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ON THE GAUSS MAPS OF SINGULAR PROJECTIVE VARIETIES

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Abstract

Here we study the dimension $\delta(m, X)$ of the general fibers of the m-Gaussian map of a singular n-dimensional variety $X \subset \mathbf{P}^N$. We show that for all integers a, b, c, d with $n \le a < b \le c < d \le N - 1$ and $a + d = b + c$ we have $\delta(a, X) + \delta(d, X) \ge \delta(b, X) + \delta(c, X)$. If $\delta(X, N - 1)$ is very large we give some classification results which extend to the singular case some results of Ein.

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1. Introduction

We work over an algebraically closed field $\mathbf{K}$. We are mainly interested in the case $\text{char}(\mathbf{K}) = 0$. Let $G(m + 1, N + 1)$ be the Grassmannian of all m-dimensional projective subspaces of $\mathbf{P}^N$. Let $X \subset \mathbf{P}^N$ be an integral variety with $\dim(X) = n$. For every integer m with $n \le m \le N - 1$ the m-Gauss map $\gamma_{m,X}$ of X and the m-defect $\delta(m, X)$ of X are defined in the following way. Set $A'(m, X) := \{(P, L) \in X_{\text{reg}} \times G(m + 1, N + 1) : T_P X \subseteq L\}$ and let $A(m, X)$ be the closure of $A(m, X)$ in $X \times G(m+1, N+1) \subset \mathbf{P}^N \times G(m+1, N+1)$. Let $\pi_1 : X \times G(m+1, N+1) \to X$ and $\pi_2 : X \times G(m+1, N+1) \to G(m+1, N+1)$ be the projection on the first (respectively second) factor. Set $\gamma_{m,X} := \pi_2 | A(m, X)$ and call $\gamma_{m,X}(X) = \pi_2(A(m, X))$ the m-Gauss image of X. Let $\delta(m, X)$ be the dimension of the general fiber of $\gamma_{m,X}$, that is, the dimension of the contact locus of a general m-dimensional linear subspace of $\mathbf{P}^N$ which is tangent to X at some smooth point. In particular, $\gamma_{N-1,X}(X) \subset \mathbf{P}^{N*}$ is the dual variety X^* of X and (with the identification of $A'(n, X)$ with X_{reg}) the morphism

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$\gamma_{m,X}|X_{\text{reg}}$ is the usual Gauss map from X_{reg} to $G(m+1, N+1)$. For the background on these topics, see [16] or [13].

In Section 2 we study the finite sequence of all $\delta(m, X)$, $n \leq m \leq N-1$, and prove that for all integers a, b, c, d with $n \leq a < b \leq c < d \leq N-1$ and $a+d = b+c$ we have $\delta(a, X) + \delta(d, X) \geq \delta(b, X) + \delta(c, X)$ (Remark 2) and that if $\delta(m, X) > \delta(n, X)$, then $\delta(m, X) > \delta(m-1, X)$ (Proposition 2). We give examples which, in our opinion, show that not much more is true. Furthermore, we study the relations between the Gauss defects of X and the Gauss defects of the intersection of X with a general hypersurface of degree d. In Section 3 we extend some of the results of [3] and [2] to the singular case; roughly speaking, we have a full extension if $\dim(\text{Sing}(X)) = 0$ (excluding cones). In the last section we discuss not the dimension of a general fiber of a Gaussian map, but the existence of positive dimensional fibers of the ordinary Gauss map $\gamma_{n,X}$. Of course, by Zak's Tangency Theorem ([16, Theorem 1.7]) the variety X cannot be smooth.

2. The sequence of Gauss defects

In this section we study the sequence $\delta(m, X)$, $n \leq m \leq N-1$, of all Gauss defects of an integral n-dimensional variety $X \subset \mathbf{P}^N$. Just by the definition of the Gauss defects we have $\delta(m, X) \leq \delta(m+1, X)$ for every X and every integer m with $n \leq m \leq N-2$. Something more can be said (see Remark 2, Proposition 2 and Example 1), but in our opinion, not much more even for smooth manifolds (see Example 1, Example 2 and Example 3), without making some very strong restrictions on the geometry of X. Then we consider the relations between the Gauss defects of X and the Gauss defects of the intersection of X with a general hypersurface of degree $d \geq 1$.

LEMMA 1. *Let n, s and N be fixed integers with $n < s < N$. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety and let $Z \subset \mathbf{P}^s$ be a general projection of X. X is reflexive if and only if Z is reflexive.*

PROOF. Use the Hessian criterion for reflexivity ([11, Theorem 12], or [6, Theorem 3.2 and Corollary 3.3]). □

REMARK 1. If X is reflexive (and in particular if char(**K**) $= 0$), then for any n-dimensional X and any m the general fiber of $\gamma_{m,X}$ is a linear space (see [11, page 173] for the case $m = N-1$ and reduce to this case taking a general projection into $\mathbf{P}^{m+1}$; the general projection of a reflexive X into $\mathbf{P}^{m+1}$ is reflexive by Lemma 1); alternatively, if char(**K**) $= 0$, see [13, 5.2 (ii)].

PROPOSITION 1. *Let $X \subset \mathbf{P}^N$ be an integral n-dimensional reflexive variety. Then for all integers m with $n \le m \le N-3$ we have*

$$\delta(m+2, X) + \delta(m, X) \ge 2\delta(m+1, X).$$

PROOF. Fix a general fiber F of $\gamma_{m,X}$ and two general fibers F', F'' of $\gamma_{m+1,X}$ with $F \subset F'$ and $F \subset F''$. Thus for dimensional reasons we have $F = F' \cap F''$. Set $G := \langle F' \cup F'' \rangle$. Hence G (respectively F' and F'') is a sufficiently general fiber of $\gamma_{m+2,X}$ (respectively $\gamma_{m+1,X}$). Let T (T', T'', R respectively) be the contact locus of F (F', F'', G respectively). By the reflexivity of X the sets T, T', T'' and R are linear spaces with dimension respectively $\delta(m+2, X)$, $\delta(m+1, X)$, $\delta(m+1, X)$ and $\delta(m, X)$ (Remark 1). Since $T = T' \cap T''$ and $T' \cup T'' \subseteq R$, we conclude the proof. □

REMARK 2. Proposition 1 is a convexity result. Indeed, it implies that for all integers a, b, c and d with $n \le a < b \le c < d \le N-1$ and $a+d = b+c$ we have

$$\delta(a, X) + \delta(d, X) \ge \delta(b, X) + \delta(c, X).$$

PROPOSITION 2. *For fixed integers n, m and N with $0 < n < m < N$, let X be an integral n-dimensional subvariety of $\mathbf{P}^N$ with $\delta(m, X) > \delta(n, X)$. Then $\delta(m, X) > \delta(m-1, X)$.*

PROOF. Fix a general $P \in X_{\text{reg}}$ and a general m-dimensional linear space M with $\dim(M) = m$ and $T_P X \subset M$. Let C (respectively C') be the contact locus of M (respectively $T_P X$). Since $\dim(C) = \delta(m, X) > \delta(n, X) = \dim(C')$, there is a hyperplane A of M containing $T_P X$ but no irreducible component of C, that is, with contact locus, C'', of dimension at most $\dim(C) - 1$. Hence we conclude the proof by the generality of P, M and A. □

The following example shows that in general in the statement of Proposition 2 we cannot replace the assumption $\delta(m, X) > \delta(n, X)$ with the assumption $\delta(m, X) > 0$.

EXAMPLE 1. Let $X \subset \mathbf{P}^N$ be the n-dimensional cone over an $(n-1)$-dimensional variety $Y \subset \mathbf{P}^{N-1}$ with $\delta(n, Y) = 0$. Then $\delta(n+1, X) = \delta(n, X) = 1$.

PROPOSITION 3. *Let X be an integral n-dimensional reflexive subvariety of $\mathbf{P}^N$ with* $\mathrm{Sing}(X)$ *finite. X is a cone if and only if $\delta(n, X) > 0$.*

PROOF. If X is a cone we have $\delta(n, X) > 0$. Assume $\delta(n, X) > 0$. Since X is reflexive, the general contact locus of an n-dimensional tangent space is a positive dimensional linear subspace (Remark 1). Since $\mathrm{Sing}(X)$ is finite, either X is a cone with $\mathrm{Sing}(X)$ as a vertex or the general contact locus is contained in X_{reg}. The latter case is excluded by Zak's Theorem of Tangency ([16, Theorem 1.7]). □

In our opinion the following two examples show that if we do not make some restrictive assumptions the finite sequence of all Gauss defects of a smooth manifold does not satisfy restrictions much stronger that the ones given by Remark 2 and Proposition 2. However, since smooth manifolds with 'bad' Gauss maps are very particular, it should be possible to make reasonable assumptions and obtain better results. For instance, if $3n \leq 2N$ all smooth n-dimensional manifolds $X \subset \mathbf{P}^N$ with $\dim(X) = \dim(X^*)$ are classified ([3, Theorem 4.5]).

EXAMPLE 2. Let $X \subset \mathbf{P}^N$ be a smooth n-fold which is a $\mathbf{P}^{n-1}$-bundle over a smooth curve C. Call $\pi : X \to C$ the projection. For every $P \in X$, we have $\pi^{-1}(\pi(P)) \cong \mathbf{P}^{n-1}$ and $\pi^{-1}(\pi(P)) \subset T_P X$. If X is reflexive, then $\delta(N-1, X) = n-2$ ([10, page 360]). For the reverse when X is reflexive, see [2, Theorem 3.1], or, if $\mathrm{char}(\mathbf{K}) = 0$, [3, Theorem 3.2]. Furthermore, for general $P \in X$ and $H, R \in X^*$ with $T_P X \subset H \cap R$, the contact loci, L_H and L_R, of H and R are hyperplanes of $\pi^{-1}(\pi(P))$. Since $L_H \cap L_R$ is the contact locus of $H \cap R$, we obtain that either $\delta(N-2, X) = \delta(N-1, X)$ or $\delta(N-2, X) = \delta(N-1, X) - 1$. By Lemma 2 we must have $\delta(N-2, X) = \delta(N-1, X) - 1$. Hence, using Proposition 1, we obtain $\delta(m, X) = \max\{0, n-1-N+m\}$ for every integer $m \geq 0$.

The following example shows a trivial way to obtain singular non-normal varieties, X, which have large $\delta(N-1, X)$.

EXAMPLE 3. Let $Y \subset \mathbf{P}^{N+1}$ be a smooth $\mathbf{P}^{m-1}$-bundle over a smooth curve (Example 2). Fix general $P', P'' \in Y$ and take a general point P on the line $\langle P', P'' \rangle$ spanned by P' and P''. Let $f : \mathbf{P}^{N+1} \setminus \{P\} \to \mathbf{P}^N$ be the projection from P. Set $X := f(Y)$. Since $\langle P', P'' \rangle \cap Y = \{P', P''\}$ for large N, Y is the normalization of X. By construction X has a non-normal point $f(P') = f(P'')$ and for large N (for fixed n) this is usually the only singular point of X. If Y is reflexive, then X is reflexive. Hence by [10, page 360], we have $\delta(N-1, X) \geq n-2$ and (except again trivial cases like $N = 3$ and Y a smooth quadric surface) we have $\delta(N-1, X) = n-2$. But we have similar 'non-normal scrolls', X', whose normalization $g : Y' \to X'$ has $h^0(X', g^*(\mathcal{O}_{Y'}(1))) = N+1$, that is, the map $Y' \to \mathbf{P}^N$ corresponds to a complete linear system; even in these cases [10, page 360], shows that $\delta(Y', N-1) \geq n-2$ if Y' is reflexive. More generally, if the n-dimensional reflexive variety $Z \subset \mathbf{P}^N$ is uniruled by t-dimensional linear spaces we have $\delta(N-1, Z) \geq t-1$. If $W = \{f(x_0, \ldots, x_N) = 0\}$ is a hypersurface of $\mathbf{P}^N$, there is a necessary and sufficient condition (at least if $\mathrm{char}(\mathbf{K}) = 0$) to have $\delta(N-1, W) > 0$: a certain projective Hessian matrix must be divisible by f (for the case $N = 3$ this is an old theorem of Schläfli, see [5, page 23]). Hence there are non-normal reflexive surfaces $W \subset \mathbf{P}^3$ uniruled by lines with $\delta(2, W) > 0$.

Example 2 shows why in the statements of Proposition 7 and Proposition 8 we only classified the normalization of X and why in the statement of Theorem 1 we have to assume that X is normal.

Now we study the relation between the Gauss defects of X and the Gauss defects of the intersection of X with a general hypersurface of degree $d \geq 1$.

LEMMA 2. *Fix integers n, m and N with $2 \leq n \leq m < N$. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety and Y a general hyperplane section of X. Then* $\delta(m-1, Y) \geq \delta(m, X) - 1$.

PROOF. We may assume $\delta(m, X) \geq 2$. Fix a general $P \in X$ and a general m-dimensional linear subspace M of $\mathbf{P}^N$ with $T_P X \subset M$. Let C be the contact locus of M. Take a general hyperplane H containing P and set $Y := X \cap H$. By the generality of P, Y is a general hyperplane section of X. By the generality of H, we have $\dim(M \cap H) = m - 1$ and $\dim(C \cap H) = \dim(C) - 1$. Furthermore, fixing H and varying P in Y we see that P may be considered a general point of Y. Since $C \cap H$ is contained in the contact locus of $M \cap H$ with Y, we obtain the lemma. □

REMARK 3. Assume $\mathrm{char}(\mathbf{K}) \neq 2$. Let $X \subset \mathbf{P}^N$ be an integral variety with $\dim(X) \geq 2$. For a general hyperplane H of $\mathbf{P}^N$, we have $\delta(N-2, X \cap H) = \max\{0, \delta(N-1, X) - 1\}$ ([6, Theorem 5.9]).

PROPOSITION 4. *Fix integers n, m and N with $2 \leq n \leq m < N$. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety and Y a general hyperplane section of X. Then*

$$\delta(m-1, Y) = \max\{\delta(m, X) - 1, 0\}.$$

PROOF. The case $m = N - 1$ of the result was proved in [6, 5.9 and 5.12]. Use Lemma 1 to reduce the general statement to the case $m = N - 1$. □

REMARK 4. Fix integers d, n and N with $2 \leq n < N$ and $d \geq 2$. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety and Y the intersection of X with a general hypersurface of degree d. By [6, Theorem 5.6], if either $\mathrm{char}(\mathbf{K}) \neq 2$ or n is even, then Y is reflexive and $\delta(N-1, Y) = 0$. It is easy to check directly that $\delta(N-1, Y) = 0$ even in the remaining case $\mathrm{char}(\mathbf{K}) = 2$ and n odd using a Bertini type argument. Hence for every integer m with $n \leq m < N$ we have $\delta(m, Y) = 0$.

In positive characteristic it is natural to give criteria for the separability of a Gauss map $\gamma_{m,X}$. In the case $m = N - 1$ this is exactly the reflexivity of X (see [10, Theorem 4]). There are at least two natural definitions for the inseparability degree of Gaussian maps ([12, page 2]); for their coincidence if X is a curve, see [9]; for a discussion of the general case, see [12] and the last part of the introduction of [9].

Using Lemma 1 for $s = m + 1$ and the corresponding result for $\gamma_{N-1,X}$ ([11, Theorem 4]) we obtain the following result.

PROPOSITION 5. *Fix integers n, m and N with $n \leq m < N$. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety. The map $\gamma_{m,X}$ is separable if and only if $\gamma_{N-1,X}$ is separable, that is, if and only if X is reflexive.*

3. Adjunction theory and Ein's papers

In this section we will try to follow as closely as possible [3] and [2] to obtain results on a singular variety with $\delta(N-1, X)$ large. Roughly speaking, we are able to handle isolated singularities, while in the general case we need to add an additive factor dim(Sing(X)) to the assumptions of every statement. If A is a closed subvariety of the variety B, let $N_{A/B}$ be the normal sheaf of A in B.

REMARK 5. Let $X \subset \mathbf{P}^N$ be an n-dimensional integral reflexive variety with $\delta(N-1, X) > 0$. Let L_H be the contact locus of a general $H \in X^*$. Since X is reflexive, L_H is a linear space of dimension $\delta(N-1, X)$ ([11, page 173]). Hence if Sing(X) is finite, then either $L_H \subset X_{\text{reg}}$ or X is a cone.

DEFINITION 1. Let X be an integral n-dimensional variety, $n \geq 2$, with $\delta(N-1, X) > 0$. Let Π be the covering family of all $\delta(N-1, X)$-dimensional linear spaces which are limits of the family of all 'general' $\delta(N-1, X)$-dimensional contact loci. Since X is closed in $\mathbf{P}^N$, every $R \in \Pi$ is contained in X. Since X^* is irreducible, Π is irreducible. We will say that X *satisfies Condition* (\$) if for a general $R \in \Pi$ we have $R \subset X_{\text{reg}}$. Let $\mathbf{T}$ be the covering family of lines obtained as closure of all lines in the contact loci of general $M \in X^*$, that is, let $\mathbf{T}$ be the set of all lines contained in some $R \in \Pi$. We will say that X *satisfies Condition* (£) if for a general $D \in \mathbf{T}$ we have $D \subset X_{\text{reg}}$.

By Remark 5 Condition (\$) is satisfied if Sing(X) is finite and X is not a cone. Obviously, Condition (\$) implies Condition (£).

REMARK 6. Let $X \subset \mathbf{P}^N$ be an n-dimensional integral reflexive variety with $\delta(N-1, X) > 0$. We assume char($\mathbf{K}$) $\neq 2$. Here we assume that the contact locus, L, of a general $H \in X^*$ is contained in X_{reg}. We will follow quite closely [3] and [2] and obtain some informations on X. In [3] and [2] the variety X was assumed to be smooth. Since X is reflexive, L is a linear space of dimension $\delta(N-1, X)$ ([11, page 173]). Set $k := \delta(N-1, X)$. Since $L \subset X_{\mathrm{reg}}$, the normal sheaf $N_{L/X}$ is locally free of rank $n-k$. Since $L \subset X_{\mathrm{reg}}$, X is reflexive and $\mathrm{char}(\mathbf{K}) \neq 2$, the proof of [3, Theorem 2.1] works verbatim and gives [3, Theorem 2.2], that is, $N_{L/X} \cong N^*_{L/X}(1)$. The proof of [3, Theorem 2.3] works verbatim and gives that for every line $T \subseteq X$ we have

$$N_{L/X}|T \cong \mathcal{O}_T^{\oplus(n-k)/2} \oplus \mathcal{O}_T(1)^{\oplus(n-k)/2},$$

in particular, $N_{L/X}$ is a uniform vector bundle and if $\mathrm{rank}(N_L/X) < \dim(L)$, that is, if $n < 2k$, then

$$N_{L/X} \cong \mathcal{O}_L^{\oplus(n-k)/2} \oplus \mathcal{O}_L(1)^{\oplus(n-k)/2}$$

by a theorem of Tango on uniform vector bundles on $\mathbf{P}^k$ valid in arbitrary characteristic ([14] and [15]). As in [3, Theorem 2.3, part (b)] we obtain the existence of an irreducible family of dimension $(n+k-2)/2$ of lines on X and that for any $P \in T$ there is a family of dimension $(n+k-2)/2$ of lines in X through P. Using the extension of [3, Theorem 2.3], we obtain Landman's parity criterion given in [3, Theorem 2.4], that is, we obtain that $n - \delta(N-1, X)$ is even (of course, only under the assumption $L \subset X_{\mathrm{reg}}$). For the same reason we have parts (a), (b), (c) and (d) of [3, Theorem 2.4], but not (a priori) part (e) of [3, Theorem 4], because of $\dim \mathrm{Sing}(X)$ we cannot apply the part of Lefschetz theorem stating that if $2n > N+2$, then $\mathrm{Pic}(X)$ is generated by the hyperplane class.

PROPOSITION 6. *Assume* $\mathrm{char}(\mathbf{K}) \neq 2$. *Let* $X \subset \mathbf{P}^N$ *be a two-dimensional integral reflexive variety with* $\delta(N-1, X) > 0$ *and* $\mathrm{Sing}(X)$ *finite and not empty. Then* X *is a cone.*

PROOF. We have $0 < \delta(N-1, X) < 2$. If the contact locus of a general $H \in X^*$ intersects $\mathrm{Sing}(X)$, X is a cone by Remark 5. If the contact locus of a general $H \in X^*$ does not intersect $\mathrm{Sing}(X)$, then $2 - \delta(N-1, X)$ is even by Landman's parity criterion extended in Remark 5, contradiction. □

PROPOSITION 7. *Assume* $\mathrm{char}(\mathbf{K}) \neq 2$. *Let* $X \subset \mathbf{P}^N$, $N \geq 5$, *be a reflexive integral variety with* $\dim(X) = 3$, $\mathrm{Sing}(X)$ *finite and* $\delta(N-1, X) > 0$. *Let* $g : Z \to X$ *be the normalization. Then either* X *is a cone or* Z *is a smooth scroll, say* $\pi : Z \to C$, *over a smooth curve* C *and* g *sends every fiber of* π *isomorphically onto a plane contained in* X.

PROOF. By [2, Corollary 3.2 (a)] we may assume $\mathrm{Sing}(X) \neq \emptyset$. Since X is reflexive, the generic contact locus is a linear space. Assume that X is not a cone. Hence there is an irreducible covering family $\mathbf{T}$ of lines such that for a general $L \in \mathbf{T}$ we have $L \subset X_{\mathrm{reg}}$, L is a contact locus and $N_{L/X} \cong \mathcal{O}_L(1) \oplus \mathcal{O}$. Fix a general $L \in \mathbf{T}$

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and a general $P \in L$. Hence P may be considered as a general point of X. Since X is not a cone and $\mathrm{Sing}(X)$ is finite, no line containing P intersects $\mathrm{Sing}(X)$. Hence by the deformation theory there is an irreducible one-dimensional family $\mathbf{T}(P) \subset \mathbf{T}$ formed by the lines containing P. Since $h^1(L, N_{L/X}(-P)) = 0$, $\mathbf{T}(P)$ is smooth at L. Set $F(P) := \{Q \in X : Q \in D$ for some $D \in \mathbf{T}(P)\}$. Since two distinct lines through P intersect only at P, $F(P)$ is a two-dimensional subvariety of X. We have $F(P) \cap \mathrm{Sing}(X) = \emptyset$ for general P. $F(P)$ is a cone with vertex P and hence T_PX contains $F(P)$.

First assume that for general P the variety $F(P)$ is not a plane. Let $R \subset \mathbf{P}^N$ be a general hyperplane containing P. Set $Y := X \cap R$. Since P is general, Y may be considered as a general hyperplane section of X. Thus $\delta(N-2, Y) = \delta(N-1, X) - 1 = 0$ (Remark 3). Hence $T_PY \cap Y$ has P as a unique singular point and this singularity is an ordinary quadratic singularity ([6, Theorem 3.5], or [11, Theorem 17, page 179]). Thus $R \cap F(P)$ has at most two irreducible components, that is, $F(P)$ is a quadratic cone. Since $N \geq 5$ and $F(P) \subset T_PX$, for a general $H \in X^*$ with $T_PX \subset H$ the scheme $H \cap X$ contains at least two irreducible components, one of them (that is, $F(P)$) being singular. Hence $Y \cap H = X \cap H \cap R$ does not have an ordinary quadratic singularity as a unique singular point, contradiction.

Now assume that $F(P)$ is a plane. Since $F(P) \cap F(P') = \emptyset$ for general $P, P' \in X$, we have $N_{F(P)/X} \cong \mathcal{O}_{F(P}$. Since the Grassmannian $G(2, N+1)$ is complete, for every $Q \in \mathrm{Sing}(X)$, there is at least a plane $V(Q) \in \mathbf{T}$ with $Q \in V(Q)$. Fix any such $V(Q)$ and a plane $F \in \mathbf{T}$ with $F \cap \mathrm{Sing}(X) = \emptyset$. First assume $F \cap V(Q) \neq \emptyset$. Since $F \cap \mathrm{Sing}(X) = \emptyset$ and both F and $V(Q)$ are planes, $D := F \cap V(Q)$ is a line. For every $P \in D$, we have $F \cup V(Q) \subset T_PX$. Hence T_PX is the 3-dimensional linear space M spanned by $F \cup V(Q)$, that is, M is tangent along D to X. Since $D \subset F \subset X_{\mathrm{reg}}$, this contradicts Zak's Tangency Theorem ([16, Theorem 1.7]). Hence $V(Q) \cap F = \emptyset$ for every $F \in \mathbf{T}$ with $F \cap \mathrm{Sing}(X) = \emptyset$. For the same reason, if $Q' \in \mathrm{Sing}(X)$ and $Q' \notin V(Q)$, then $V(Q) \cap V(Q') = \emptyset$ for any $V(Q')$ with $Q \notin V(Q')$. By construction we have a smooth affine curve A parametrizing an open subset of planes in X, an open subset Ω of X_{reg} and a morphism $\pi' : \Omega \to A$ with fibers as planes of $\mathbf{T}$ not intersecting $\mathrm{Sing}(X)$. Fix $B \in X_{\mathrm{reg}}$. For a general hyperplane R through B, the surface $X \cap R$ is smooth and it is a $\mathbf{P}^1$-bundle $\pi_R : X \cap R \to C_R$ over a smooth curve C_R. If $\mathrm{char}(\mathbf{K}) = 0$ and Z is assumed to have only locally complete intersection singularities, then Z is a $\mathbf{P}^2$-bundle over C_R by a theorem of Badescu ([1, Theorem 5.5.3]); in the general case we need to work more. Using π' we see that $C_R = C_{R'}$ for any R, R' and that in this way we define a fibration $\pi_{\mathrm{reg}} : X_{\mathrm{reg}} \to C_R$ such that $\pi_{\mathrm{reg}}/U = \pi'$. Set $C := C_R$. Varying the hyperplane R we obtain that for every plane $V \in \mathbf{T}$ we have $\mathrm{Card}(V \backslash V \cap \mathrm{Sing}(X)) = 1$. Let $\Gamma \subset X \times G(3, N+1)$ be the closure of the restriction to the fibers of π' of the incidence correspondence and let Φ be the normalization of Γ. Since $\mathrm{Card}(V \backslash V \cap \mathrm{Sing}(X)) = 1$ for every plane

$V \in \mathbf{T}$, we obtain a morphism $\pi'' : \Phi \to C$. The projection $\Phi \to X$ factors through a birational morphism $\alpha : \Phi \to Z$. Since $\dim(X) = 3$ and X is not a cone, for every $Q \in \mathrm{Sing}(X)$ there are only finitely many planes containing Q and contained in X. Hence α is finite. Since Z is normal, α is an isomorphism and hence Z is a $\mathbf{P}^2$-bundle over the smooth curve C. In particular, Z is smooth. □

PROPOSITION 8. *Let $X \subset \mathbf{P}^N$, $N \geq 6$, be an integral variety with* $\dim(X) = 4$, $\mathrm{Sing}(X)$ *finite and* $\delta(N-1, X) > 0$. *Let $g : Z \to X$ be the normalization. Then either X is a cone or Z is a smooth scroll over a smooth curve, say $\pi : Z \to C$, and g sends every fiber of π isomorphically onto a 3-dimensional linear subspace contained in X.*

PROOF. By [2, Corollary 3.3 (b)], we may assume $\mathrm{Sing}(X) \neq \emptyset$. By Remark 5 either X is a cone or Condition ($) is satisfied and in particular $\dim(X) - \delta(N-1, X)$ is even (Remark 6). Hence, $\delta(N-1, X) = 2$. Fix $Q \in X$ and let Z be a general hyperplane section of X containing Q. By Bertini's theorem we have $\mathrm{Sing}(Z) \subseteq \{Q\}$. We have $\delta(N-2, Z) \geq \delta(N-1, X) - 1 = 1$. Hence we may apply Proposition 7 to Z and then apply the same proof taking Z instead of $X \cap R$. □

THEOREM 1. *Assume* $\mathrm{char}(\mathbf{K}) \neq 2$. *Let $X \subset \mathbf{P}^N$ be an irreducible normal reflexive n-dimensional variety with* $\mathrm{Sing}(X) \neq \emptyset$ *and* $\delta(N-1, X) > 0$. *Assume* $2\delta(N-1, X) \geq n + \dim(\mathrm{Sing}(X))$. *Then* $\mathrm{Sing}(X)$ *is a linear space and X is a cone with* $\mathrm{Sing}(X)$ *as its vertex.*

PROOF. Set $k := \delta(N-1, X)$. Using Remark 3 and the preservation of reflexivity for general hyperplane sections ([10, Theorem 22 (i)], or [6, 5.9 and 5.12]), we reduce the general case to the case in which $\mathrm{Sing}(X)$ is finite; it is quite subtle (but true in arbitrary characteristic) that if a general hyperplane section of X is a cone, then X is a cone (see the second part titled 'When is the general hyperplane section of a variety a cone?' of [7]). In order to obtain a contradiction we assume that X is not a cone. Since $\mathrm{Sing}(X)$ is finite and X is not a cone we may assume Condition ($) (Proposition 2). By Remark 6 and Proposition 6 we may assume $n \geq 5$. Fix a general $P \in X$ and a general contact locus L_0 with $P \in L_0$. We will follow the proof of [2, Theorem 4.2]. Let $F(P)$ (or just Q_0 as in [2, page 903]) be the connected component containing L_0 of the set of all k-dimensional linear spaces in X which are deformations of L_0 and contain P; parts (a), (b) and (c) of [2, Lemma 4.2] work verbatim by Condition ($) because these parts concern only a general element of F(P); part (d) of [2, Lemma 4.2] works for the elements $L \in F(P)$ with $l \cap \mathrm{Sing}(X) = \emptyset$; since $\mathrm{Sing}(X)$ is finite, part (d) of [2, Lemma 4.2] is true by Proposition 2. Thus we may obtain [2, Lemma 4.3], that is, the existence of a linear space $D_0 \subset X$ with $L_0 \subset D_0$ and $\dim(D_0) = (n+k)/2$; D_0 is the union of all k-planes in the family

$F(P)$. Since D_0 depends only on P (assumed to be general), we will set $D(P) := D_0$. Then we conclude as in the proof of Proposition 7. □

If char(**K**) $= 0$ and Z is assumed to have only locally complete intersection singularities, Theorem 1 follows from a theorem of Sommese ([1, Theorem 5.5.2]).

4. Zak's Tangency Theorem

In this section we discuss the existence of positive dimensional fibers of the ordinary Gauss map $\gamma_{n,X}$. Of course, by Zak's Tangency Theorem ([16, Theorem 1.7]) the variety X cannot be smooth.

REMARK 7. Let $X \subset \mathbf{P}^N$ be an integral n-dimensional variety, $P \in X$ and let $V \subset \mathbf{P}^N$ be a linear space with $\dim(V) = n$ which is J-tangent to X at P in the sense of [16, Definition 1.6]. Then X must be smooth at P and $V = T_PX$. This trivial observation shows why in the case of the ordinary Gauss map Theorem 1.7 of [16] covers only the case in which the contact locus is contained in X_{reg}. This observation was one of the motivations for this paper.

EXAMPLE 4. Here we make no restriction on char(**K**). Fix homogeneous coordinates $x_0, \dots, x_3$ of $\mathbf{P}^3$ and set $H := \{x_0 = 0\}$. Fix an integer $s \geq 1$ and integers $a_1, \dots, a_s, m_1, \dots, m_s$ with $a_i \geq 1$ and $m_i \geq 2$ for every i. Fix an integer $d \geq \sum_{i=1}^{s} a_i m_i$. In the plane H we fix s distinct integral curves $R_1, \dots, R_s$ with $\deg(R_i) = a_i$. Set $R := \bigcup_{i=1}^{s} R_i$. We want to find a degree d normal surface $A \subset \mathbf{P}^3$ such that H is tangent to A at each point of R_{reg} and such that the scheme $A \cap H$ contains each curve R_i with multiplicity m_i. Let B be any reduced curve contained in H with $\deg(B) = d - \sum_{i=1}^{s} a_i m_i$ and $\text{card}(B \cap R)$ finite; if $d = \sum_{i=1}^{s} a_i m_i$ we take $B = \emptyset$. Set $C := B \cup \left(\bigcup_{i=1}^{s} m_i R_i\right)$. Hence C is a degree d plane curve. We will show that we may find such a normal surface A with $H \cap A$ containing C (as schemes), A smooth at every point not on H and smooth at every point of $B \cup \text{Sing}(R)$. Let W be the linear system of all degree d surfaces in $\mathbf{P}^3$ containing C and hence either containing H or with C as a scheme-theoretic intersection with H. By the definition of W and Bezout theorem every $A' \in W$ not containing H is smooth at each point of B_{reg}. Taking reducible surfaces $H \cup F' \in W$ with F' as any degree $d - 1$ surface we see that the linear system W has no base point outside C and separates the tangent vectors at each point of $\mathbf{P}^3 \backslash H$. Hence by Bertini's theorem ([8, Theorem 6.3, part 4]) general $F \in W$ is smooth outside H. Since $\text{Sing}(R) \cup \text{Sing}(B)$ is finite, we easily see that general $A' \in W$ is smooth at each point of $\text{Sing}(R) \cup \text{Sing}(B)$. Let $X = \{f = 0\}$ be an irreducible degree d surface containing C. Since $H = \{x_0 = 0\}$ and X contains every irreducible component of R with multiplicity at least 2, the Euler sequence of

$T\mathbf{P}^3$ shows that $\mathrm{Sing}(X) \cap R = H \cap \{\partial f/\partial x_0\}$. Hence we see that such a general X has exactly $(d-1)\left(\sum_{i=1}^{s} a_i\right)$ singular points, each of them on R_{reg}, exactly $(d-1)a_i$ of them on R_i, $1 \leq i \leq s$, and that each of these singular points is an ordinary double point. However, taking particular equations f we may find X with a smaller number of singular points, although these singular points may be non-ordinary or with higher multiplicity. For instance, take $s = 1$ and R smooth. Fix a degree $d-1$ homogeneous polynomial $g(x_1, x_2, x_3)$ in 3 variables such that $w := \mathrm{card}(R \cap \{g(x_1, x_2, x_3) = 0\})$ is as small as possible. For $\deg(R) \leq 3$, we may take $w = 1$. There exists a degree d polynomial f with $\{f = 0\} \cap H = C$ and with $\partial f/\partial x_0 \equiv g(x_1, x_2, x_3) \bmod (x_0)$. At least in some cases (for instance when R is a line) for the general polynomial f with these properties the surface $\{f = 0\}$ is smooth outside R.

Now we will show that, at least for non-normal varieties, we cannot extend Zak's Tangency Theorem making assumptions on their birational model, for instance to be very ample.

EXAMPLE 5. Fix an integer $n \geq 2$, a smooth n-dimensional variety Z and an effective Cartier divisor $C \subset Z$. Let D be an effective divisor such that the line bundle $\mathcal{O}_Z(2C + D)$ is very ample. Consider the complete embedding ϕ of Z into $\mathbf{P}^s := \mathbf{P}(H^0(Z, \mathcal{O}_Z(2C + D)))$ and let H be the hyperplane of $\mathbf{P}^s$ corresponding to the divisor $2C + D$. By construction H is tangent to $\phi(Z)$ along $\phi(C)$. Assume that for a general linear subspace M of H with $\dim(M) = s - n - 2$, the linear projection of $\mathbf{P}^s$ from M into $\mathbf{P}^{n+1}$ induces a birational map of $\phi(Z)$ and of C. Let X and C' be the corresponding images. Assume that X is not singular at the general point of every irreducible component of C'. Let Π be the hyperplane of $\mathbf{P}^{n+1}$ image of H through the projection from M. By construction Π is tangent to X along C' and X is birational to Z.

References

[1] M. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties* (Walter de Gruyter, Berlin, 1995).

[2] L. Ein, 'Varieties with small dual varieties, II', *Duke Math. J.* **52** (1985), 895–907.

[3] ———, 'Varieties with small dual varieties, I', *Invent. Math.* **86** (1986), 63–74.

[4] T. Fujita, 'Remarks on quasi-polarized varieties', *Nagoya Math. J.* **115** (1989), 105–123.

[5] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants* (Birkhäuser, Boston, 1994).

[6] A. Hefez and S. L. Kleiman, 'Notes on duality for projective varieties', in: *Geometry today* (eds. E. Arbarello, C. Procesi and E. Strickland), Progr. Math. 60 (Birkhäuser, Boston, 1985) pp. 143–185.

[7] M. L. Johnson, *Higher secant varieties* (Ph.D. Thesis, Brandeis University, 1994).

[8] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progr. Math. 42 (Birkhäuser, Boston, 1983).
[9] H. Kaji, 'On the inseparable degrees of the Gauss map and the projection of the conormal variety to the dual of higher order for space curves', *Math. Ann.* **292** (1992), 529–532.
[10] S. L. Kleiman, 'The enumerative theory of singularities', in: *Real and Complex Singularities, Oslo, 1976* (ed. P. Holm) (Sijitoff and Noordhoof, Alphen aan den Rijn, 1977) pp. 297–396.
[11] ———, 'Tangency and duality', in: *Proc. 1984 Vancouver Conference in Algebraic Geometry*, CMS-AMS Conference Proceedings 6 (Amer. Math. Soc., Providence, RI, 1986).
[12] S. L. Kleiman and R. Piene, 'On the inseparability of the Gauss map', in: *Enumerative Algebraic Geometry, Proc. Zeuthen Symposium*, Contemp. Math. 123 (Amer. Math. Soc., Providence, RI, 1991) pp. 149–160.
[13] J. Landsberg, 'Algebraic geometry and projective differential geometry—Seoul National University concentrated lecture series', preprint math.AG/9809184, 1997.
[14] H. Tango, 'On $(n-1)$-dimensional projective spaces contained in the Grassmann variety $\mathrm{Gr}(n, 1)$', *J. Math. Kyoto Univ.* **14** (1974), 415–460.
[15] ———, 'On morphisms from projective space $\mathbf{P}^n$', *J. Math. Kyoto Univ.* **16** (1976), 201–207.
[16] F. L. Zak, *Tangents and secants of algebraic varieties*, Transl. Math. Monographs 127 (Amer. Math. Soc., Providence, RI, 1983).

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