\}$ on L^2, that is, a family of symmetric, continuous, linear transformations on L^2 to L^2 such that $0 \leqq u \leqq 1$ a.e. implies $0 \leqq \lambda G_\lambda u \leqq 1$ a.e. and

$$G_\lambda - G_\mu = (\mu - \lambda) G_\lambda G_\mu \quad (\lambda, \mu > 0). \tag{1.1}$$

The G_λ's satisfy the equation

$$\mathscr{E}_\lambda(f, G_\lambda g) = (f, g). \tag{1.2}$$

Conversely, each symmetric submarkovian pseudo-resolvent $\{G_\lambda : \lambda > 0\}$ on L^2 is associated with a symmetric Dirichlet form by the formulas

$$\mathscr{E}(f, g) = \lim_{\lambda \to \infty} \lambda(f, g - \lambda G_\lambda g), \tag{1.3}$$

$$F = \{f \in X : \sup_{\lambda > 0} \lambda(f, f - \lambda G_\lambda f) < \infty\}. \tag{1.4}$$

The pair $(F, \mathscr{E})$ is called a Dirichlet space relative to $L^2[X, dm]$.

The concept of a Dirichlet space is due to Beurling and Deny [**1**]. Their definition is more general than the one given above, since they do not assume their spaces contained in an L^2 space. The Dirichlet spaces relative to L^2 were

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used by Fukushima [**2**] to classify all the symmetric conservative Brownian motion processes on a bounded closed domain in $\mathbf{R}^n$.

Since unsymmetric submarkovian resolvents frequently arise in the applications, it is important to extend the notion of Dirichlet form so as to be able to handle the more general situation. One such extension has been given by H. Kunita [**3**]. He has used this extension to generalize Fukushima's results to the classification of unsymmetric multidimensional diffusion processes.

Let us briefly outline some of his results which are pertinent to the present paper. Let B be a Hilbert lattice which is contained in an enlarged lattice B_1. It is assumed that there exists a positive element $e \in B_1$ such that

$$f \in B \Rightarrow (f \wedge e) \in B.$$

Consider a bilinear form $\mathscr{E}$ (not necessarily symmetric) defined on a set $F \times F$ where F is a linear subspace of B such that:

B1: $\mathscr{E}$ is bounded from below, i.e. there exists $\beta_0 \geqq 0$ such that

$$\mathscr{E}(f,f) + \beta_0(f,f) \geqq 0 \quad (f \in F);$$

B2: $|\mathscr{E}(f,g)| \leqq K[\mathscr{E}(f,f) + \beta_0(f,f)]^{1/2}[\mathscr{E}(g,g) + \beta_0(g,g)]^{1/2}$ for all $f, g \in F$;

B3: F is dense in B and complete relative to each of the norms

$$[\mathscr{E}(f,f) + \alpha(f,f)]^{1/2} \quad (\alpha > \beta_0);$$

B4: F is a vector sublattice of B such that

$$f \in F \Rightarrow (f \wedge e) \in F$$

and

$$\mathscr{E}((f - ce)^+, f \wedge ce) \geqq 0$$

for all $c \in \mathbf{R}^+$ and all $f \in F$. A bilinear form satisfying B1-B4 is called a Dirichlet bilinear form.

To see why this new definition extends the notion of symmetric Dirichlet form, we must first specify what "submarkovian" means in this context.

Definition. A linear operator $T : B \to B$ is called submarkovian if and only if

$$0 \leqq u \leqq e \Rightarrow 0 \leqq Tu \leqq e.$$

The following theorem is proved in Kunita's paper [**3**]:

THEOREM 1.1. *If B is a Hilbert lattice, then to any bilinear form satisfying* B1-B3 *there is associated a unique semigroup $\{T_t : t > 0\}$ such that $||T_t|| \leqq \exp(\beta_0 t)$, whose generator A satisfies*

$$\mathscr{E}(f,g) = -(f, Ag) \quad (f \in F).$$

The semigroup is submarkovian (meaning that each T_t is) if and only if B4 *is satisfied.*

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The semigroup $\{T_t\}$ is called the semigroup associated with $\mathscr{E}$. A submarkovian resolvent family $\{G_\lambda : \lambda > 0\}$ is also associated with $\mathscr{E}$ and satisfies

$$G_\lambda f = \int_0^\infty e^{-\lambda t} T_t f dt. \tag{1.5}$$

This resolvent then has the property that

$$||\lambda G_\lambda|| \leqq \exp(\beta_0 t)$$

and (1.2) and (1.3) hold.

Thus each Dirichlet form in the sense of Kunita's definition gives rise to a submarkovian pseudo-resolvent exactly as in the symmetric case. However, the converse is not true, i.e. there are submarkovian pseudo-resolvents whose corresponding bilinear forms fail to satisfy B2 and B3. (See section 6 of the present note for an example.) The 1-1 correspondence between symmetric submarkovian pseudo-resolvents and Dirichlet forms that exists in the symmetric case does not hold here.

In this note we give a more general definition of Dirichlet form which recaptures the 1-1 correspondence between submarkovian resolvents and Dirichlet forms. We also extend Theorem 1 to the case where B is a more general Banach lattice than a Hilbert lattice.

We prefer to work with the pseudo-resolvents rather than the semigroups. The following theorem (see [**3**] for a proof) allows us to pass from the pseudo-resolvent to the semigroup:

THEOREM 1.2. *Let $\{G_\alpha : \alpha > \beta_0\}$ be a submarkovian pseudo-resolvent such that*

$$(\alpha - \beta_0)^m ||G_\alpha||^m \leqq k \quad \textit{for } \alpha > \beta_0.$$

Then there exists a unique submarkovian semigroup $\{T_t : t > 0\}$ in L = the smallest strongly closed vector lattice including $R(G_\alpha)$ = range G_α, *such that for each $f \in L$ the function $t \to T_t f$ is right continuous relative to the w^*-topology of $\bar{L}$ = the completion of L in the weak topology of L, and* (1.5) *holds for all $f \in L$.* (Note: In the statement of the theorem we regard $\bar{L}$ as being embedded in the bidual L^{**}. The w^*-topology on $\bar{L}$ then means the topology induced by the w^*-topology on L^{**}.)

Our general definition is given in section 3. In sections 4 and 5 we prove that to each of our Dirichlet forms there corresponds a submarkovian pseudo-resolvent, and conversely. In section 6 we discuss an example of a resolvent whose corresponding Dirichlet form does not satisfy B2 of Kunita's definition. We also indicate in section 6 a type of application which will be exploited more fully in a subsequent paper.

2. Preliminaries. Let B be a Banach lattice and B^* its dual. That is, B is a complete normed vector lattice such that

$$|y| \leqq |x| \Rightarrow ||y|| \leqq ||x||.$$

We shall denote the duality between B and B^* by $(\cdot, \cdot)$. If $u \in B$ let

$$Ju = \{u' \in B^* : (u', u) = ||u||_B^2 = ||u'||_{B^*}^2\}. \tag{2.1}$$

Using the usual notation $u^+ = u \vee 0$ and $u^- = -(u \wedge 0)$ we have

$$([\phi']^-, \phi) \leqq 0 \quad (\phi \in B, \phi' \in J(\phi)). \tag{2.2}$$

We usually express conditions such as (2.2) in the notationally simpler form

$$([J\phi]^-, \phi) \leqq 0. \tag{2.3}$$

We also use the fact that

$$|| \, |\phi| \, || = ||\phi|| \geqq ||\phi^+||. \tag{2.4}$$

Assumptions about the lattice B. Although we do not assume that B is a Hilbert lattice, we shall need some conditions on B. We shall always assume, as in [**3**], that B is contained in a larger lattice B_1 in which there exists a positive element e ($\neq 0$) with the property that

$$u \in B \rightarrow (e \wedge u) \in B. \tag{2.5}$$

Here $B \subset B_1$ only as a vector lattice. We assume no topology on B_1. A sub-markovian mapping is then defined as in the definition of section 1.

Notation. If c is a real number we shall write $\mathbf{c}$ to denote the element $ce \in B_1$. Any element $w \in B$ can be written as

$$w = (w - \mathbf{c})^+ + (w \wedge \mathbf{c}), \tag{2.6}$$

so (2.5) implies

$$w \in B \rightarrow (w - \mathbf{c})^+ \in B \quad (c \geqq 0). \tag{2.7}$$

Finally, we assume that B has the following property:

$$v \leqq \mathbf{c} \rightarrow (J[(w - \mathbf{c})^+], v - (w \wedge \mathbf{c})) \leqq 0 \quad (w \in B, c \geqq 0). \tag{2.8}$$

Note that this implies

$$v \leqq \mathbf{c} \rightarrow (J[(w - \mathbf{c})^+], v - w) \leqq 0 \quad (w \in B, c \geqq 0). \tag{2.9}$$

Furthermore, (2.8) implies

$$\begin{aligned} &v \leqq \mathbf{c} \quad \text{and} \quad (J[(w - \mathbf{c})^+], v - w) = 0 \\ &\Rightarrow (w - \mathbf{c})^+ = 0 \quad (w \in B, c \geqq 0). \end{aligned} \tag{2.10}$$

In particular, if $c = 0$ we get

$$(J[w^+], w) = 0 \Rightarrow w^+ = 0 \quad (w \in B). \tag{2.11}$$

3. Dirichlet forms. In what follows B is a Banach lattice satisfying the conditions of section 1. Let F be a linear subspace of B such that $\bar{F}$ (the norm closure of F) is a sublattice of B satisfying

$$\phi \in \bar{F} \Rightarrow (\phi \wedge e) \in \bar{F}. \tag{3.1}$$

Let $F^* = [J(\bar{F})]$ = the linear space generated by $J(\bar{F})$. We assume that

$$[\psi \in F^* \quad \text{and} \quad (\psi, \phi) = 0, \quad \text{for} \quad \phi \in \bar{F}] \Rightarrow [\psi = 0]. \tag{3.2}$$

Definition 3.1. A Dirichlet form $\mathscr{E}$ on $F^* \times F$ is a bilinear form satisfying (3.3), (3.4), and (3.6) below:

$$\mathscr{E}(J\phi, \phi) \geqq 0 \quad (\phi \in F); \tag{3.3}$$

$$\mathscr{E}(J\{(\phi - \mathbf{c})^+\}, \phi) \geqq 0 \quad (c \geqq 0, \phi \in F). \tag{3.4}$$

For each $\alpha > 0$ define

$$\mathscr{E}_\alpha(\psi, \phi) = \mathscr{E}(\psi, \phi) + \alpha(\psi, \phi). \tag{3.5}$$

Given $\alpha > 0$

$$[\lim_\beta \mathscr{E}_\alpha(\psi_\beta, \phi) = 0, \text{ for all } \phi \in F] \to [\lim_\beta (\psi_\beta, f) = 0, \text{ for all } f \in \bar{F}]. \tag{3.6}$$

If $\mathscr{E}_\alpha(\psi, \phi) = 0$ for all $\phi \in F$, and some $\psi \neq 0$, then (3.6) implies that $(\psi, f) = 0$ for all $f \in \bar{F}$, which is impossible by (3.2). On the other hand, given $\phi \in F$, $\phi \neq 0$, we have $J\phi \in F^*$. Choosing $\phi' \in J\phi$ we then get from (3.3)

$$\mathscr{E}_\alpha(\phi', \phi) \geqq \alpha(\phi', \phi) = \alpha||\phi||^2 \neq 0.$$

Thus for each $\alpha > 0$, the form $\mathscr{E}_\alpha(\psi, \phi)$ defines a separated duality between F and F^*. Let the corresponding weak topologies be denoted by $\sigma_\alpha(F, F^*)$ and $\sigma_\alpha(F^*, F)$, respectively, on F and F^*. By similar arguments, we see that the bilinear form $(\cdot, \cdot)$ also defines a separated duality between $\bar{F}$ and F^*. Let us denote the corresponding weak topologies for this duality by $w(\bar{F}, F^*)$ and $w(F^*, \bar{F})$. Then condition (3.6) states simply that the $\sigma_\alpha(F^*, F)$ topology is stronger than the $w(F^*, \bar{F})$ topology on F^*.

It is to be noted, in comparing our definition with that of Kunita, that we have normalized for simplicity so that $\beta_0 = 0$.

4. The existence of submarkovian resolvents. In this section we assume we are given a Dirichlet form $\mathscr{E}$ on a set $F^* \times F \subset B^* \times B$, as defined in the previous section.

THEOREM 4.1. *Given $f \in \bar{F}$ and $\lambda > 0$, there exists a unique $G_\lambda f \in F$ such that the family of linear mappings $f \to G_\lambda f$ $(\lambda > 0)$ is a submarkovian resolvent satisfying*

(i) $||\lambda G_\lambda f|| \leqq ||f||$ $(f \in \bar{F})$;

(ii) *for $f \in \bar{F}$, $\psi \in F^*$ we have*

$$\mathscr{E}_\lambda(\psi, G_\lambda f) = (\psi, f) \tag{4.1}$$

and $f \to G_\lambda f$ is thus a continuous mapping from $\bar{F}$ with the $w(\bar{F}, F^*)$ topology to F with the $\sigma_\lambda(F, F^*)$ topology;

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(iii) $\mathscr{R} = G_\lambda \bar{F}$ *is* $\sigma_\lambda(F, F^*)$*-dense in* F *for each* $\lambda > 0$;

(iv) $\mathscr{E}(\psi, \phi) = \lim_{\lambda\to\infty} \lambda(\psi, \phi - \lambda G_\lambda \phi) = \lim_{\lambda\to\infty} \mathscr{E}^\lambda(\psi, \phi)$

for all pairs $(\psi, \phi) \in F^* \times G_\lambda(\overline{\mathscr{R}})$.

Proof. By condition (3.6), for each $f \in \bar{F}$ and $\lambda > 0$ the linear functional $\psi \to (\psi, f)$ is a continuous linear functional on F^* with the $\sigma_\lambda(F^*, F)$ topology. Hence, given $f \in \bar{F}$ and $\lambda > 0$ there exists a unique $G_\lambda f \in F$ such that (4.2) holds. The linearity and the resolvent equation follow in standard fashion from the uniqueness of the representation of the linear functional $\psi \to (\psi, f)$. Putting $\psi = [G_\lambda f]' \in J[G_\lambda f]$ in (4.2) gives

$$||\lambda G_\lambda f||^2 \leqq ([G_\lambda f]', f), \tag{4.2}$$

from which (i) follows. Note that (4.1) also implies the continuity of the mapping $f \to G_\lambda f$ from $\bar{F}$ with the $w(\bar{F}, F^*)$ topology to F with the $\sigma_\lambda(F, F^*)$ topology.

To prove (iii), note that if $\mathscr{R}$ were not $\sigma_\lambda(F, F^*)$-dense in F then there would exist a $\psi \in F^*$ such that $\psi \neq 0$ and $(\psi, \phi) = 0$ for all $\phi \in F$. But this is impossible by (3.2).

Let us next show that G_λ is submarkovian. This will follow immediately if we can prove

$$[\phi \leqq \mathbf{c}] \Rightarrow [\lambda G_\lambda \phi \leqq \mathbf{c}]$$

by taking in turn $\mathbf{c} = 0$ and $\mathbf{c} = e$. If $0 \leqq \mathbf{c}$, then from (3.2)

$$0 \leqq \mathscr{E}(J[(\lambda G_\lambda \phi - \mathbf{c})^+], \lambda G_\lambda \phi) = \lambda(J[(\lambda G_\lambda \phi - \mathbf{c})^+], \phi - \lambda G_\lambda \phi).$$

On the other hand, (2.9) implies that the right side vanishes. By (2.10) we must have $(\lambda G_\lambda \phi - \mathbf{c})^+ = 0$. Hence, $\lambda G_\lambda \phi \leqq \mathbf{c}$.

Finally, let us prove (iv). If $\phi = G_\mu \eta$ for some $\mu > 0$ and $\eta \in \overline{\mathscr{R}}$, we have

$$\mathscr{E}(\psi, \phi) = (\psi, \eta - \mu G_\mu \eta).$$

On the other hand,

$$\lambda(\psi, G_\mu \eta - \lambda G_\lambda G_\mu \eta) = \lambda(\psi, G_\lambda \eta - \mu G_\mu G_\lambda \eta).$$

Thus as $\lambda \to \infty$ this expression tends to $\mathscr{E}(\psi, \phi)$, since $\eta \in \overline{\mathscr{R}}$.

5. The unsymmetric Dirichlet form associated with a given resolvent. In this section we assume we are given a Banach lattice B satisfying the conditions of section 1, and a submarkovian resolvent G_λ on a closed sublattice Y of B with $\mathscr{R}$ = range G_λ dense in Y, and $||\lambda G_\lambda y|| \leqq ||y||$ for all $y \in Y$. We put $F = \mathscr{R}$ and assume that (3.1) and (3.2) are satisfied. Then $F^* = [J(Y)]$ = the linear space generated by $J(Y)$ in B^*. Let

$$\mathscr{E}^\lambda(\psi, \phi) = \lambda(\psi, \phi - \lambda G_\lambda \phi) \tag{5.1}$$

for all $\psi \in F^*$ and $\phi \in Y$. Define

$$\mathscr{E}(\psi, \phi) = \lim_{\lambda\to\infty} \lambda(\psi, \phi - \lambda G_\lambda \phi) \tag{5.2}$$

whenever the limit exists.

THEOREM 5.1. *The bilinear form of* (5.2) *is a Dirichlet form on* $F^* \times F$.

Proof. We check the defining properties in Definition 3.1. First, let us verify that $\mathscr{E}_\alpha$ is defined on $F^* \times F$ for each α. If $\phi \in F$, then $\phi = G_\alpha \eta$ for some $\eta \in \mathscr{R} = Y$. We have for $\psi \in F^*$:

$$\begin{aligned}\mathscr{E}_\alpha(\psi, \phi) &= \lim_{\lambda\to\infty} \lambda(\psi, G_\alpha\eta - \lambda G_\lambda G_\alpha \eta) + \alpha(\psi, G_\alpha\eta) \\ &= \lim_{\lambda\to\infty} (\psi, \lambda G_\lambda \eta - \lambda G_\lambda \cdot \alpha G_\alpha \eta) + \alpha(\psi, G_\alpha \eta) = (\psi, \eta).\end{aligned} \tag{5.3}$$

Condition (3.3) holds since if $\phi' \in J\phi$

$$(\phi', \phi - \lambda G_\lambda \phi) = ||\phi||^2 - (\phi', \lambda G_\lambda \phi) \geqq ||\phi||^2 - ||\phi||\,||\lambda G_\lambda \phi|| \geqq 0. \tag{5.4}$$

To verify condition (3.4) we note that

$$\begin{aligned}\mathscr{E}^\lambda(J[(\phi - \mathbf{c})^+], \phi) = \mathscr{E}^\lambda&(J[(\phi - \mathbf{c})^+], (\phi - \mathbf{c})^+) \\ &+ \mathscr{E}^\lambda(J[(\phi - \mathbf{c})^+], \phi \wedge \mathbf{c}),\end{aligned} \tag{5.5}$$

where $\mathscr{E}^\lambda$ is the form defined in (5.1). The first term is positive by (5.4), and the second by the submarkovian property of G_λ combined with (2.8). Finally, we showed in (5.3) that

$$\mathscr{E}_\alpha(\psi, G_\alpha \eta) = (\psi, \eta), \tag{5.6}$$

so that (3.6) holds.

Standard Dirichlet triples. In Definition 3.1 the domain of a Dirichlet form $\mathscr{E}$ is a set $F^* \times F$ where F contains the range $\mathscr{R}$ of the corresponding resolvent. In the case that $\bar{\mathscr{R}} = \bar{F}$ the restriction of $\mathscr{E}$ to $F^* \times \mathscr{R}$ is also a Dirichlet form and is associated with the same resolvent. In order to avoid this slight degree of arbitratiness in the domain we make the following definition:

Definition 5.1. Suppose $\mathscr{E}$ is a Dirichlet form on $F^* \times F$. Then $(\mathscr{E}, F^*, F)$ is called a standard Dirichlet triple if F is the range of the corresponding resolvent on the lattice $\bar{F}$.

We then have as a corollary to Theorems 4.1 and 5.1:

COROLLARY 5.1. *There is a* 1-1 *correspondence between standard Dirichlet triples* $(\mathscr{E}, F^*, F)$ *and submarkovian resolvents* G_λ *defined on a closed sublattice* $\bar{F}$ *of* B *such that:*

(i) $F = \text{range } G_\lambda$;
(ii) $\lambda||G_\lambda\phi|| \leqq ||\phi||$ *for all* $\phi \in \bar{F}$;
(iii) *the lattice* $\bar{F}$ *satisfies* (3.1) *and* (3.2).

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6. Some examples. The first example we give is a Dirichlet form by our definition, but does not satisfy the conditions of Kunita's definition.

(a) *Translations in L^2.* Let the Banach lattice B be $L^2(-\infty, \infty)$ using the ordinary Lebesgue measure on the line. Let $\{T_t : t \geqq 0\}$ be the translation semigroup

$$T_t f(x) = f(x + t)$$

and G_λ its resolvent satisfying (1.5). Clearly L^2 is contained in the larger lattice of measurable functions on $(-\infty, \infty)$, and $f \in L^2$ implies $f \wedge 1 \in L^2$, so we put e = the equivalence class of functions equal to 1 a.e. on the line. It is well-known that $\mathscr{R}$ = range G_λ is dense in B and, in fact, contains all the C^1 functions with compact support on $(-\infty, \infty)$. We shall therefore put $F = \mathscr{R}$ and $F^* = B^*$. The form $\mathscr{E}$ is then defined by (5.2) on $F^* \times F = B^* \times \mathscr{R}$, and $(\mathscr{E}, B^*, \mathscr{R})$ is a standard Dirichlet triple as defined in Definition 5.1. Now let us ask if $\mathscr{E}$ is a Dirichlet form on $\mathscr{R} \times \mathscr{R}$ by Kunita's definition. Let $\phi \in C^1(-\infty, \infty)$ with compact support. Then,

$$\mathscr{E}(\phi, \phi) = -\int_{-\infty}^{\infty} \phi\phi' dx = 0.$$

However, it is clear the ψ can be chosen so that $\psi \in \mathscr{R}$ and

$$\mathscr{E}(\psi, \phi) = -\int_{-\infty}^{\infty} \psi\phi' dx \neq 0.$$

Thus, condition B2 of Kunita's definition fails to hold here. The above argument shows, in fact, that there exists no extension of $\mathscr{E}$ to a larger subset $D(B) \times D(B) \supset \mathscr{R} \times \mathscr{R}$ of $B \times B$ such that condition B2 of Kunita's definition holds. In this example we have $\beta_0 = 0$.

The second example involves a type of application that will be exploited more fully in a subsequent article; the details are too lengthy to present here.

(b) Let $B = L^2(X, dm)$, where (X, m) is a measure space. Each Dirichlet form $\mathscr{E}$ arises as a limit of the form

$$\lim_{\lambda\to\infty} \lambda \int_X \psi(x)[\phi(x) - \lambda G_\lambda\phi(x)]dm = \mathscr{E}(\psi, \phi), \tag{6.1}$$

where $\{G_\lambda : \lambda > 0\}$ is a submarkovian resolvent family. However, Dirichlet forms may also arise naturally as limits of the type

$$\lim_{\lambda\to\infty} \int_X \psi(x)[\phi(x) - S_\lambda\phi(x)] f_\lambda(x)dm. \tag{6.2}$$

where the S_λ's are submarkovian operators, but not necessarily pseudo-resolvents. Although the general theory of Dirichlet forms shows that the limit can be rewritten in the form (6.1), it often occurs originally in the form (6.2).

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