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A CLASS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS IN SEVERAL VARIABLES

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ABSTRACT. We study a class of generalized hypergeometric functions in several variables introduced by A. Korányi. It is shown that the generalized Gaussian hypergeometric function is the unique solution of a system partial differential equations. Analogues of some classical results such as Kummer relations and Euler integral representations are established. Asymptotic behavior of generalized hypergeometric functions is obtained which includes some known estimates.

0. **Introduction.** In the case of positive definite matrices, generalized hypergeometric functions (with a definition based on Laplace transforms) were introduced by C. Herz [5], and their series expansion is due to A. Constantine [1]. Further properties and applications in statistics were given by A. James and R. Muirhead [11]. The case of positive Hermitian or quaternion matrices was studied by K. Gross and D. Richards [4]. Generalized hypergeometric functions associated with arbitrary symmetric cones were considered by J. Faraut and A. Korányi [3]. A more general class of hypergeometric functions was introduced by A. Korányi [7]. In this paper we shall study that class of generalized hypergeometric functions.

In §2 we prove that ${}_2F_1^{(d)}(a,b;c;x_1,\dots,x_r)$ is the unique solution of the system of the partial differential equations

$$\begin{aligned}x_i(1-x_i)\frac{\partial^2 F}{\partial x_i^2}+\Big\{c-\frac{d}{2}(r-1)-\Big[a+b+1-\frac{d}{2}(r-1)\Big]x_i+\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{x_i(1-x_i)}{x_i-x_j}\Big\}\frac{\partial F}{\partial x_i}\\ -\frac{d}{2}\sum_{j=1,j\neq i}^{r}\frac{x_j(1-x_j)}{x_i-x_j}\frac{\partial F}{\partial x_j}=abF\quad i=1,\dots,r\end{aligned}\tag{1}$$

subject to the conditions that

(a) F is a symmetric function of $x_1,\dots,x_r$ and

(b) F is analytic at $x_1=\cdots=x_r=0$ and $F(0)=1$

(1) is a generalization of the classical hypergeometric equation. This result was claimed in [7], but the proof was incomplete.

In §3 we obtain some analogues of classical results about hypergeometric functions and, in particular, establish integral representations of the generalized hypergeometric functions. In §4 we obtain the asymptotic behavior of ${}_{p+1}F_p^{(d)}$. As an application, we get

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the generalized Rudin-Forelli inequalities in function theory on a bounded symmetric domain, which are due to J. Faraut and A. Korányi for ${}_2F_1^{(d)}(a, b; c; t_1, \ldots, t_r)$ with some special a, b and c [2]. Our results also include, in a unified way, the estimates obtained by J. Mitchell and G. Sampson [9], [10].

Some other results are announced in [13].

1. **Notation, definitions and basic facts.** A *partition* is any finite or infinite sequence

$$\kappa = (k_1, k_2, \ldots, k_r, \ldots) \tag{2}$$

of non-negative integers in decreasing order $k_1 \geq k_2 \geq \cdots \geq k_r \geq \cdots$ containing only finitely many non-zero terms. The non-zero k_i in (2) are called the *parts* of κ. The number of parts is called the *length* of κ, denoted by $l(\kappa)$; and the sum of the parts is the *weight* of κ, denoted by $|\kappa| = k_1 + k_2 + \cdots + k_{l(\kappa)}$. When $l(\kappa) \leq r$, we simply write κ as $\kappa = (k_1, \ldots, k_r)$. We say that κ is a *partition of* k if $|\kappa| = k$. For a partition κ, hereafter, we use k to denote $|\kappa|$. The partitions of k are ordered lexicographically, that is, if $\kappa = (k_1, k_2, \ldots)$, $\lambda = (l_1, l_2, \ldots)$, we write $\kappa > \lambda$ if $k_i > l_i$ for the first index i for which the parts are unequal. Let $y_1, \ldots, y_r$ be r variables; if $\kappa > \lambda$ and $l(k), l(\lambda) \leq r$, we say that the monomial $y_1^{k_1} \cdots y_r^{k_r}$ is of higher weight than the monomial $y_1^{l_1} \cdots y_r^{l_r}$.

For a partition κ, we define its *diagram* by

$$G(\kappa) = \{(i, j) : 1 \leq i \leq l(\kappa), 1 \leq j \leq k_i\}$$

If λ, κ are partitions, then we write $\lambda \subseteq \kappa$ if $\lambda_i \leq k_i$ for all i. If $\lambda \subseteq \kappa$, then κ/λ is defined to be the difference $\kappa - \lambda$ of diagrams.

For each j, $j = 1, 2, \ldots, k_i$, let

$$k_j' = \max\{i \mid (i, j) \in G(\kappa)\}.$$

For $s = (i, j) \in G(\kappa)$, and a parameter α, let

$$\begin{aligned} a(s) &= k_i - j \\ l(s) &= k_j' - i \\ h_\kappa^*(s) &= l(s) + (1 + a(s))\alpha \\ h_*^\kappa(s) &= l(s) + 1 + a(s)\alpha \end{aligned}$$

We simply write $s \in \kappa$ instead of $s \in G(\kappa)$.

Let $\bigwedge_r$ be the vector space of symmetric polynomials in $x_1, \ldots, x_r$, $p_k = \sum_{i=1}^r x_i^k$ and $P_\kappa = p_{k_1} \cdots p_{k_{l(\kappa)}}$; then $\{P_\kappa$, for all $\kappa\}$ forms a basis of $\bigwedge_r$. For each $\alpha > 0$, one defines an inner product on $\bigwedge_r$ by

$$\langle P_\kappa, P_\lambda \rangle_\alpha = \delta_{\kappa\lambda} z_\kappa \alpha^{l(\kappa)}$$

where $z_\kappa = (1^{m_1} 2^{m_2} \cdots) m_1! \, m_2! \cdots$ and m_j = the number of k_i which are equal to j. Let $J_\kappa(y_1, \ldots, y_r; \alpha)$ be the Jack polynomial indexed by the partition κ and parameter α. The

J_κ are gotten by orthogonalizing the monomial symmetric polynomials with respect to $\langle\, , \,\rangle_\alpha$. Notations are as in [8], [12].

The following results about Jack polynomials are known. See [12].

(i) $J_\kappa(y_1,\ldots,y_r;\alpha)=0$ if $l(\kappa)>r$.

(ii) $J_\kappa(y_1,\ldots,y_r;\alpha)=J_\kappa(y_1,\ldots,y_r,0;\alpha)$

(iii) $(y_1+\cdots+y_r)^k=\sum\alpha^\kappa k!\,J_\kappa(y_1,\ldots,y_r;\alpha)j_\kappa^{-1}$

(iv) $J_\kappa(1,\ldots,1;\alpha)=\prod_{(i,j)\in\kappa}\big(r-(i-1)+\alpha(j-1)\big)$

(v) Let $\nu_{\kappa\kappa}(\alpha)=\prod_{s\in\kappa}h_*^\kappa(s)$; then, $\nu_{\kappa\kappa}(\alpha)y_1^{k_1}\cdots y_r^{k_r}$ is the term of the highest weight in $J_\kappa(y_1,\ldots,y_r;\alpha)$.

(vi) $J_\kappa(y_1,\ldots,y_r;\alpha)$ is an eigenfunction of the differential operator

$$\Delta_r=\sum_{i=1}^r y_i^2\frac{\partial^2}{\partial y_i^2}+\frac{2}{\alpha}\sum_{i=1}^r\sum_{j=1,j\neq i}^r\frac{y_i^2}{y_i-y_j}\frac{\partial}{\partial y_i}\tag{3}$$

with the eigenvalue $\mu_\kappa=\rho_\kappa+k(\frac{2}{\alpha}r-1)$, where $\rho_\kappa=\sum_{i=1}^r k_i(k_i-\frac{2}{\alpha}i)$, if $l(\kappa)\le r$.

(vii) $j_\kappa=\langle J_\kappa,J_\kappa\rangle=\prod_{s\in\kappa}h_*^\kappa(s)h_\kappa^*(s)$

One defines, for a partition κ and a positive number d,

$$C_\kappa^{(d)}(y_1,\ldots,y_r)=(2/d)^k k!\,J_\kappa(y_1,\ldots,y_r;2/d)j_\kappa^{-1}.$$

DEFINITION. For $a_1,\ldots,a_p,b_1,\ldots,b_q\in\mathbf{C}$, such that $(b_j)_\kappa\neq 0$, for all κ, j, the *hypergeometric functions* associated with the parameter $d>0$ are defined by

$$\begin{aligned}{}_pF_q^{(d)}&(a_1\ldots,a_p;b_1,\ldots,b_q;y_1,\ldots,y_r)\\&=\sum_{k=0}^\infty\sum_\kappa\frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\frac{C_\kappa^{(d)}(y_1,\ldots,y_r)}{k!}\end{aligned}\tag{4}$$

where $\sum_\kappa$ denotes the summation over all partitions of k,

$$(a)_\kappa=\prod_{i=1}^{l(\kappa)}(a-(i-1)d/2)_{k_i}$$

and

$$(a)_m=a(a+1)\cdots(a+m-1),\quad (a)_0=1.$$

REMARK 1. From (i), we have $C_\kappa^{(d)}(y_1,\ldots,y_r)=0$ for κ with $l(\kappa)>r$; therefore the summation in (4) is only over those partitions with length not greater than r.

REMARK 2. Let Y be an $r\times r$ symmetric matrix with latent roots $y_1,\ldots,y_r$; then it is known that the zonal polynomial $C_\kappa(Y)$ of Y corresponding to a partition κ, defined in [11], is equal to $C_\kappa^{(1)}(Y)$.

Throughout this paper, we denote $(y_1,\ldots,y_r)$ by Y_r or simply by Y whenever no confusion is caused.

2. **Partial differential equations for hypergeometric functions.** It is well known that the classical Gaussian hypergeometric function ${}_2f_1(a, b; c; z)$ is the unique solution of the second order differential equation

$$z(1-z)\frac{d^2f}{dz^2} + [c-(a+b+1)z]\frac{df}{dz} = abf$$

subject to the conditions that

(a) f is analytic at 0

(b) $f(0) = 1$

For the hypergeometric functions of a real matrix argument, a generalization of this classical result was given by Muirhead [11]. A more general result is the following (*cf.* [7]):

THEOREM 2.1. *${}_2F_1^{(d)}(a, b, c; y_1, \ldots, y_r)$ is the unique solution of the system of r partial differential equations*

$$y_i(1-y_i)\frac{\partial^2 F}{\partial y_i^2} + \left\{c - \frac{d}{2}(r-1) - \left[a+b+1-\frac{d}{2}(r-1)\right]y_i + \frac{d}{2}\sum_{j=1, j\neq i}^{r} \frac{y_i(1-y_i)}{y_i - y_j}\right\}\frac{\partial F}{\partial y_i}$$
$$- \frac{d}{2}\sum_{j=1, j\neq i}^{r} \frac{y_j(1-y_j)}{y_i - y_j}\frac{\partial F}{\partial y_j} = abF \qquad i = 1, \ldots, r$$

(5)

subject to the conditions that

(a) F is a symmetric function of $y_1, \ldots, y_r$ and

(b) F is analytic at $y_1 = \cdots = y_r = 0$ and $F(0) = 1$.

The remainder of this section is devoted to the proof of Theorem 2.1. Our proof follows closely that of Muirhead with some modification and clarification.

Let

$$\Delta_r = \sum_{i=1}^{r} y_i^2 \frac{\partial^2}{\partial y_i^2} + d\sum_{i=1}^{r}\sum_{j=1, j\neq i}^{r} \frac{y_i^2}{y_i - y_j}\frac{\partial}{\partial y_i} \tag{6}$$

$$\delta_r = \sum_{i=1}^{r} y_i \frac{\partial^2}{\partial y_i^2} + d\sum_{i=1}^{r}\sum_{j=1, j\neq i}^{r} \frac{y_i}{y_i - y_j}\frac{\partial}{\partial y_i} \tag{7}$$

$$E_r = \sum_{i}^{r} y_i \frac{\partial}{\partial y_i} \tag{8}$$

$$\varepsilon_r = \sum_{i}^{r} \frac{\partial}{\partial y_i} \tag{9}$$

For simplicity, we denote $(y_1, \ldots, y_r)$ and $(1, \ldots, 1) \in \mathbf{R}^r$ by Y_r and I_r respectively.

We define the generalized binomial coefficients by

$$\frac{C_\kappa^{(d)}(I_r + Y_r)}{C_\kappa^{(d)}(I_r)} = \sum_{s=0}^{k}\sum_{\sigma, |\sigma|=s} \binom{\kappa}{\sigma}_r \frac{C_\sigma^{(d)}(Y_r)}{C_\sigma^{(d)}(I_r)} \tag{10}$$

where $k = |\kappa|$, $r \geq l(\kappa)$.

REMARK. We note that the generalized binomial coefficients depend on r by the definition. But in the case of symmetric cones, one can readily show that they are independent of r. In the following, we prove that it is still true for some special generalized binomial coefficients. We expect such a result in the general case.

For a partition $\kappa = (k_1, \ldots, k_r)$ of k, $r \geq l(\kappa)$, let

$$\kappa_i^{(r)} = (k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_r)$$
$$\kappa_{(r)}^i = (k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$$

whenever these are partitions of $k+1$ and $k-1$ respectively. Since we can also write κ as $(k_1, \ldots, k_r, 0)$, $\kappa_i^{(r)}$ depends on r. But when $r \geq l(\kappa) + 1$, $\kappa_i^{(r)} = \kappa_i^{(l(\kappa)+1)}$, then, we simply write κ_i instead of $\kappa_i^{(r)}$. It is easy to see that $\kappa_{(r)}^i$ does not depend on r, thus, we omit the subscript r.

As a consequence of (vi) in §1, we have

LEMMA 2.2.

$$\Delta_r C_\kappa^{(d)}(Y_r) = [\rho_\kappa + k(dr-1)] C_\kappa^{(d)}(Y_r)$$

The following two lemmas can be proved in the same way as in [11].

LEMMA 2.3. *For κ with $l(\kappa) \leq r$,*

$$\delta_r \frac{C_\kappa^{(d)}(Y_r)}{C_\kappa^{(d)}(I_r)} = \sum_i \binom{\kappa}{\kappa^i}_r \left[k_i - 1 + \frac{d}{2}(r-i)\right] \frac{C_{\kappa^i}^{(d)}(Y_r)}{C_{\kappa^i}^{(d)}(I_r)}, \tag{11}$$

$$\varepsilon_r \frac{C_\kappa^{(d)}(Y_r)}{C_\kappa^{(d)}(I_r)} = \sum_i \binom{\kappa}{\kappa^i}_r \frac{C_{\kappa^i}^{(d)}(Y_r)}{C_{\kappa^i}^{(d)}(I_r)}. \tag{12}$$

LEMMA 2.4. *For κ with $l(\kappa) \leq r$,*

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r C_{\kappa_i^{(r)}}^{(d)}(I_r) = r(k+1) C_\kappa^{(d)}(I_r), \tag{13}$$

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r \left[k_i - \frac{d}{2}(i-1)\right] C_{\kappa_i^{(r)}}^{(d)}(I_r) = k(k+1) C_\kappa^{(d)}(I_r), \tag{14}$$

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r \left[k_i - \frac{d}{2}(i-1)\right]^2 C_{\kappa_i^{(r)}}^{(d)}(I_r) = (k+1)\left[\rho_\kappa + \frac{d}{2}k(r+1)\right] C_\kappa^{(d)}(I_r). \tag{15}$$

PROPOSITION 2.5. *The function ${}_2F_1^{(d)}(a, b; c; y_1, \ldots, y_r)$ satisfies the differential equation*

$$\delta_r F - \Delta_r F + \left[c - \frac{d}{2}(r-1)\right] \varepsilon_r F - \left[a + b + 1 - \frac{d}{2}(r-1)\right] E_r F = rabF. \tag{16}$$

PROOF. Let

$$F(Y_r) = \sum_\kappa \alpha_\kappa C_\kappa^{(d)}(Y_r)$$

Substituting the series into (16), applying Lemma 2.3 and equating the coefficients of $C_\kappa^{(d)}(Y_r)$ on both sides, we can see that if for all κ, α_κ satisfies

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r \Big[c + k_i - \frac{d}{2}(i-1)\Big] C^{(d)}_{\kappa_i^{(r)}}(I_r)\alpha_{\kappa_i^{(r)}} = \Big[rab + k(a+b) + \rho_\kappa + \frac{d}{2}k(r+1)\Big] C_\kappa^{(d)}(I_r)\alpha_\kappa \tag{17}$$

then $F(Y_r)$ satisfies (16).

Now, it suffices to show that

$$\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$$

is a solution of (17). We note that $(a)_{\kappa_i^{(r)}} = (a)_\kappa[a + k_i - \frac{d}{2}(i-1)]$.

The problem is reduced to showing that

$$\sum_i \binom{\kappa_i^{(r)}}{\kappa}_r \Big[a + k_i - \frac{d}{2}(i-1)\Big]\Big[b + k_i - \frac{d}{2}(i-1)\Big] C^{(d)}_{\kappa_i^{(r)}}(I_r) = (k+1)\Big[rab + \rho_\kappa + ka + kb + \frac{d}{2}k(r+1)\Big] C_\kappa^{(d)}(I_r). \tag{18}$$

This is an immediate consequence of Lemma 2.4.

In the following, for simplicity, 1 stands for the partition $(1, 0, \ldots, 0)$ in the subscripts when partitions are involved.

LEMMA 2.7. *If κ is a partition of k, then, for all $r \geq l(\kappa)$, and $i = 1, \ldots, r$,*

$$\frac{J_\kappa(I_{r+1})}{J_{\kappa^i}(I_{r+1})}\binom{\kappa}{\kappa^i}_{r+1} = \frac{J_\kappa(I_r)}{J_{\kappa^i}(I_r)}\binom{\kappa}{\kappa^i}_r + G^\kappa_{\kappa^i 1} \tag{19}$$

where $G^\kappa_{\sigma\tau} = g^\kappa_{\sigma\tau} j_\sigma^{-1} j_\tau^{-1}$ and $g^\kappa_{\sigma\tau} = \langle J_\sigma J_\tau, J_\kappa \rangle$.

PROOF. Let $X = (x_1, \ldots, x_r)$, by Proposition 4.2 in [12], we have

$$\begin{aligned} J_\kappa(x_1, \ldots, x_r, x_{r+1}) &= \sum_\nu J_\nu(X; 2/d)\Big(\sum_\alpha j_\alpha^{-1} g^\kappa_{\nu\alpha} J_\alpha(x_{r+1}; 2/d)\Big) j_\nu^{-1} \\ &= J_\kappa(X; 2/d) + \Big[\sum_i j_{\kappa^i}^{-1} j_1^{-1} g^\kappa_{\kappa^i 1} J_{\kappa^i}(X; 2/d)\Big] x_{r+1} + P(X, x_{r+1}) x_{r+1}^2 \end{aligned}$$

where $P(X, x_{r+1})$ is a polynomial of $x_1, \ldots, x_r, x_{r+1}$.

Then, using (12) and §1 (ii), we have

$$\begin{aligned} J_\kappa(I_{r+1}) \sum_i \binom{\kappa}{\kappa^i}_{r+1} \frac{J_{\kappa^i}(X_r)}{J_{\kappa^i}(I_{r+1})} &= \varepsilon_{r+1} J_\kappa(X, x_{r+1}) \,|_{x_{r+1}=0} \\ &= J_\kappa(I_r) \sum_i \binom{\kappa}{\kappa^i}_r \frac{J_{\kappa^i}(X)}{J_{\kappa^i}(I_r)} + \sum_i G^\kappa_{\kappa^i 1} J_{\kappa^i}(X) \end{aligned}$$

Hence

$$\frac{J_\kappa(I_{r+1})}{J_{\kappa^i}(I_{r+1})}\binom{\kappa}{\kappa^i}_{r+1} = \frac{J_\kappa(I_r)}{J_{\kappa^i}(I_r)}\binom{\kappa}{\kappa^i}_r + G^\kappa_{\kappa^i 1}.$$

LEMMA 2.8. *Suppose $l(\kappa) = n$, then*

$$\binom{\kappa}{\kappa^n}_r = G^\kappa_{\kappa^n 1}, \tag{20}$$

for all $r \geq n$.

PROOF. Since $l(\kappa) = n, \kappa = (k_1, \ldots, k_n), k_n \geq 1$, by (19) and §1(iv), we only have to prove that

$$\binom{\kappa}{\kappa^n}_n = G^\kappa_{\kappa^n 1}.$$

For a partition λ of length $\leq n$, let m_λ be the symmetric polynomial

$$m_\lambda(x_1, \ldots, x_n) = \sum x^\alpha.$$

The summation is over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_n)$.

On the one hand, §1 (v) gives

$$\begin{aligned} J_\kappa(X_n + I_n) &= \nu_{\kappa\kappa}\Big((x_1+1)^{k_1}\cdots(x_n+1)^{k_n} + \cdots\Big) + \cdots \\ &= \text{terms of degree } k + \text{terms of degree } k-1 \\ &\qquad + \text{terms of lower degree} \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

In II, the term of highest weight is $k_n \nu_{\kappa\kappa} x_1^{k_1} \cdots x_n^{k_n - 1}$.

On the other hand, by definition and (iv) in §1

$$J_\kappa(X_n + I_n) = \sum_{s=0}^{k} \sum_{\sigma, |\sigma| = s} \binom{\kappa}{\sigma}_n \frac{J_\kappa(I_n)}{J_\sigma(I_n)} J_\sigma(X_n)$$

and

$$\frac{J_\kappa(I_n)}{J_{\kappa^n}(I_n)} = 1 + (k_n - 1)\frac{2}{d}.$$

Equating the coefficients of $J_{\kappa^n}(X_n)$ and applying §1 (v), we get

$$k_n \frac{\nu_{\kappa\kappa}(2/d)}{\nu_{\kappa^n\kappa^n}(2/d)} = \binom{\kappa}{\kappa^n}_n \Big[1 + (k_n - 1)\frac{2}{d}\Big].$$

Now it is enough to show that

$$k_n \frac{\nu_{\kappa\kappa}(2/d)}{\nu_{\kappa^n\kappa^n}(2/d)} = G^\kappa_{\kappa^n 1}\Big[1 + (k_n - 1)\frac{2}{d}\Big].$$

Theorem 6.1 in [12] gives

$$g^{\kappa}_{\kappa^n 1} = \frac{2}{d} \prod_{s\in\kappa^n} A_{\kappa\kappa^n}(s) \prod_{s\in\kappa} B_{\kappa\kappa^n}(s)$$

where $A_{\kappa\kappa^n}(s)$ is defined to be $h^{\kappa^n}_*(s)$, if κ/κ^n does not contain an element in the same column as s, $h^*_{\kappa^n}(s)$, otherwise; and $B_{\kappa\kappa^n}(s)$ is defined to be $h^*_\kappa(s)$, if κ/κ^n does not contain an element in the same column as s, $h^\kappa_*(s)$, otherwise.

A direct computation yields

$$\frac{g^{\kappa}_{\kappa^n 1}[1+(k_n-1)\frac{2}{d}]}{(2/d)\prod_{s\in\kappa^n} h^*_{\kappa^n}(s)} = k_n \prod_{s\in\kappa} h^\kappa_*(s)$$

Hence, by §1 (v), we have

$$\frac{g^{\kappa}_{\kappa^n 1}[1+(k_n-1)\frac{2}{d}]}{j_{\kappa^n} j_1} = k_n \frac{\prod_{s\in\kappa} h^\kappa_*(s)}{\prod_{s\in\kappa_n} h^{\kappa^n}_*(s)} = k_n \frac{\nu_{\kappa\kappa}(2/d)}{\nu_{\kappa^n\kappa^n}(2/d)}$$

finishing the proof.

Let $N(k)$ denote the number of partitions of k. When κ runs over all partitions of k, κ_i, $i=1,\ldots,l(\kappa)+1$, run over all partitions of $(k+1)$. We note that $2N(k) \geq N(k+1)$. For $n \geq k+1$, let

$$H(n,\kappa,\kappa_i) = \frac{J_{\kappa_i}(I_n)}{J_\kappa(I_n)} \binom{\kappa_i}{\kappa}_n$$

We consider the system of linear equations

$$\begin{aligned} \sum_i G^{\kappa_i}_{\kappa 1} x_{\kappa_i} &= a_\kappa \\ H(n,\kappa,\kappa_i) x_{\kappa_i} &= b_\kappa \end{aligned} \tag{21}$$

where the x_λ are independent variables indexed by partitions of $k+1$, κ runs over all partitions of k, and a_κ, b_κ are given constants.

LEMMA 2.9. *The $2N(k) \times N(k+1)$ matrix formed from the coefficients of the left hand side of (21) has rank $N(k+1)$.*

PROOF. Let $\lambda(1) < \lambda(2) < \cdots < \lambda\big(N(k+1)\big)$ and $x_j = x_{\lambda(j)}$, where $\lambda(j)$ is a partition of $k+1$. In the following, we want to produce a system of linear equations which is equivalent to (21) and whose coefficient matrix has the form

$$\begin{pmatrix} c_1 & * & * & \cdots & * \\ 0 & c_2 & * & \cdots & * \\ 0 & 0 & c_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N(k+1)} \\ * & * & * & \cdots & * \end{pmatrix}$$

with $c_j \neq 0, j=1,\ldots,N(k+1)$.

For each j, $j=1,\ldots,N(k+1)$, there are two possible cases for $\lambda(j)$.

CASE 1.

$$\lambda(j) = (l_1, \ldots, l_{s-1}, 1, 0, \ldots, 0).$$

Set

$$\kappa = (l_1, \ldots, l_{s-1}, 0, \ldots, 0.)$$

then

$$\kappa_s = (l_1, \ldots, l_{s-1}, 1, 0, \ldots, 0) = \lambda(j).$$

Since κ_i is not a partition for $i > s$, $\sum_i G_{\kappa 1}^{\kappa_i} x_{\kappa_i} = a_\kappa$ becomes

$$G_{\kappa 1}^{\kappa_s} x_{\kappa_s} + \sum_{i<s} G_{\kappa 1}^{\kappa_i} x_{\kappa_i} = a_\kappa \tag{22}$$

Set $c_j = G_{\kappa 1}^{\kappa_s}$, then c_j is positive.

Now we can write (22) as

$$c_j x_j + \sum_{m>j} c_{mj} x_m = a_\kappa$$

by observing that $\lambda(j) = \kappa_s < \kappa_{s-1} < \cdots < \kappa_1$. So there is nothing to change; the j-th equation is already "triangular".

CASE 2.

$$\lambda(j) = (l_1, \ldots, l_{s-1}, l_s, 0, \ldots, 0)$$

with $l_s \geq 2$. Let

$$\kappa = (l_1, \ldots, l_{s-1}, l_s - 1, 0, \ldots, 0).$$

Then

$$\kappa_{s+1} = (l_1, \ldots, l_{s-1}, l_s - 1, 1, 0, \ldots, 0) = \lambda(j-1)$$
$$\kappa_s = (l_1, \ldots, l_{s-1}, l_s, 0, \ldots, 0) = \lambda(j).$$

From (21), we have two equations

(a) $G_{\kappa 1}^{\kappa_{s+1}} x_{\kappa_{s+1}} + G_{\kappa 1}^{\kappa_s} x_{\kappa_s} + \sum_{i<s} G_{\kappa 1}^{\kappa_i} x_{\kappa_i} = a_\kappa$

(b) $H(n, \kappa, \kappa_{s+1}) x_{\kappa_{s+1}} + H(n, \kappa, \kappa_s) x_{\kappa_s} + \sum_{i<s} H(n, \kappa, \kappa_i) x_{\kappa_i} = b_\kappa$.

By Lemma 2.8 and §1 (iv)

$$H(n, \kappa, \kappa_{s+1}) = \frac{J_{\kappa_{s+1}}(I_n)}{J_\kappa(I_n)} G_{\kappa 1}^{\kappa_{s+1}} = (n-s) G_{\kappa 1}^{\kappa_{s+1}}$$
$$H(n, \kappa, \kappa_s) = \frac{J_{\kappa_s}(I_n)}{J_\kappa(I_n)} G_{\kappa 1}^{\kappa_s} = \left[(n-s+1) + \frac{2}{d}(l_s - 1)\right] G_{\kappa 1}^{\kappa_s}.$$

In the system of equations formed by (a) and (b) we can equivalently replace (b) by the following equation

$$\left[1 + \frac{2}{d}(l_s - 1)\right] G_{\kappa 1}^{\kappa_s} x_{\kappa_s} + \sum_{i<s} [H(n, \kappa, \kappa_i) - (n-s) G_{\kappa 1}^{\kappa_i}] x_{\kappa_i} = b_\kappa - (n-s) a_\kappa$$

Let $c_j = [1 + \frac{2}{d}(l_s - 1)]G_{\kappa 1}^{\kappa_s}$, then $c_j > 0$, we can write the above equation as

$$c_j x_j + \sum_{m>j} c_{mj} x_m = d_\kappa.$$

Thus we have proved the lemma.

LEMMA 2.10. *If a sequence $\{A_\kappa\}$ indexed by all partitions satisfies*

$$\sum_i \binom{\kappa_i}{\kappa}_{k+1+r} \frac{J_{\kappa_i}(I_{k+1+r})}{J_\kappa(I_{k+1+r})} A_{\kappa_i} = \frac{d}{2(k+1)}\Big[(k+1+r)ab + \rho_\kappa + k(a+b) + \frac{d}{2}k(k+r+2)\Big]A_\kappa \tag{23}$$

for all positive integer $r \geq 2$, then $\{A_\kappa\}$ is uniquely determined by A_0.

PROOF. Applying Lemma 2.7, we have

$$\Big[\sum_i \binom{\kappa_i}{\kappa}_{k+1} \frac{J_{\kappa_i}(I_{k+1})}{J_\kappa(I_{k+1})} + rG_{\kappa 1}^{\kappa_i}\Big]A_{\kappa_i} = \frac{d}{2(k+1)}\Big[(k+1+r)ab + \rho_\kappa + k(a+b) + \frac{d}{2}k(k+r+2)\Big]A_\kappa \tag{24}$$

for all $r \geq 1$. Equating coefficients of r on both sides of (24) gives

$$\sum_i G_{\kappa 1}^{\kappa_i} A_{\kappa_i} = \frac{d}{2(k+1)}\Big(ab + \frac{d}{2}k\Big)A_\kappa \tag{25}$$

and equating constant terms gives

$$\sum_i \binom{\kappa_i}{\kappa}_{k+1} \frac{J_{\kappa_i}(I_{k+1})}{J_\kappa(I_{k+1})} A_{\kappa_i} = \frac{d}{2(k+1)}\Big[(k+1)ab + \rho_\kappa + k(a+b) + \frac{d}{2}k(k+2)\Big]A_\kappa. \tag{26}$$

By Lemma 2.9, we see that A_κ is uniquely determined by A_0.

THEOREM 2.11. *There exists a unique sequence $\{\alpha_\kappa\}$ indexed by all partitions with $\alpha_0 = 1$ such that for $r = 2, 3, \ldots$*

$$F_r(y_1, \ldots, y_r) = \sum_\kappa \alpha_\kappa C_\kappa^{(d)}(y_1, \ldots, y_r) \tag{27}$$

satisfies

$$\delta_r F - \Delta_r F + \Big[c - \frac{d}{2}(r-1)\Big]\varepsilon_r F - \Big[a + b + 1 - \frac{d}{2}(r-1)\Big]E_r F = rabF. \tag{28}$$

Moreover, $\alpha_\kappa = \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa k!}$.

REMARK. By §1 (i), we know that the summation in (27) is only over the partitions with $l(\kappa) \leq r$.

PROOF. Let $\alpha_\kappa = \frac{(a)_\kappa(b)_\kappa}{(c)_\kappa k!}$; then Proposition 2.5 shows that for $r = 2, 3, \ldots$, $\sum_\kappa \alpha_\kappa C_\kappa^{(d)}(y_1, \ldots, y_r)$ satisfies (28).

Next, suppose that $\{\alpha_\kappa\}$ is such a sequence. From the proof of Proposition 2.5, we see that for all κ, all $r \geq l(\kappa) + 1$

$$\sum_i \binom{\kappa_i}{\kappa}_r \Big[c + k_i - \frac{d}{2}(i-1)\Big] C_{\kappa_i}^{(d)}(I_r)\alpha_{\kappa_i} = \Big[rab + k(a+b) + \rho_\kappa + \frac{d}{2}k(r+1)\Big] C_\kappa^{(d)}(I_r)\alpha_\kappa.$$

Let $\alpha_\kappa = \frac{\beta_\kappa}{(c)_\kappa}$; then the above becomes

$$\sum_i \binom{\kappa_i}{\kappa}_r C_{\kappa_i}^{(d)}(I_r)\beta_{\kappa_i} = \Big[rab + k(a+b) + \rho_\kappa + \frac{d}{2}k(r+1)\Big] C_\kappa^{(d)}(I_r)\beta_\kappa. \tag{29}$$

Since $C_\kappa^{(d)}(y_1, \ldots, y_r) = \left(\frac{2}{d}\right)^k k!\, J_\kappa(y_1, \ldots, y_r; 2/d) j_\kappa^{-1}$, we have

$$\sum_i \binom{\kappa_i}{\kappa}_r \frac{J_{\kappa_i}(I_r)}{J_\kappa(I_r)} \beta_{\kappa_i} j_{\kappa_i}^{-1} = \frac{d}{2(k+1)} \Big[rab + \rho_\kappa + k(a+b) + \frac{d}{2}k(r+1)\Big] \beta_\kappa j_\kappa^{-1}. \tag{30}$$

By Lemma 2.10, $\beta_\kappa j_\kappa^{-1}$ is uniquely determined by $\beta_{(0)} j_{(0)}^{-1}$, therefore α_κ is uniquely determined by $\alpha_{(0)}$.

The following theorem can be proved in the same way as the case $d = 1$ in [11].

THEOREM 2.12. *There exists a unique function F which satisfies the system of r partial differential equations*

$$\begin{aligned} & y_i(1-y_i)\frac{\partial^2 F}{\partial y_i^2} + \Big\{c - \frac{d}{2}(r-1) - \Big[a + b + 1 - \frac{d}{2}(r-1)\Big]y_i \\ & \quad + \frac{d}{2}\sum_{j=1, j\neq i}^r \frac{y_i(1-y_i)}{y_i - y_j}\Big\}\frac{\partial F}{\partial y_i} - \frac{d}{2}\sum_{j=1, j\neq i}^r \frac{y_j(1-y_j)}{y_i - y_j}\frac{\partial F}{\partial y_j} = abF \end{aligned} \tag{31}$$

$i = 1, \ldots, r$, subject to the conditions that

(a) F is a symmetric function of $y_1, \ldots, y_r$ and

(b) F is analytic at $y_1 = \cdots = y_r = 0$ and $F(0) = 1$.

THEOREM 2.13. *There exists a unique sequence $\{A_\kappa\}$ with $A_{(0)} = 1$ such that $F_r(y_1, \ldots, y_r) = \sum_\kappa A_\kappa C_\kappa^{(d)}(y_1, \ldots, y_r)$ satisfies (31) for $r = 2, 3, \ldots$. Moreover, $A_\kappa = \frac{(a)_\kappa(b)_\kappa}{(c)_\kappa k!}$.*

PROOF. If such a sequence $\{A_\kappa\}$ exists, then $A_\kappa = \frac{(a)_\kappa(b)_\kappa}{(c)_\kappa k!}$ since the sum of the r partial differential equations of (31) is (28).

Therefore, we only need to establish the existence of $\{A_\kappa\}$. By Theorem 2.12 , there exist F_n and F_{n+1} which are solutions of (27) subject to (a) and (b) for $r = n$ and $r = n+1$ respectively. Then, we have

$$F_n(y_1, \ldots, y_n) = \sum_\kappa B_\kappa C_\kappa^{(d)}(y_1, \ldots, y_n), \quad l(\kappa) \leq n,$$

$$F_{n+1}(y_1, \ldots, y_{n+1}) = \sum_\kappa D_\kappa C_\kappa^{(d)}(y_1, \ldots, y_{n+1}), \quad l(\kappa) \leq n+1.$$

Now it is enough to show that $B_\kappa = D_\kappa$, if $l(\kappa) \leq n$.

Let

$$G_n(y_1, \ldots, y_n) = F_{n+1}(y_1, \ldots, y_n, 0).$$

We note that

$$\frac{\partial F_{n+1}}{\partial y_i}(y_1, \ldots, y_n, 0) = \frac{\partial G_n}{\partial y_i}(y_1, \ldots, y_n), \quad 1 \leq i \leq n$$

$$\frac{\partial^2 F_{n+1}}{\partial y_i^2} = \frac{\partial^2 G_n}{\partial y_i^2}(y_1, \ldots, y_n), \quad 1 \leq i \leq n.$$

For $i = 1, \ldots, n$, we have

$$\begin{aligned} y_i(1-y_i)\frac{\partial^2 F_{n+1}}{\partial y_i^2} &+ \left\{c - \frac{d}{2}n - \left[a+b+1-\frac{d}{2}n\right]y_i \right. \\ &\left. + \frac{d}{2}\sum_{j=1,j\neq i}^{n} \frac{y_i(1-y_i)}{y_i - y_j} + \frac{d}{2}\frac{y_i(1-y_i)}{y_i - y_{n+1}}\right\}\frac{\partial F_{n+1}}{\partial y_i} \\ &- \frac{d}{2}\sum_{j=1,j\neq i}^{n}\frac{y_j(1-y_j)}{y_i-y_j}\frac{\partial F_{n+1}}{\partial y_j} - \frac{d}{2}\frac{y_{n+1}(1-y_{n+1})}{y_i - y_{n+1}}\frac{\partial F_{n+1}}{\partial y_{n+1}} = abF_{n+1}. \end{aligned}$$

Suppose $y_j \neq 0, j = 1, \ldots, n$, let $y_{n+1} \longrightarrow 0$. We have

$$\begin{aligned} y_i(1-y_i)\frac{\partial^2 F_{n+1}}{\partial y_i^2}(y_1, \ldots, y_n, 0) &+ \left\{c - \frac{d}{2}n - \left[a+b+1-\frac{d}{2}n\right]y_i \right. \\ &\left. + \frac{d}{2}\sum_{j=1,j\neq i}^{n} \frac{y_i(1-y_i)}{y_i - y_j} + \frac{d}{2}(1-y_i)\right\}\frac{\partial F_{n+1}}{\partial y_i}(y_1, \ldots, y_n, 0) \\ &- \frac{d}{2}\sum_{j=1,j\neq i}^{n}\frac{y_j(1-y_j)}{y_i-y_j}\frac{\partial F_{n+1}}{\partial y_j}(y_1, \ldots, y_n, 0) = abF_{n+1}(y_1, \ldots, y_n, 0). \end{aligned} \tag{32}$$

This is true for all $y_i \neq y_j, i, j = 1, \ldots, n$. (32) says that $G_n(y_1, \ldots, y_n)$ is a solution of (31) for $r = n$ with $G_n(0, \ldots, 0) = 1$. By the uniqueness statement of Theorem 2.12, we have

$$G_n(y_1, \ldots, y_n) = F_n(y_1, \ldots, y_n).$$

So $B_\kappa = D_\kappa$ for all $\kappa, l(\kappa) \leq n$.

As a corollary of Theorem 2.13, we have Theorem 2.1 .

3. **Generalized hypergeometric functions and their integral representations.**

In this section, we shall establish some properties of generalized hypergeometric functions and their integral representations.

Two special cases of the hypergeometric functions are given in the next proposition.

PROPOSITION 3.1. *We have*

$$ {}_0F_0^{(d)}(y_1, \ldots, y_r) = e^{y_1 + \cdots + y_r} \tag{33}$$

$$ {}_1F_0^{(d)}(a; y_1, \ldots, y_r) = \prod_{i=1}^{r}(1-y_i)^{-a} \tag{34}$$

PROOF. (33) follows from the definition and (iii) in §1.

Let $b = c = 1 + \frac{d}{2}(r-1)$ in (5). Since both ${}_1F_0^{(d)}(a; y_1, \ldots, y_r)$ and $\prod_{i=1}^r (1-y_i)^{-a}$ satisfy (5), (34) follows from the uniqueness of the solution of (5).

Similarly we can establish analogues of the classical Kummer relations.

PROPOSITION 3.2. *We have*

$$
\begin{aligned}
&{}_2F_1^{(d)}(a,b;c;y_1,\ldots,y_r)\\
&= \prod_{i=1}^r (1-y_i)^{-a}\, {}_2F_1^{(d)}\Big(a, c-b; c; -\frac{y_1}{1-y_1}, \ldots, -\frac{y_r}{1-y_r}\Big) \qquad (35)\\
&= \prod_{i=1}^r (1-y_i)^{c-a-b}\, {}_2F_1^{(d)}(c-a, c-b; c; y_1, \ldots, y_r) \qquad (36)
\end{aligned}
$$

The remainder of this section is to establish integral representations for the generalized hypergeometric functions.

For $a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbf{C}$, such that $(b_j)_\kappa \neq 0$ for all κ, j, we define

$$
\begin{aligned}
&{}_p\mathcal{F}_q^{(d)}(a_1,\ldots,a_p; b_1,\ldots,b_q; x_1,\ldots,x_r \mid y_1,\ldots,y_r)\\
&= \sum_\kappa \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C_\kappa^{(d)}(x_1,\ldots,x_r)}{k!} \frac{C_\kappa^{(d)}(y_1,\ldots,y_r)}{C_\kappa^{(d)}(1,\ldots,1)} \qquad (37)
\end{aligned}
$$

REMARK. When $r = 1$, ${}_p\mathcal{F}_q^{(d)}(a_1,\ldots,a_p; b_1,\ldots,b_q; x \mid y)$ becomes the classical hypergeometric function ${}_pf_q(a_1,\ldots,a_p; b_1,\ldots,b_q; xy)$, in particular, ${}_0\mathcal{F}_0^{(d)}(x \mid y) = e^{xy}$ and ${}_1\mathcal{F}_0^{(d)}(a; x \mid y) = (1-xy)^{-a}$.

In the following, we simply denote $\prod_{1 \le i < j \le r} |x_i - x_j|^d \, dx_1 \cdots dx_r$ by $dV(X,d,r)$.

The following conjecture of Macdonald has been proved in [6].

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 J_\kappa(X; 2/d) \prod_{i=1}^r x_i^{a-1} \prod_{i=1}^r (1-x_i)^{b-1}\, dV(X,d,r)\\
&= J_\kappa(I_r; 2/d) \prod_{i=1}^r \frac{\Gamma(k_i + a + \frac{d}{2}(r-i))\Gamma(b + \frac{d}{2}(r-i))\Gamma(\frac{d}{2}i+1)}{\Gamma(k_i + a + b + \frac{d}{2}(2r-i-1))\Gamma(\frac{d}{2}+1)}. \qquad (38)
\end{aligned}
$$

We define, for every $\mathbf{s} = (s_1, \ldots, s_r)$,

$$\Gamma_d(\mathbf{s}) = (2\pi)^{\frac{r(r-1)}{4}d} \prod_{i=1}^r \Gamma\Big(s_i - (i-1)\frac{d}{2}\Big). \qquad (39)$$

For $\mathbf{s} = (s, \ldots, s)$, we write $\Gamma_d(s)$ instead of $\Gamma\big((s,\ldots,s)\big)$. We also define

$$c_0 = (2\pi)^{\frac{r(r-1)}{4}d} \prod_{i=1}^r \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(i\frac{d}{2}+1)}, \qquad (40)$$

$$q_0 = 1 + \frac{d}{2}(r-1). \qquad (41)$$

PROPOSITION 3.3. *If $p \leq q+1$, we have, for $a_{p+1} > \frac{d}{2}(r-1)$, $b_{q+1} - a_{p+1} > \frac{d}{2}(r-1)$,*

$$
\begin{aligned}
{}_{p+1}F^{(d)}_{q+1}(a_1,\ldots,a_{p+1};b_1,\ldots,b_{q+1};Y) = c_0 \frac{\Gamma_d(b_{q+1})}{\Gamma_d(a_{p+1})\Gamma_d(b_{q+1}-a_{p+1})} \\
\cdot \int_0^1 \cdots \int_0^1 {}_p\mathcal{F}^{(d)}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;X \mid Y) \\
\cdot \prod_{i=1}^r x_i^{a_{p+1}-q_0} \prod_{i=1}^r (1-x_i)^{b_{q+1}-a_{p+1}-q_0} \prod_{1\leq i<j\leq r} |x_i - x_j|^d \, dx_1 \cdots dx_r.
\end{aligned}
\tag{42}
$$

PROOF. (38) implies that the integral on the right side in (42) is equal to

$$
\begin{aligned}
&\sum_\kappa \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C^{(d)}_\kappa(Y)}{k!} \cdot \int_0^1 \cdots \int_0^1 \frac{C^{(d)}_\kappa(X)}{C^{(d)}_\kappa(I)} \prod_{i=1}^r x_i^{a_{p+1}-q_0} \prod_{i=1}^r (1-x_i)^{b_{q+1}-a_{p+1}-q_0} \, dV(X,d,r) \\
&= \sum_\kappa \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C^{(d)}_\kappa(Y)}{k!} \\
&\cdot \prod_{i=1}^r \frac{\Gamma(k_i + a_{p+1} - \frac{d}{2}(r-1) + \frac{d}{2}(r-i))\Gamma(b_{q+1} - a_{p+1} - \frac{d}{2}(r-1) + \frac{d}{2}(r-i))\Gamma(\frac{d}{2}i+1)}{\Gamma(k_i + b_{q+1} - (r-1)d + \frac{d}{2}(2r-i-1))\Gamma(\frac{d}{2}+1)} \\
&= \sum_\kappa \frac{(a_1)_\kappa \cdots (a_p)_\kappa}{(b_1)_\kappa \cdots (b_q)_\kappa} \frac{C^{(d)}_\kappa(Y)}{k!} \\
&\cdot \prod_{i=1}^r \frac{\Gamma(k_i + a_{p+1} - \frac{d}{2}(i-1))\Gamma(b_{q+1} - a_{p+1} - \frac{d}{2}(i-1))\Gamma(\frac{d}{2}i+1)}{\Gamma(k_i + b_{q+1} - \frac{d}{2}(i-1))\Gamma(\frac{d}{2}+1)}.
\end{aligned}
$$

From (39) and (40) we have

$$
\begin{aligned}
&\int_0^1 \cdots \int_0^1 {}_p\mathcal{F}^{(d)}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;X \mid Y) \\
&\qquad \cdot \prod_{i=1}^r x_i^{a_{p+1}-q_0} \prod_{i=1}^r (1-x_i)^{b_{q+1}-a_{p+1}-q_0} \, dV(X,d,r) \\
&\quad = \prod_{i=1}^r \frac{\Gamma(a_{p+1} - \frac{d}{2}(i-1))\Gamma(b_{q+1} - a_{p+1} - \frac{d}{2}(i-1))\Gamma(\frac{d}{2}i+1)}{\Gamma(b_{q+1} - \frac{d}{2}(i-1))\Gamma(\frac{d}{2}+1)} \\
&\qquad \cdot {}_{p+1}F^{(d)}_{q+1}(a_1,\ldots,a_{p+1};b_1,\ldots,b_{q+1};y_1,\ldots,y_r) \\
&\quad = \frac{1}{c_0} \frac{\Gamma_d(a_{p+1})\Gamma_d(b_{q+1}-a_{p+1})}{\Gamma_d(b_{q+1})} {}_{p+1}F^{(d)}_{q+1}(a_1,\ldots,a_{p+1};b_1,\ldots,b_{q+1};y_1,\ldots,y_r).
\end{aligned}
$$

In the classical case, there are the following well-known Euler integrals for ${}_1f_1$ and ${}_2f_1$

$$
{}_1f_1(a;b;y) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{xy} x^{a-1}(1-x)^{b-a-1} \, dx
$$

for $a > 0, b - a > 0$.

$$
{}_2f_1(a,b;c;y) = \frac{\Gamma(c)}{\Gamma(a) \mid \Gamma(c-a)} \int_0^1 (1-xy)^{-b} x^{a-1}(1-x)^{c-a-1} \, dx
$$

for $a > 0, c - a > 0$.

As remarked after (37), two special cases of Proposition 3.3 give the following generalizations of Euler integrals.

PROPOSITION 3.4. *We have*

$$
\begin{aligned}
&{}_1F_1^{(d)}(a;b;y_1,\ldots,y_r)\\
(43)\quad &= c_0\frac{\Gamma_d(b)}{\Gamma_d(a)\Gamma_d(b-a)}\cdot\int_0^1\cdots\int_0^1 {}_0\mathcal{F}_0^{(d)}(X\mid Y)\prod_{i=1}^r x_i^{a-q_0}\prod_{i=1}^r(1-x_i)^{b-a-q_0}\,dV(X,d,r)
\end{aligned}
$$

if $a>\frac{d}{2}(r-1)$, $b-a>\frac{d}{2}(r-1)$, *and*

$$
\begin{aligned}
&{}_2F_1^{(d)}(a,b;c;y_1,\ldots,y_r)\\
(44)\quad &= c_0\frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)}\cdot\int_0^1\cdots\int_0^1 {}_1\mathcal{F}_0^{(d)}(b;X\mid Y)\prod_{i=1}^r x_i^{a-q_0}\prod_{i=1}^r(1-x_i)^{c-a-q_0}\,dV(X,d,r)
\end{aligned}
$$

if $a>\frac{d}{2}(r-1)$, $c-a>\frac{d}{2}(r-1)$.

As a consequence of (34), (37), (38) and Proposition 3.4, we have the following generalized Gaussian summation formula

COROLLARY 3.5. *If* $a>\frac{r-1}{2}d$, $c-a-b>\frac{r-1}{2}d$, *then*

$$
{}_2F_1{}^{(d)}(a,b;c;I_r)=\frac{\Gamma_d(c)\Gamma_d(c-a-b)}{\Gamma_d(c-a)\Gamma_d(c-b)}.
$$

Once we have Proposition 3.4, it is interesting to express ${}_0\mathcal{F}_0$ and ${}_1\mathcal{F}_0$ explicitly. In the case of $r=2$, we can express ${}_0\mathcal{F}_0$ and ${}_1\mathcal{F}_0$ in terms of classical hypergeometric functions. See [14]. For general r, we have

PROPOSITION 3.6. *We have*

$$
{}_1\mathcal{F}_0^{(d)}\Big(\frac{rd}{2};x_1,\ldots,x_r\mid y_1,\ldots,y_r\Big)=\prod_{i,j=1}^r(1-x_iy_j)^{-d/2}.
$$

PROOF. On the one hand, by Proposition 2.1 in [12], we have

$$
\prod_{i,j=1}^r(1-x_iy_j)^{-d/2}=\sum_\kappa J_\kappa(X;2/d)J_\kappa(Y;2/d)j_\kappa^{-1}
$$

On the other hand, by §1 (iv), we have

$$
J_\kappa(I_r;2/d)=(2/d)^k\Big(\frac{rd}{2}\Big)_\kappa.
$$

Hence, by the definitions, we have

$$
\begin{aligned}
{}_1\mathcal{F}_0^{(d)}\Big(\frac{rd}{2};x_1,\ldots,x_r\mid y_1,\ldots,y_r\Big)&=\sum_\kappa\Big(\frac{rd}{2}\Big)\frac{(2/d)^k}{J_\kappa(I_r;2/d)}J_\kappa(X;2/d)J_\kappa(Y;2/d)j_\kappa^{-1}\\
&=\prod_{i,j=1}^r(1-x_iy_j)^{-d/2}.
\end{aligned}
$$

As a corollary, we have

COROLLARY 3.7.

$$
{}_2F_1^{(d)}\Big(a, \frac{rd}{2}; c; y_1, \dots, y_r\Big)
= c_0 \frac{\Gamma_d(c)}{\Gamma_d(a)\Gamma_d(c-a)} \cdot \int_0^1 \cdots \int_0^1 \prod_{i,j=1}^r (1 - x_i y_j)^{-d/2} \prod_{i=1}^r x_i^{a-q_0} \prod_{i=1}^r (1 - x_i)^{c-a-q_0}\, dV(X, d, r)
$$

if $a > \frac{d}{2}(r-1), c - a > \frac{d}{2}(r-1)$.

4. **Asymptotic behavior of** ${}_{p+1}F_p^{(d)}$**.** It is known that ${}_{p+1}F_p(Y)$ is convergent for Y with $|y_i| < 1, i = 1, \dots, r$. In this section, we study the asymptotic behavior of ${}_{p+1}F_p$ as $Y \longrightarrow I$. It turns out that some new phenomena appear when $r > 1$.

Let

$$
d_\kappa = \prod_{1 \le i < j \le r} \frac{k_i - k_j + \frac{d}{2}(j-i)}{\frac{d}{2}(j-i)} \frac{B(k_i - k_j, \frac{d}{2}(j-i-1)+1)}{B(k_i - k_j, \frac{d}{2}(j-i+1))}
$$

for all κ with $l(\kappa) \le r$.

PROPOSITION 4.1.

$$
\Big(\frac{2}{d}\Big)^k J_\kappa(1, \dots, 1; 2/d) j_\kappa^{-1} = \frac{d_\kappa}{(q_0)_\kappa} \tag{45}
$$

PROOF. For a partition κ, let $s(\kappa)$ be the positive integer such that

$$
k_1 \ge \cdots \ge k_{s(\kappa)} > k_{s(\kappa)+1} = \cdots = k_r = 0.
$$

We will prove (45) by induction on $s(\kappa)$.

Let $s = s(\kappa)$. When $s = 1$, a direct calculation gives (45).

Now we assume that (45) is true for all partition λ with $s(\lambda) \le s - 1$.

Suppose $s > 1$. Let

(a) $l_i = k_i - k_s$, if $i \le s$,

(b) $l_i = 0$, if $i > s$,

and $\lambda = (l_1, \dots, l_r)$.

Then λ is a partition of $k - sk_s$ with $s(\lambda) \le s - 1$, hence

$$
\Big(\frac{2}{d}\Big)^l J_\lambda(1, \dots, 1; 2/d) j_\lambda^{-1} = \frac{d_\lambda}{(q_0)_\lambda} \tag{46}
$$

with $l = k - sk_s$.

Let

$$
A = \prod_{i=1}^s \prod_{j=1}^{k_s} [1 + (r-i)d/2 + k_i - j],
$$

$$
B = \prod_{1 \le i < s+1 \le j \le r} \left[\frac{k_i + \frac{j-i}{2} d}{k_i - k_s + \frac{j-i}{2} d} \cdot \prod_{n=1}^{k_s} \frac{k_i - n + \frac{j-i+1}{2} d}{k_i - n + \frac{j-i-1}{2} d + 1} \right].
$$

CLAIM 1.

$$
(q_0)_\kappa = (q_0)_\lambda A. \tag{47}
$$

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PROOF. A direct calculation.

CLAIM 2.

$$d_\kappa = d_\lambda B. \tag{48}$$

PROOF.

$$\begin{aligned}
d_\lambda &= \prod_{1\le i<j\le r} \frac{l_i - l_j + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(l_i - l_j, \frac{j-i-1}{2}d + 1)}{B(l_i - l_j, \frac{j-i+1}{2}d)} \\
&= \prod_{1\le i<j\le s} \frac{k_i - k_j + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i - k_j, \frac{j-i-1}{2}d + 1)}{B(k_i - k_j, \frac{j-i+1}{2}d)} \\
&\quad \cdot \prod_{1\le i<s+1\le j\le r} \frac{k_i - k_s + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i - k_s, \frac{j-i-1}{2}d + 1)}{B(k_i - k_s, \frac{j-i+1}{2}d)} \\
d_\kappa &= \prod_{1\le i<j\le s} \frac{k_i - k_j + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i - k_j, \frac{j-i-1}{2}d + 1)}{B(k_i - k_j, \frac{j-i+1}{2}d)} \\
&\quad \cdot \prod_{1\le i<s+1\le j\le r} \frac{k_i + \frac{j-i}{2}d}{\frac{j-i}{2}d} \cdot \frac{B(k_i, \frac{j-i-1}{2}d + 1)}{B(k_i, \frac{j-i+1}{2}d)} \\
&= d_\lambda \prod_{1\le i<s+1\le j\le r} \frac{k_i + \frac{j-i}{2}d}{k_i - k_s + \frac{j-i}{2}d} \\
&\quad \cdot \frac{B(k_i, \frac{j-i-1}{2}d + 1)}{B(k_i, \frac{j-i+1}{2}d)} \frac{B(k_i - k_s, \frac{j-i+1}{2}d)}{B(k_i - k_s, \frac{j-i-1}{2}d + 1)} \\
&= d_\lambda \prod_{1\le i<s+1\le j\le r} \left[\frac{k_i + \frac{j-i}{2}d}{k_i - k_s + \frac{j-i}{2}d} \cdot \prod_{n=1}^{k_s} \frac{k_i - n + \frac{j-i+1}{2}d}{k_i - n + \frac{j-i+1}{2}d + 1} \right] \\
&= d_\lambda B.
\end{aligned}$$

Let

$$\begin{aligned}
C_1 &= \prod_{i=1}^{s} \prod_{j=1}^{k_s} \left[r - (i-1) + \frac{2}{d}(k_i - k_s + j - 1) \right], \\
C_2 &= \prod_{i=1}^{s} \prod_{j=1}^{k_s} \left[s - i + \frac{2}{d}(1 + k_i - j) \right], \\
C_3 &= \prod_{i=1}^{s} \prod_{j=1}^{k_s} \left[s - i + 1 + \frac{2}{d}(k_i - j) \right].
\end{aligned}$$

From (iv) in §1 and [8], we have

$$J_\kappa(I_r; 2/d) = J_\lambda(I_r; 2/d) C_1. \tag{49}$$

For a partition κ, let

$$\begin{aligned}
h^*(\kappa) &= \prod_{s\in\kappa} h^*_\kappa(s), \\
h_*(\kappa) &= \prod_{s\in\kappa} h^\kappa_*(s).
\end{aligned}$$

Then, a computation yields

$$h^*(\kappa) = h^*(\lambda)C_2,$$
$$h_*(\kappa) = h_*(\lambda)C_3.$$

By §1 (vii), we have

$$j_\kappa = h^*(\kappa)h_*(\kappa).$$

Hence

$$j_\kappa = j_\lambda C_2 C_3,$$

(50)
$$\frac{J_\lambda(I_r; 2/d)}{j_\lambda} = \frac{J_\kappa(I_r; 2/d)}{j_\kappa}\frac{C_2C_3}{C_1}.$$

By Claim 1, Claim 2, (46) and (50), we have

$$\begin{aligned}\frac{d_\kappa}{(q_0)_\kappa} &= \frac{d_\lambda}{(q_0)_\lambda}\cdot\frac{B}{A}\\ &= (2/d)^l\frac{J_\lambda(I_r;2/d)}{j_\lambda}\frac{B}{A}\\ &= (2/d)^{k-sk_s}\frac{J_\kappa(I_r;2/d)}{j_\kappa}\frac{C_2C_3}{C_1}\frac{B}{A}.\end{aligned}$$

Therefore, it is enough to show that

$$\frac{B}{A} = (2/d)^{sk_s}\frac{C_1}{C_2C_3}.$$

In fact, a computation shows that

$$B = \frac{\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + 1 + \frac{r-i}{2}d]\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + \frac{r+1-i}{2}d]}{\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + 1 + \frac{s-i}{2}d]\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + \frac{s+1-i}{2}d]}.$$

Thus,

$$\begin{aligned}\frac{B}{A} &= \frac{\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + \frac{r+1-i}{2}d]}{\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + 1 + \frac{s-i}{2}d]\prod_{i=1}^s\prod_{j=1}^{k_s}[k_i - j + \frac{s+1-i}{2}d]}\\ &= (2/d)^{sk_s}C_1C_2^{-1}C_3^{-1}.\end{aligned}$$

This finishes the proof.

COROLLARY 4.2.

$$\begin{aligned}&{}_pF_q^{(d)}(a_1,\ldots,a_p;b_1,\ldots,b_p;y_1,\ldots,y_r)\\ &\qquad= \sum_\kappa\frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(a_1)_\kappa\cdots(a_p)_\kappa}\frac{d_\kappa}{(q_0)_\kappa}\cdot\frac{C_\kappa^{(d)}(y_1,\ldots,y_r)}{C_\kappa^{(d)}(1,\ldots,1)}\end{aligned}$$

Once having Corollary 4.2, we can use the Γ-function to give the asymptotic behavior of ${}_{p+1}F_p^{(d)}$.

Set

$$I_\alpha(t) = \sum_\kappa \Big[\prod_{j=1}^{l}(k_j+1)^\alpha\Big]\Big[\prod_{1\le p<q\le l}(k_p-k_q+1)^d\Big]t^k$$

with $l(\kappa)\le l$, $0<t<1$.

First we have the following lemma.

LEMMA 4.3. *For* $0<t<1$,

(i) If $\alpha+(l-1)d+1<0$, *then*, $I_\alpha(t)$ *is bounded;*

(ii) If $\alpha+1>0$,

$$I_\alpha(t)\approx(1-t)^{-[l\alpha+l+(l-1)\frac{l}{2}d]};$$

(iii) If $\alpha+(l-j)d+1=0$,

$$I_\alpha(t)\approx(1-t)^{-(j-1)\frac{j}{2}d}\log\frac{1}{1-t};$$

(iv) If $\alpha+(l-j)d+1>0>\alpha+(l-j-1)d+1$,

$$I_\alpha(t)\approx(1-t)^{-j[\alpha+1+ld-\frac{j+1}{2}d]}.$$

(By $A(x)\approx B(x)$, we mean that there exist two positive numbers C_1 and C_2 such that $C_1\le\frac{A(x)}{B(x)}\le C_2$ as x varies.)

PROOF. On the one hand, we have

$$\begin{aligned}
I_\alpha(t) &\ge \sum_{k_l=0}^{\infty}\Bigg\{\sum_{\substack{k_1\ge\cdots\ge k_{l-1}\\ k_{l-1}\ge k_l}}\Big(\prod_{j=1}^{l-1}(k_j-k_l+1)^\alpha\Big)\\
&\quad\cdot\Big(\prod_{1\le p<q\le l-1}[(k_p-k_l)-(k_q-k_l)+1]^d\Big)\\
&\quad\Big(\prod_{j=1}^{l-1}(k_j-k_l+1)^d\Big)t^{(k_1-k_l)+\cdots+(k_{l-1}-k_l)+(l-1)k_l}\Bigg\}k_l^\alpha t^{k_l}\\
&=\sum_{k_l=0}^{\infty}\Big[\sum_{k_1\ge\cdots\ge k_{l-1}\ge0}\prod_{j=1}^{l-1}(k_j+1)^{\alpha+d}\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d t^{k_1+\cdots+k_{l-1}}\Big]k_l^\alpha t^{lk_l}\\
&=\Big[\sum_{k_1\ge\cdots\ge k_{l-1}\ge0}\prod_{j=1}^{l-1}(k_j+1)^{\alpha+d}\\
&\quad\cdot\prod_{1\le p<q\le l-1}(k_p-k_q+1)^d t^{k_1+\cdots+k_{l-1}}\Big]\Big[\sum_{k_l=0}^{\infty}k_l^\alpha t^{lk_l}\Big]\\
&\;\;\vdots\\
&\ge C\Big(\sum_{k_1=1}^{\infty}k_1^{\alpha+(l-1)d}t^{k_1}\Big)\\
&\quad\cdot\Big(\sum_{k_2=1}^{\infty}k_2^{\alpha+(l-2)d}t^{k_2}\Big)\cdots\Big(\sum_{k_{l-1}=1}^{\infty}k_{l-1}^{\alpha+d}t^{k_{l-1}}\Big)\Big(\sum_{k_l=1}^{\infty}k_l^\alpha t^{k_l}\Big).
\end{aligned}$$

On the other hand, we can similarly show that

$$I_\alpha(t) \leq \Big(\sum_{k_1=0}^{\infty} k_1^{\alpha+(l-1)d} t^{k_1}\Big)\Big(\sum_{k_2=0}^{\infty} k_2^{\alpha+(l-2)d} t^{k_2}\Big)\cdots\Big(\sum_{k_{l-1}=0}^{\infty} k_{l-1}^{\alpha+d} t^{k_{l-1}}\Big)\Big(\sum_{k_l=0}^{\infty} k_l^{\alpha} t^{k_l}\Big).$$

Hence

(51) $$I_{\alpha(t)} \approx \Big(\sum_{m=1}^{\infty} m^{\alpha+(l-1)d} t^m\Big)\Big(\sum_{m=1}^{\infty} m^{\alpha+(l-2)d} t^m\Big)\cdots\Big(\sum_{m=1}^{\infty} m^{\alpha} t^m\Big).$$

Let

$$I_{\alpha,j}(t) = \sum_{m=1}^{\infty} m^{\alpha+(l-j)d} t^m.$$

For $-1 < t < 1$, we have

(a) if $\alpha + (l-j)d + 1 < 0$, then, $I_{\alpha,j}(t)$ is bounded;

(b) if $\alpha + (l-j)d + 1 > 0$, then,

$$I_{\alpha,j}(t) \approx (1-t)^{-[\alpha+(l-j)d+1]};$$

(c) if $\alpha + (l-j)d + 1 = 0$, then,

$$I_{\alpha,j}(t) \approx \log \frac{1}{1-t}.$$

Now the lemma follows immediately from (51), (a), (b) and (c).

PROPOSITION 4.4. *Let $\gamma = \sum_{i=1}^{p+1} a_i - \sum_{i=1}^{p} b_i$. Suppose for all κ*

$$\frac{(a_1)_\kappa \cdots (a_{p+1})_\kappa}{(b_1)_\kappa \cdots (b_p)_\kappa} > 0.$$

We have, for $-1 < y_i < 1$, $i = 1, \ldots, r$,

(i) if $\gamma > (r-1)d/2$, then

$$_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; y_1, \ldots, y_r) \approx \prod_{i=1}^{r} (1-y_i)^{-\gamma};$$

(ii) if $\gamma < -(r-1)d/2$, then there exists a constant C such that

$$_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; y_1, \ldots, y_r) \leq C;$$

(iii) if $\gamma = d(-\frac{r-1}{2} + j - 1)$, $j = 1, \ldots, r$, then, for $y_1 = \cdots = y_r = t$, $-1 < t < 1$,

$$_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; t, \ldots, t) \approx (1-t)^{-(j-1)\frac{i}{2}d} \log \frac{1}{1-t};$$

(iv) if $d(-\frac{r-1}{2} + j - 1) < \gamma < (-\frac{r-1}{2} + j)d$, $j = 1, \ldots, r-1$, then, for $y_1 = \cdots = y = t$, $-1 < t < 1$,

$$_{p+1}F_p(a_1, \ldots, a_{p+1}; b_1, \ldots, b_p; t, \ldots, t) \approx (1-t)^{-j[\gamma+(r-j)d/2]}.$$

PROOF. By Corollary 4.2,

$$ {}_{p+1}F_p(a_1,\ldots,a_{p+1};b_1,\ldots,b_p;Y)=\sum_\kappa \frac{(a_1)_\kappa\cdots(a_{p+1})_\kappa}{(b_1)_\kappa\cdots(b_p)_\kappa}\frac{d_\kappa}{(q_0)_\kappa}\frac{C_\kappa(Y)}{C_\kappa(I_r)}. $$

First,

$$ d_\kappa=\prod_{1\le i<j\le r}\frac{\Gamma((j-i-1)d/2+1)}{\Gamma(d/2(j-i+1))} \cdot\prod_{1\le i<j\le r}\frac{k_i-k_j+(j-i)d/2}{(j-i)d/2}\frac{\Gamma(k_i-k_j+(j-i+1)d/2)}{\Gamma(k_i-k_j+(j-i-1)d/2+1)}. $$

By Stirling's formula, as κ varies

$$ d_\kappa\approx\prod_{1\le i<j\le r}(k_i-k_j+1)^d. \tag{52} $$

Secondly, if $|\frac{(A)_\kappa}{(B)_\kappa}|>0$, again by Stirling's formula, as κ varies

$$ \left|\frac{(A)_\kappa}{(B)_\kappa}\right|\approx\prod_{j=1}^r(k_j+1)^{A-B}. $$

Hence, as κ varies,

$$ \frac{(a_1)_\kappa\cdots(a_{p+1})_\kappa}{(b_1)_\kappa\cdots(b_p)_\kappa(q_0)_\kappa}\approx\prod_{j=1}^r(k_j+1)^{\gamma-(r-1)d/2d-1}. \tag{53} $$

(a) If $\gamma>(r-1)d/2$, then

$$ \frac{(\gamma)_\kappa}{(q)_\kappa}\approx\prod_{j=1}^r(k_j+1)^{\gamma-(r-1)d/2-1}. $$

Thus

$$ {}_{p+1}F_p(a_1,\ldots,a_{p+1};b_1,\ldots,b_p;Y)\approx\sum_\kappa\frac{(\gamma)_\kappa}{(q_0)_\kappa}d_\kappa\frac{C_\kappa(Y)}{C_\kappa(I_r)}=\prod_{i=1}^r(1-y_i)^{-\gamma}. $$

(b) If $\gamma<-(r-1)d/2$, let $t=\max\{|y_1|,\ldots,|y_r|\}$, then

$$ |C_\kappa^{(d)}(y_1,\ldots,y_r)|\le C_\kappa^{(d)}(t,\ldots,t). $$

So, by (i) in Lemma 4.3,

$$ |{}_{p+1}F_p(a_1,\ldots,a_{p+1};b_1,\ldots,b_p;Y)|=I_{\gamma-\frac{d}{2}(r-1)-1}\quad(t)\le C. $$

That is (ii).

(c) If $-\frac{d}{2}(r-1)\le\gamma\le\frac{d}{2}(r-1)$, then, Lemma 4.3 gives (iii) and (iv).

REMARK. In the case of $r \geq 2$, we note that as γ varies in the interval $[-\frac{r-1}{2}d, \frac{r-1}{2}d]$, the asymptotics of ${}_{p+1}F_p$ varies in such a way as described in the proposition, these features are not shared by the $r = 1$ case in which the interval $[-\frac{r-1}{2}d, \frac{r-1}{2}d]$ is degenerated to the point 0.

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