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Framed Stratified Sets in Morse Theory

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Abstract. In this paper, we present a smooth framework for some aspects of the "geometry of CW complexes", in the sense of Buoncristiano, Rourke and Sanderson [3]. We then apply these ideas to Morse theory, in order to generalize results of Franks [5] and Iriye-Kono [8].

More precisely, consider a Morse function f on a closed manifold M. We investigate the relations between the attaching maps in a CW complex determined by f, and the moduli spaces of gradient flow lines of f, with respect to some Riemannian metric on M.

1 Introduction

Let $f\colon M \to \mathbf{R}$ be a Morse function on a closed Riemannian manifold M. If γ is a flow line of $\operatorname{grad}(f)$, (*i.e.*, $\dot\gamma = -\operatorname{grad}(f)$) and a a critical point of f, then recall that

$$W^u(a) = \{x \in M \mid \lim_{t\to-\infty} \gamma_x(t) = a\}$$

is the so-called *unstable manifold* and

$$W^s(a) = \{x \in M \mid \lim_{t\to\infty} \gamma_x(t) = a\}$$

is the *stable manifold* of $\operatorname{grad}(f)$ at a. It is a well-known fact that both $W^u(a)$ and $W^s(a)$ are submanifolds of M, diffeomorphic to Euclidean spaces, with $\dim W^u(a) = \lambda_a = \operatorname{codim} W^s(a)$. Here λ_a is the index of a.

Such a Morse function f is said to be *Morse-Smale* if it satisfies the following *transversality condition*: for any pair a, b of critical points of f, $W^u(a)$ is transverse to $W^s(b)$. This is a generic condition [11].

When f is Morse-Smale, one can consider the manifold

$$W(a,b) = W^u(a) \cap W^s(b).$$

There is a smooth and free action of the real numbers on this manifold given the gradient flow of f:

$$W(a,b) \times \mathbf{R} \longrightarrow W(a,b) \quad (x,t) \longmapsto \gamma_x(t).$$

Any regular value t of f in the interval $\big(f(b), f(a)\big)$ gives rise to the submanifold $W(a,b) \cap f^{-1}(t)$ of M. There is an obvious diffeomorphism

$$\big(W(a,b) \cap f^{-1}(t)\big) \times \mathbf{R} \longrightarrow W(a,b),$$

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and therefore the quotient space $W(a,b)/\mathbf{R} = M(a,b)$ is a manifold of dimension $\lambda_a - \lambda_b - 1$, diffeomorphic to $W(a,b) \cap f^{-1}(t)$, and called the *moduli space of flow lines from a to b*. It can clearly be embedded in the unstable sphere $S^u(a) = W^u(a) \cap f^{-1}\bigl(f(a) - \epsilon\bigr)$ (for ϵ small) of the critical point a.

The main goal of this paper is to exploit the properties of the moduli spaces in classical Morse theory. More precisely, we wish to generalize results of [5] and [8] concerning the attaching maps in the Morse CW complex and begin the study of the cup product in terms of Morse-theoretic data.

The main tool will be *framed stratified sets*. It is an adaptation of the concept of *framified set*, introduced by Buoncristiano, Rourke and Sanderson [3] in the PL category. A framed stratified set is a natural extension of the idea of a framed manifold. There is an analogue of the Pontryagin-Thom correspondence where the target sphere is replaced by a *transverse CW complex X* and the framed manifolds by framed stratified sets "modelled" on X. More details are given in Sections 2 and 3.

Let $[M, X]$ denotes the set of homotopy classes of maps from the manifold M to the transverse CW complex X and $\Omega_X(M)$ the set of cobordism classes of embedded framed stratified subsets of M modelled on X. Denote the cobordism class of the framed stratified set F by $[F]$. The following theorem (see Section 3) is a smooth version of a result in [3].

Theorem I *The correspondence* $[\phi] \mapsto [F(\phi, X)]$ *is a bijection from* $[M, X]$ *to* $\Omega_X(M)$.

Let a and b be two critical points of f which are *successors* in the Smale order, *i.e.*, such that $a \succ b$ ($a \succ b$ means that there is a flow line from a to b) and there is no critical point z satisfying $a \succ z \succ b$. We will denote this relation by $a \succ_s b$ (the same notation will be used for successors in other posets).

Now denote by $F(a)$ the union of all moduli spaces of the form $M(a, z)$, where $a \succ z$ and z is not a minimum. It is easily seen that these moduli spaces are the strata of a stratified set in $S^u(a)$. Moreover each stratum has a framing in $S^u(a)$. It will be shown in Section 4 that $F(a)$ has the structure of an embedded framed stratified set of $S^u(a)$.

Franks shows in [5] how to construct a CW complex X_f, well-defined up to cell equivalence, out of a Morse-Smale function f, when the gradient field of f is linear about the critical points. The complex X_f is homotopy equivalent to M and will be called the *Morse complex* of f.

Recall that in the CW decomposition of M induced by f, one attaches a cell D^{λ_a} to the $(\lambda_a - 1)$-skeleton $X_f^{\lambda_a - 1}$ of X_f by a map

$$\phi_a \colon \partial D^{\lambda_a} = S^u(a) \longrightarrow X_f^{\lambda_a - 1},$$

the *attaching map* of a.

Let $F(a)$ be the framed stratified set constructed above, and let $F(\phi_a, X_f)$ be the framed stratified set constructed out of the attaching map of the critical point a (it exists, by Pontryagin-Thom).

Theorem II *One can assume that* X_f *is transverse and that* $F(a)$ *is modelled on a subcomplex of* X_f. *Morever,* $[F(a)] = [F(\phi_a, X_f)]$ *in* $\Omega_{X_f}\bigl(S^u(a)\bigr)$.

In other words, the framed stratifications by moduli spaces of flow lines determine the attaching maps in the Morse complex X_f. This generalizes Franks' main result in [5]. Full details are given in Section 5.

In Section 6, we will see that the CW complex X_f contains more information about the flow of $\operatorname{grad}(f)$. It will be shown that the way two closed moduli spaces are linked in an unstable sphere is reflected in the cohomology ring of X_f.

Presumably, the results of this paper hold in a more general setting. For example, it would be interesting to see if the theory extends to Goresky's π-fibre Morse functions on stratified sets [6].

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2 Framed Stratified Sets

In this section, we indicate how to generalize the concept of framed submanifold. For technical reasons, we need to work with manifolds with faces. The reader is referred to [12] for background material.

Throughout this paper, and unless explicitly stated to the contrary, the word *manifold* will mean a compact smooth manifold with faces, and the word *closed manifold* a compact smooth manifold without boundary. All maps will be *compatible with the faces* [12, p. 31]. The latter condition is not an important restriction since a simple application of the homotopy extension property shows that any map is homotopic to such a map.

We introduce the notion of a transverse map from a manifold to a CW complex. It is a natural generalization of the idea of a Pontryagin-Thom map associated to a framed submanifold.

Let X be a CW complex with a fixed cellular decomposition. Then X is a disjoint union of cells $X = \bigcup_{\lambda\in\Lambda} e_\lambda$ and each cell has a characteristic map $h_\lambda \colon D_\lambda \to X$. The definition of a transverse map is by induction on the number of cells in X.

A smooth map f from a manifold M to a sphere $S^p = * \cup e$ is *transverse* if M can be written as a union of two compact manifolds $M_0 \cup_{\delta T} T$ such that T is the total space of a smooth bundle $t\colon T \to D$ over a closed disc D, f restricted to M_0 is the constant map to the base-point $*$ and $f|_T$ is given by

$$f|_T\colon T \xrightarrow{t} D \xrightarrow{h} S^p.$$

Here, h is the obvious characteristic map and δT is a common face of M_0 and T. It is clear that $T = \operatorname{cl}[f^{-1}(e)]$.

Now, assume that the concept of a transverse map has been defined for all CW complexes with less than n cells and consider a complex X with exactly n cells: $X = X_0 \cup e$, where e is a top cell of dimension p. Here again, h will denote the characteristic map of the cell e.

The map $f\colon M \to X$ is *transverse* if M can be written as a union of two compact manifolds $M_0 \cup_{\delta T} T$ such that $f|_{M_0}\colon M_0 \to X_0$ is transverse and there is a smooth bundle map $t\colon T \to D$ such that

$$f|_T\colon T \xrightarrow{t} D \xrightarrow{h} X.$$

If $f\colon M \to X$ is transverse then for every $\lambda \in \Lambda$, there is a factorization

$$f|_{T_\lambda}\colon T_\lambda \xrightarrow{t_\lambda} D_\lambda \xrightarrow{h_\lambda} X$$

where $T_\lambda = \mathrm{cl}[f^{-1}(e_\lambda)]$ and $t_\lambda\colon T_\lambda \to D_\lambda$ is a smooth bundle map. Notice that the latter can be characterized as follows: it is the unique map whose restriction to $f^{-1}(e_\lambda)$ is equal to $h_\lambda^{-1} \circ f$.

We will say that X is a *transverse CW complex* if all the attaching maps in X are transverse. The proof of the following has been deferred to the last section.

Proposition 1 *Let $f\colon M \to X$ be a map from a compact manifold M to a transverse CW complex X. Suppose that the restrictions of f to the faces of M are transverse maps. Then f is homotopic rel ∂M to a transverse map.*

We are now ready to generalize the concept of framed submanifold. While formulating our definitions, we relied heavily on [2], [3] and on [12].

We will not repeat here the rather involved definition of a *stratified set with faces* $(A, \mathcal{S})$ (see [12]). Let us simply recall that A is a space, $\mathcal{S}$ a set of *strata*, such that A is the disjoint union of the strata and the strata are manifolds.

More importantly for us, each stratum X of A has *tubular neighbourhood* T_X. More precisely, each T_X is an open neighbourhood of X in A, and there is a continuous retraction $\Pi_X\colon T_X \to X$ (*i.e.*, Π_X restricted to X is the identity), and a continuous *tubular function* $\rho_X\colon T_X \to [0,\infty)$ with $\rho_X^{-1}(0) = X$.

These objects must satisfy the usual set of axioms [12]. There is also a partial order on the set of strata: We write $X < Y$ if $X \subset \mathrm{cl}_A Y$ and $X \neq Y$. The relation $\leq$ defines a poset structure on the set of strata.

Let A be a compact stratified set. The *open cone* C_0A on A is by definition the set $A \times [0,\infty)/\sim$, where $(x,0) \sim (x',0)$, with the natural stratification.

A *trivial stratified set* is a stratified set A such that for each stratum X, there is an open cone $D(X)$ and a homeomorphism $h_X\colon T_X \to X \times D(X)$, such that $\Pi_X = p_1 \circ h_X$. The maps h_X are called *trivialisations*. Here, $D(X)$ is the cone $C_0L(X)$ on a compact space $L(X)$ called the *link* of the stratum X.

Our goal now is to define a "subcategory" of the category of trivial stratified sets, in which the objects have coherent systems of trivialisations. We will call these special stratified sets *framed stratified sets*. They have been studied by Buoncristiano and Dedo under the name "trivialised sets". These authors work with manifolds with boundary but all the basic constructions and definitions of [2] are valid for manifolds with faces.

Let A be a trivial stratified set. Recall that for each stratum X, one has chosen a homeomorphism

$$h_X\colon T_X \longrightarrow X \times D(X).$$

The trivial stratified set A is a *framed stratified set* if the following five conditions hold. If $X \leq Y$, then

1. The open cone $D(X)$ associated to the stratum X of A is a trivial stratified set with one stratum for each stratum of Y of A satisfying $X \leq Y$.

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We will denote the stratum corresponding to Y by $D(X)_Y$. Thus the set of strata of $D(X)$ is $\{D(X)_Y \mid X \leq Y\}$.

2. The homeomorphism h_X maps $T_X \cap Y$ into $X \times D(X)_Y$ and is furthermore a diffeomorphism between these two manifolds.
3. The cone associated to Y is equal to the cone associated to $D(X)_Y$, *i.e.*, $D(Y) = D(D(X)_Y)$.
4. $h_X(T_X \cap T_Y) = X \times T_{D(X)_Y}$.
5. On the intersection $T_X \cap T_Y$, the following hold:

   (i) $(1_X \times \Pi_{D(X)_Y}) \circ h_X = h_X \circ \Pi_Y$

   (ii) $(1_X \times h_{D(X)_Y}) \circ h_X = (h_X \times 1_{D(Y)}) \circ h_Y$

   (iii) $\rho_Y = \rho_{D(X)_Y} \circ p_2 \circ h_X$

   (iv) $\rho_{D(X)_Y} = \rho_0 \circ p_2 \circ h_{D(X)_Y}$.

   Here, p_2 is the projection on the second factor and ρ_0 is the tubular function of $D(Y)$.

A note of warning: these equalities need to be interpreted. For example, the inclusion $\Pi_Y(T_X \cap T_Y) \subset T_X \cap Y$, necessary to give a meaning to the first equation, follows from the so-called "control conditions" which are assumed to hold in any stratified set.

An *isomorphism of framed stratified sets* is an isomorphism of stratified sets f: $(A, \mathcal{S}) \to (B, \mathcal{S}')$ which satisfies the following conditions. For any stratum X of A, corresponding under f to the stratum X' of B, there is a commutative diagram

$$\begin{array}{ccc} T_X & \xrightarrow{h_X} & X \times D(X) \\ {\scriptstyle f|}\downarrow & & \downarrow{\scriptstyle f \times f_{X,X'}} \\ T_{X'} & \xrightarrow{h_{X'}} & X' \times D(X') \end{array}$$

where $f_{X,X'}$ is an isomorphism of stratified sets. Moreover, f and $f_{X,X'}$ induce isomorphisms between the commutative diagrams corresponding to the four equalities of item 5 above and the corresponding diagrams for $(B, \mathcal{S}')$, *i.e.*, one has commutative "cubes".

3 A Pontryagin-Thom Correspondence

We are now going to consider framed stratified sets up to framed cobordism. This will allow us to prove a Pontryagin-Thom correspondence for maps from manifolds with faces to CW complexes.

Let M be a compact manifold of dimension m and $A \subset M$, an *embedded framed stratified set*. This means that each stratum of A is a proper framed submanifold of M and that M has a natural structure of framed stratified set whose $(m-1)$ - skeleton is equal to A. Obviously, all the links are then framed stratified spheres. We will use the notation (M, A) to denote M with this particular framed stratification.

We first show how to construct out of a map ϕ from M to a CW complex X an embedded framed stratified set denoted $F(\phi, X)$.

Proposition 2 *Let $\phi\colon M \to X$ be a transverse map from a compact manifold M to a transverse CW complex X. There is an embedded framed stratified set $F(\phi, X)$ in M, determined up to isomorphism by the map ϕ.*

The proof is by induction on the number of cells in X. If $X = * \cup e$, the result is trivial. Write X as $X_0 \cup e$ where X_0 is the subcomplex $X - e$. By transversality, one has a factorization:

$$\phi|_T\colon T \xrightarrow{t} D \xrightarrow{h} X$$

where $T = \mathrm{cl}[\phi^{-1}(e)]$ and h is the characteristic map of the cell e. Then, construct $F(\phi, X)$ starting with the closed stratum $N = t^{-1}(0)$, a compact submanifold of M.

Now, extend $t\colon T \to D$ to $t'\colon T' \to D'$ where T' is T with a small collar in M and D' is an open ball containing the closed unit ball D. Set $T_N = T'$ and $D(N) = D'$. Note that this construction can be achieved because N is compact.

Any strong trivialisation $h_N\colon T_N \to N \times D(N)$ (by strong, we mean that h_N^{-1} restricted to $N \times \{0\}$ is the inclusion map) will determine in an obvious manner a retraction $\Pi_N\colon T_N \to N$ and a tubular map $\rho_N\colon T_N \to [0, \infty)$ satisfying all the requirements.

Let $\phi_0 = \phi|_{M_0}$. By induction, ϕ_0 is a transverse map from M_0 to X_0 and therefore M_0 has a framed stratification $F(\phi_0, X_0)$, well-defined up to isomorphism. Let $\{N_\alpha\}$ be the set of strata of $F(\phi_0, X_0)$. Notice that a stratum N_α has a face of the form $N_\alpha \cap \delta T$.

The map h_N determines an identification $\theta\colon N \times \partial D \to \delta T$ such that

$$N \times \partial D \xrightarrow{\theta} \delta T \xrightarrow{t|} \partial D$$

is the projection p_2 on the second factor. By induction, the attaching map $h|\colon \partial D \to X$ of the cell e determines a framed stratified set $F(h|, X_0)$ in ∂D, unique up to isomorphism. Now, it is easy to see that θ determines an isomorphism of framed stratified sets

$$\big(\delta T, F(\phi_0|, X_0)\big) \simeq \big(N \times \partial D, N \times F(h|, X_0)\big).$$

Let $(\partial D)_\alpha$ be any stratum of $F(h|, X_0)$. In order to construct $F(\phi, X)$, one simply extends smoothly the strata $\{N_\alpha\}$ of $F(\phi_0, X)$ by taking products of the form $N \times C(\partial D)_\alpha - N \times \{\text{cone point}\}$, where $C(\partial D)_\alpha$ is the cone over $(\partial D)_\alpha$. The framings of these strata are defined in the obvious way. ■

It follows from this proof that when a framed stratification is of the form $F(\phi, X)$, one can assume that $t_\lambda\colon T_\lambda \to D_\lambda$ (in the definition of transverse map) is given by the restriction of $p_2 \circ h_{N_\lambda}$, for all λ.

Let X be any CW complex. There is a natural partial order on the set of cells of X defined as follows: given two cells e and e' in X, we write $e' \le e$ if $e' \cap \bar{e} \neq \varnothing$. The transitive closure of $\le$ will also be denoted by $\le$. If e is a cell of X, the *base* of e is by definition the subcomplex $X(e)$ of X given by $X(e) = \bigcup_{e' \le e} e'$.

Let M be a manifold and X a transverse CW complex. Consider an embedded framed stratified set F. We will say that F is *modelled on X* if

1. there is a injective map c

$$\{\text{strata of } (M,F)\} \xrightarrow{c} \{\text{cells of } X\}$$

which for any stratum N of F restricts to an anti-isomorphism of posets

$$\{\text{strata of } (T_N,F)\} \longrightarrow \Big\{ \text{cells of } X\big(c(N)\big) \Big\} .$$

2. For each stratum N of F, there is an isomorphism of framed stratified sets

$$\big(L(N),F\big) \longrightarrow \big(S^n, F(h,X^n)\big)$$

where $h\colon S^n \to X^n$ is the attaching map of the cell $c(N)$.

Note that since (M,F) is a framed stratified set, the link $L(N)$ of any stratum is canonically a framed stratified set. It is clear that $F(\phi,X)$ has these two properties, *i.e.*, is modelled on X. Conversely:

Proposition 3 *Any embedded framed stratified set F of M modelled on X is of the form $F(\phi,X)$ for some transverse map ϕ.*

The proof is again by induction on the number of cells. The case $X = * \cup e$ is easily dealt with using the ordinary Pontryagin-Thom correspondence.

Now, let us write $M = M_0 \cup T$, where $T = \rho_N^{-1}([0,1])$ and N is a closed stratum. Set $\delta T = \rho_N^{-1}(1)$. It is clear that $c\big(\{\text{strata of } (M,F)\}\big)$ is a subcomplex X' of X and that N corresponds to a top cell: $X' = X'_0 \cup e \subset X$. We can clearly assume that $X' = X$. Since the restriction F_0 of F to M_0 is clearly modelled on X_0, it follows by induction that there is a map

$$\phi_0 \colon M_0 \longrightarrow X_0$$

such that $F_0 = F(\phi_0, X_0)$. Also, we know that there is an isomorphism of framed stratified sets

$$h_N|\colon \delta T \longrightarrow N \times L(N)$$

and also that there is a map $l\colon L(N) \to X_0$ such that the induced framed stratification of $L(N)$ coincides with $F(h,X_0)$ (here, h is the attaching map of e in X).

Thus,

$$\delta T \xrightarrow{h_N|} N \times L(N) \xrightarrow{p_2} L(N) \xrightarrow{l} X_0$$

induces a framed stratification of δT, modelled on X_0. But by definition $h_N|$ identifies the framed stratified set coming from ϕ_0 with the one coming from $l \circ p_2$. Notice that the latter map extends to

$$T \longrightarrow N \times D \longrightarrow D \longrightarrow X.$$

Gluing ϕ_0 and this composite gives the required map. ■

Let (M,F_0) and (M,F_1) be two framed stratified sets. We will say that (M,F_0) is *cobordant* to (M,F_1) if $(M \times \{0\}, F_0)$ and $(M \times \{1\}, F_1)$ are restrictions of

$(M \times [0,1], F)$ and that (M, F_0) is X-*cobordant* to (M, F_1) if the framed stratified set F is modelled on X. Clearly, cobordism and X-cobordism are equivalence relations. The set of X-cobordism classes will be denoted by $\Omega_X(M)$. The following theorem follows easily from Propositions 1, 2 and 3.

***Theorem 4* (Theorem I)** *Let M be a compact manifold and X a transverse CW complex. The map*

$$[M, X] \longrightarrow \Omega_X(M),$$

given by $[\phi] \mapsto [F(\phi, X)]$, is a bijection.

4 Morse Theory

Throughout this section, $f \colon M \to \mathbf{R}$ will be a Morse-Smale function on a closed Riemannian manifold M. If a and b are critical points of this function, one can consider the *modified gradient flow equation of f*

$$\frac{d\omega}{dt} = \frac{\operatorname{grad}(f)}{\|\operatorname{grad}(f)\|^2}$$

and its associated *compactified moduli space of flow lines from a to b*, denoted by $\mathcal{C}(a, b)$. It is the subspace of $\operatorname{Map}\bigl([f(b), f(a)], M\bigr)$ (equipped with the compact-open topology) consisting of all continuous curves ω which are smooth on the complement of the set of critical values of f and satisfy the modified gradient flow equation with boundary conditions $\omega\bigl(f(b)\bigr) = b, \omega\bigl(f(a)\bigr) = a$.

Notice that if one removes the critical points from the image of a curve $\omega \in \mathcal{C}(a, b)$, what is left is a finite number of genuine geometric flow lines of f. This is why these curves are often called *piecewise flow lines.*

The space $\mathcal{C}(a, b)$ admits another description, given in terms of *gluing* of flow lines. Recall that the moduli space $M(a, b)$, defined in the introduction as

$$M(a, b) = W(a, b)/\mathbf{R} = W^u(a) \cap W^s(b)/\mathbf{R},$$

can be also thought of as

$$\left\{\gamma \colon \mathbf{R} \longrightarrow M \text{ smooth} \mid \gamma(-\infty) = a, \gamma(+\infty) = b, \frac{d\gamma}{dt} = -\operatorname{grad}(f)\right\} \Big/ \sim$$

where "$\sim$" is an equivalence relation given by additive reparametrization of γ: $\gamma \sim \gamma_\tau$ where $\gamma_\tau(t) := \gamma(t + \tau)$, for $\tau \in \mathbf{R}$. The correspondence between these two versions of $M(a, b)$ is given by the evaluation map. A proof of the following lemma can be found in [1].

Lemma 5 *Let $a \succ b$ be two critical points of f and (γ_i) be any sequence in $M(a, b)$. Then, either (γ_i) converges in $M(a, b)$, or there is*

1. *a subsequence (γ_j) of (γ_i),*

2. *an ordered set of critical points* $a = a_1 \succ \cdots \succ a_k \succ a_{k+1} = b$,
3. *a finite set of real numbers* $r_1 > \cdots > r_k$,

such that the points $x_{i,j} = \gamma_j(s)$ *satisfying* $f(x_{i,j}) = r_i$ *converge to a regular point of* M *which belongs to* $W(a_i, a_{i+1}) \cap f^{-1}(r_i) \simeq M(a_i, a_{i+1})$.

Therefore $M(a,b)$ is a closed manifold if and only if $a \succ_s b$, *i.e.*, if b is a successor of a. If b is not a successor of a, the next lemma gives a local parametrization of the ends of $M(a,b)$. Proofs of this result can be found in [1], [4] and [10]. Set $\mathrm{Int}[\mathcal{C}(a,b)] = \mathcal{M}(a,b)$.

Lemma 6 *Let* $a \succ b \succ c$. *There exists an* $\epsilon > 0$ *and a map*

$$\mu\colon \mathcal{M}(a,b) \times \mathcal{M}(b,c) \times (0,\epsilon] \longrightarrow \mathcal{M}(a,c)$$

which is a diffeomorphism onto its image.

These maps are called *gluing maps*. According to Austin and Braam [1], the last two lemmas imply that $\mathcal{C}(a,b)$ is a manifold with corners. It is important to note that μ extends to a map

$$\mu\colon \mathcal{C}(a,b) \times \mathcal{C}(b,c) \times [0,\epsilon] \longrightarrow \mathcal{C}(a,c),$$

in the sense that

$$\lim_{t\to 0} \mu(\omega^1, \omega^2, t) = \mu(\omega^1, \omega^2, 0)$$

is the obvious piecewise flow from a to c. For the sake of simplicity, we will suppose that the parameter ϵ can be chosen to be 1.

The work of Cohen, Jones and Segal [4] refines this description of $\mathcal{C}(a,b)$ in the following way. First, we need some notation. Let $a \succ b \succ c \succ d$ be critical points and let us denote by μ_{abc} the gluing map

$$\mu\colon \mathcal{C}(a,b) \times \mathcal{C}(b,c) \times [0,1] \longrightarrow \mathcal{C}(a,c).$$

The *clean intersection condition* states that the various gluing maps satisfy

$$\mathrm{im}[\mu_{abd}] \cap \mathrm{im}[\mu_{acd}] = \mathrm{im}[\mu_{abcd}] = \mathrm{im}[\mu_{acbd}]$$

where μ_{abcd} and μ_{acbd} are both maps

$$\mathcal{C}(a,b) \times \mathcal{C}(b,c) \times \mathcal{C}(c,d) \times [0,1]^2 \longrightarrow \mathcal{C}(a,d)$$

given respectively by

$$\mu_{abcd}\colon \big(x,y,z,(t,s)\big) \mapsto \mu_{acd}\big(\mu_{abc}(x,y,t), z, s\big),$$

and

$$\mu_{acbd}\colon \big(x,y,z,(t,s)\big) \mapsto \mu_{abd}\big(x, \mu_{bcd}(y,z,s), t\big).$$

The next result is from [4].

Theorem 7 *It is possible to choose the gluing maps so that they satisfy the clean intersection condition and the following associativity property:*

$$\mu\big(\mu(u,v,t),w,s\big)=\mu\big(u,\mu(v,w,s),t\big),$$

for any $(u,v,w)\in\mathcal{C}(a,b)\times\mathcal{C}(b,c)\times\mathcal{C}(c,d)$ *and* $s,t\in[0,1]$. *Moreover, if* $\operatorname{im}[\mu_{abd}]\cap\operatorname{im}[\mu_{acd}]\neq\varnothing$, *then* b *and* c *are comparable in the Smale order, i.e.,* $b\succ c$ *or* $c\succ b$.

Another crucial property of the moduli spaces of flow lines of a Morse-Smale function is that they are *framed manifolds*. To see this, notice that the Morse-Smale condition implies that the inclusion map

$$S^u(a)\overset{j}{\hookrightarrow}M$$

is transverse to the submanifold $W^s(b)$ of M. Moreover, $W^s(b)$ is contractible. A framing of $M(a,b)$ in $S^u(a)$ is obtained by "restricting" the unique framing of $W^s(b)$ in M to $M(a,b)=S^u(a)\cap W^s(b)$. It will be called the *Morse framing*.

The rest of this section is somewhat technical. Our goal is simply to prove that the unstable sphere $S^u(a)$ and its framed submanifolds $M(a,z)$ can be given the structure of a framed stratified set. Our task is to construct a system of control data and trivializations for the tubular neighbourhoods of the strata. This will be achieved using the gluing maps. Compatibility follows essentially from associativity of gluing. Full details can be found in [9].

Consider the evaluation map

$$E\colon\ \operatorname{Map}\big([f(b),f(a)],M\big)\times[f(b),f(a)]\longrightarrow M$$

and also its restriction to the subspace $\mathcal{C}(a,b)\times[f(b),f(a)]$. Let us choose a $\delta>0$ which has the following property: if $f^{-1}\big([f(a)-\delta,f(a)+\delta]\big)$ contains a critical point b not equal to a, then $f(a)=f(b)$. For the sake of simplicity, we will assume that $\delta=1$.

From now on, we will work with the *unstable disk*

$$D^u(a)=f^{-1}\big([f(a)-1,f(a)]\big)\cap W^u(a),$$

and we will identify the unstable sphere $S^u(a)$ with $\partial D^u(a)$. For each critical point a, there is a smooth function

$$t_a\colon D^u(a)\longrightarrow[0,1]$$

given by $x\mapsto f(a)-f(x)$ and satisfying $t_a^{-1}(0)=\{a\}$ and $t_a^{-1}(1)=S^u(a)$.

Consider the topological sum $\coprod_{a\succ b}\mathcal{C}(a,b)$, taken over the set of critical points b satisfying $a\succ b$. The evaluation map induces a continuous map

$$E_a\colon\coprod_{a\succ b}\mathcal{C}(a,b)\longrightarrow S^u(a)$$

given by $\omega \mapsto \omega\bigl(f(a)-1\bigr)$.

There is another map

$$I_a \colon S^u(a) \times [0,1] \longrightarrow D^u(a)$$

given by $(x,t) \mapsto \omega_x\bigl(f(a)-t\bigr)$, where ω_x is a solution of the modified gradient equation satisfying $\omega_x\bigl(f(x)\bigr) = x$. Notice that ω_x is simply a reparametrization of the unique flow line γ_x containing x. Clearly $I_a(x,0) = a$ and $I_a(x,1) = x$. Moreover, I_a is continuous. To see this, notice that $S^u(a)$ has the quotient topology induced by E_a and the continuity of I_a follows easily from the definitions and the universal property of quotient.

Consider now the map

$$\tau_b^a \colon \mathcal{C}(a,b) \times D^u(b) \longrightarrow S^u(a)$$

given by

$$\tau_b^a(\omega,x) = \begin{cases} \mu\bigl(\omega,\omega_x,t_b(x)\bigr)\bigl(f(a)-1\bigr) & \text{if } x \neq b \\ \omega\bigl(f(a)-1\bigr) & \text{if } x = b. \end{cases}$$

Note that if $x \neq b$, $\omega_x \in \mathcal{C}(b,c)$ (and is thus a solution of the modified gradient flow equation) for some critical point c of f. Clearly, $f\bigl(\tau_b^a(\omega,x)\bigr) = f(a)-1$ and τ_b^a maps into $S^u(a)$.

Lemma 8 *The map τ_b^a is continuous.*

The proof follows from the definitions and the universal property of quotient.

Let $\mathcal{C}(a,a_1,\dots,a_l,b)$ be the part of $\mathcal{C}(a,b)$ consisting of piecewise flows which "stop" at the critical points $a_1 \succ \cdots \succ a_l$.

Lemma 9 *One has the equality*

$$\tau_b^a\bigl(\mathcal{C}(a,b) \times \{b\}\bigr) = \bigcup_{a \succ z \succeq b} M(a,z).$$

Furthermore, $\tau_b^a\bigl(\mathcal{M}(a,b) \times \{b\}\bigr) = M(a,b)$ and more generally, one has that $\tau_b^a\bigl(\mathcal{C}(a,a_1,\dots,a_l,b) \times \{b\}\bigr) = M(a,a_1)$.

This is obvious since $\tau_b^a(\omega,b) = \omega\bigl(f(a)-1\bigr)$. The next lemma is a direct consequence of the associativity of the gluing maps.

Lemma 10 *If $a \succ b \succ c$, then one has the equality*

$$\tau_c^a\bigl(\mu(\omega,\omega',s),x\bigr) = \tau_b^a\Bigl(\omega, I_b\bigl(\tau_c^b(\omega',x),s\bigr)\Bigr)$$

for any flow $(\omega,\omega',s) \in \mathcal{M}(a,b) \times \mathcal{M}(b,c) \times (0,1)$ and $x \in D^u(c)$.

Remember that we constructed above a particular framing of the moduli space $M(a,b)$ in $S^u(a)$ and called it the Morse framing of $M(a,b)$.

Lemma 11 *The image of τ_b^a restricted to $\mathcal{M}(a,b) \times D^u(b)$ is a tubular neighbourhood of $M(a,b)$ in $S^u(a)$. Moreover, the induced framing is the Morse framing.*

Thus, we have constructed a tubular neighbourhood $T_{a,z}$ for each moduli space $M(a,z)$ in $S^u(a)$. Recall that $T_{a,z} = \tau_z^a\big(\mathcal{M}(a,z) \times \operatorname{Int}[D^u(z)]\big)$ and in particular, $M(a,z) = \tau_z^a\big(\mathcal{M}(a,z) \times \{z\}\big)$. There is an obvious *retraction map*

$$\Pi_{a,z}\colon T_{a,z} \longrightarrow M(a,z)$$

given by $\tau_z^a(\omega,y) \mapsto \omega\big(f(a)-1\big)$ and a continuous *tubular function*

$$\rho_{a,z}\colon T_{a,z} \longrightarrow [0,\infty)$$

given by $\tau_z^a(\omega,y) \mapsto \tan\Big(\pi/2\big(t_z(y)\big)\Big) = \tan\Big(\pi/2\big(f(z)-f(y)\big)\Big)$. Moreover, it is clear that $\rho_{a,z}^{-1}(0) = M(a,z)$.

From Lemma 10 and Theorem 7, one easily deduces the following result.

Lemma 12 *If $a \succ b \succ c$, then*

$$T_{a,b} \cap T_{a,c} = \tau_b^a\Big(\mathcal{M}(a,b) \times I_b\big(T_{b,c} \times (0,1)\big)\Big).$$

In particular, $T_{a,b} \cap M(a,c) = \tau_b^a\Big(\mathcal{M}(a,b) \times I_b\big(M(b,c) \times (0,1)\big)\Big)$.

Likewise, from Lemma 10 and the clean intersection condition, one sees that $T_{a,b} \cap T_{a,c} \neq \varnothing$ if and only if $a \succ b \succ c$ or $a \succ c \succ b$.

Let $\mathcal{S}(a)$ be the set of all moduli spaces of the form $M(a,z) \subset S^u(a)$. Checking the axioms (our reference is [12]), one sees that the pair $\big(S^u(a), \mathcal{S}(a)\big)$ is a stratified set. We can now prove that this stratification is actually a *framed stratification*.

First, one must associate with each stratum $M(a,z)$ in $S^u(a)$ an open cone $D(z)$. An obvious candidate for $D(z)$ is the open unstable disk $\operatorname{Int}[D^u(z)]$. It is clear that $D(z)$ is homeomorphic to the open cone $C\big(S^u(z)\big)$ (with z as cone point). By definition, $\tau_z^a\big(\mathcal{M}(a,z) \times D(z)\big) = T_{a,z}$. The required homeomorphism (trivialisation)

$$h_{a,z}\colon T_{a,z} \longrightarrow M(a,z) \times D(z)$$

is defined to be the restriction of the inverse of τ_z^a, followed by the obvious identification $\mathcal{M}(a,z) \simeq M(a,z)$. It follows that $S^u(a)$ is a trivial stratified set.

Proposition 13 *The pair $\big(S^u(a), F(a)\big)$ is a framed stratified set.*

This follows easily from what has been said above. The easy details are carried out in [9].

5 Attaching Maps

Let $h\colon X \to X'$ be a homotopy equivalence between two CW complexes X and X'. We will say that h is a *cell equivalence* if there is a bijection

$$\{\text{cells of } X\} \xrightarrow{b} \{\text{cells of } X'\}$$

such that h restricts to a homotopy equivalence between $X(e)$ and $X'\big(b(e)\big)$, for any cell e in X (see Section 3 for the definitions). Two CW complexes X and X' are *cell equivalent* if there is a cell equivalence $h\colon X \to X'$. Cell equivalence is an equivalence relation on finite CW complexes. Moreover, a cell equivalence preserves the partial order $\leq$.

Theorem 14 **(Franks)** *If X is a gradient-like vector field on M, there exists a CW complex Z, unique up to cell equivalence, and a homotopy equivalence $g\colon M \to Z$, such that for each rest point a of index k, $g\big(W^u(a)\big)$ is contained in the base $Z(e)$ of a single k-cell e.*

The map g establishes a bijection between rest points of X of index k and k-cells of Z. Moreover, the Smale partial order on the set of rest points of X corresponds to the natural partial order on the cells of Z, i.e., g induces a poset isomorphism.

Let us describe briefly the construction of the CW complex Z. We follow closely the exposition of Franks [5], where more details can be found.

Take a self-indexing Morse-Smale function f whose gradient is X. Its existence is ensured by an important result of Smale [11]. Set $M_k = f^{-1}([0, k + 1/2])$ and suppose that one has constructed a homotopy equivalence

$$g_{k-1}\colon M_{k-1} \longrightarrow Z^{k-1}$$

for some CW complex Z^{k-1}. Classical Morse theory tells us that there is a retraction

$$r_k\colon M_k \longrightarrow M_{k-1} \cup \{D^u(a_i)\}$$

where, for each critical point a_i of index k,

$$D^u(a_i) = W^u(a_i) \cap \mathrm{cl}[M - M_{k-1}].$$

For the construction of the retraction r_k, we refer to [7]. We identify the unstable sphere $S^u(a_i)$ of a_i with $\partial D^u(a_i)$. Clearly, g_{k-1} extends to a homotopy equivalence

$$(g_{k-1})\colon M_{k-1} \cup \{D^u(a_i)\} \longrightarrow Z^{k-1} \cup \{e_{a_i}\}$$

where the k-cells $\{e_{a_i}\}$ are attached to Z^{k-1} by the restrictions of g_{k-1} to the unstable spheres $\{S^u(a_i)\}$. Set

$$Z^k = Z^{k-1} \cup \{e_{a_i}\}.$$

Composing the retraction r_k with the homotopy equivalence (g_{k-1}) yields a homotopy equivalence $g_k = (g_{k-1}) \circ r_k \colon M_k \to Z^k$. The *attaching map* of a critical point a of index $k+1$ is the map

$$\phi_a \colon S^u(a) \longrightarrow Z^k$$

obtained by restricting g_k to $S^u(a) \subset \partial M_k$. Since there are finitely many critical points, the procedure eventually terminates and one gets a CW complex Z, together with a homotopy equivalence $g \colon M \to Z$ (Z has been denoted by X_f in the first section).

On the other hand, one knows by Pontryagin-Thom that there is a transverse CW complex $Y(a)$ and a transverse map

$$\psi_a \colon S^u(a) \longrightarrow Y(a)$$

such that $F\big(\psi_a, Y(a)\big) = F(a)$.

Proposition 15 *There exists a transverse CW complex Y^k, of dimension at most k, which has the following properties.*

1. *The complex Y^k has a filtration*

$$Y^0 \subset Y^1 \subset \cdots \subset Y^l \subset \cdots \subset Y^k$$

by transverse subcomplexes and each Y^l has dimension at most l.

2. *For any critical point z of index $\lambda_z = l+1 \leq k+1$, the transverse complex $Y(z)$ can be identified with a subcomplex of Y^l.*
3. *For any critical point z of index $\lambda_z = l+1 \leq k+1$, there is a cell equivalence $H_l \colon Z^l \to Y^l$ such that $H_l \circ \phi_z$ is homotopic to ψ_z.*

It is clear that this proposition leads immediately to:

Theorem 16 **(Theorem II)** *One can assume that X_f is transverse and that the cobordism classes $[F(\phi_a, X_f)]$ and $[F(a)]$ coincide in $\Omega_{X_f}\big(S^u(a)\big)$.*

The proposition will be proved by induction. If $k = 0$, then the definition of Y^0 is obvious (one zero cell for each critical point of index 0), and if a is a critical point of index 1, it is clear that $\phi_a \simeq \psi_a$.

Assume that the lemma holds for any critical point z with $\lambda_z < \lambda_a = k+1$. Thus, if z is such a critical point of index $l+1$, there is a cell equivalence $H_l \colon Z^l \to Y^l$ such that $H_l \circ \phi_z \simeq \psi_z$ and $F\big(\psi_z, Y(z)\big) = F(\psi_z, Y^l) = F(z)$, $Y(z) \subset Y^l$.

Let b be a critical point of index $\lambda_b = k$. For the sake of clarity, let us assume for the moment that there is no other critical point of index k. Let

$$F \colon S^u(b) \times [0,1] \longrightarrow Y^{k-1}$$

be a homotopy from $H_{k-1} \circ \phi_b$ to ψ_b, such that $F_0 = H_{k-1} \circ \phi_b$ and $F_1 = \psi_b$. This homotopy determines a homotopy equivalence $H \colon Z^{k-1} \cup_{\phi_b} e \to Y^{k-1} \cup_{\psi_b} e$ given by the following composite:

$$Z^{k-1} \cup_{\phi_b} e \xrightarrow{H'_{k-1}} Y^{k-1} \cup_{H_{k-1} \circ \phi_b} e \xrightarrow{k} Y^{k-1} \cup_{\psi_b} e$$

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Here, H'_{k-1} is the obvious extension of H_{k-1}. The map k is given by $k|_{Y^{k-1}} = \mathrm{Id}$, and

$$k(tu) = \begin{cases} 2tu & \text{if } u \in \partial e \text{ and } 0 \le t \le 1/2 \\ F_{2-2t}(u) & \text{if } u \in \partial e \text{ and } 1/2 \le t \le 1. \end{cases}$$

Let $\{b_i\}$ be the set of critical points of index k. If we repeat the above construction for each b_i, and set $Z^k = Z^{k-1} \cup_{\phi_{b_i}} \{e_{b_i}\}$ (this is actually the definition of Z^k) and $Y^k = Y^{k-1} \cup_{\psi_{b_i}} \{e_{b_i}\}$, we get a homotopy equivalence

$$H_k \colon Z^k \longrightarrow Y^k.$$

We must check that $H_k \colon Z^k \to Y^k$ satisfies the three conditions of the proposition.

The first condition trivially holds, since $Y^k = Y^{k-1} \cup_{\psi_{b_i}} \{e_{b_i}\}$ fits naturally at the end of the filtration of Y^{k-1}, which exists by induction. Now, if one refers to the construction of the framed stratification $F(a)$ of $S^u(a)$, one sees that the link of the stratum $M(a,z)$ is the unstable sphere $S^u(z)$, with framed stratification $F(z)$. By induction $F(z) = F(\psi_z, Y^l)$ if $\lambda_z = l + 1$. Thus, $F(a)$ is clearly modelled on Y^k, and this precisely means that $Y(a)$ can be identified with a subcomplex of Y^k. Thus $\psi_a \colon S^u(a) \to Y^k$. To complete the proof, it remains to show that $H_k \circ \phi_a$ is homotopic to ψ_a. The rest of this section is devoted to this problem.

Let us first recall the following concepts from Section 2, and also introduce some new terminology. Let $X = \bigcup_{\lambda\in\Lambda} e_\lambda$ be a transverse CW complex and let

$$h_\lambda \colon D_\lambda \longrightarrow X$$

be the characteristic map for the cell $e_\lambda \subset X$. To a transverse map $f \colon M \to X$ from a manifold M to X, one has associated a family of bundles

$$t_\lambda \colon T_\lambda \longrightarrow D_\lambda$$

such that $f|_{T_\lambda} = h_\lambda \circ t_\lambda$. We recall that T_λ is simply $\mathrm{cl}[f^{-1}(e_\lambda)]$. We will say that T_λ is a *block* of M and that $M = \bigcup_{\lambda\in\Lambda} T_\lambda$ is the *block decomposition* of M associated to $F(f, X)$.

The fibre $N_\lambda = t_\lambda^{-1}(0)$ of t_λ, which is a manifold with faces, will be called a *truncated stratum* of the framed stratified set $\big(M, F(f, X)\big)$. An easy induction argument shows that the truncated stratum N_λ is given by $M_\lambda \cap T_\lambda$, where M_λ is the stratum corresponding to the cell e_λ.

If f and g are two transverse maps from M to X, we will say that the block decompositions associated to $F(f, X)$ and $F(g, X)$ are *comparable* if they induce the same set-theoretic decomposition $M = \bigcup_{\lambda\in\Lambda} T_\lambda$ of M and the same truncated strata. If f and g induce comparable block decompositions, it does not of course follow that f is homotopic to g, since they could give rise to different framings of the strata. But this is the only obstruction, as we can see from the following lemma.

Lemma 17 *Let $\phi, \psi \colon M \to X$ be two transverse maps from the manifold M to the transverse CW complex X. Let us assume that ϕ and ψ induce comparable block decompositions of M. If ϕ and ψ induce equivalent framings of the truncated strata, then ϕ is homotopic to ψ.*

The proof is by induction on the number of cells in X. The case $X = S^p$ with cellular structure $S^p = * \cup e$ is trivial, since there is a unique non-open stratum, which is framed equivalently by ϕ and ψ.

If X is arbitrary then, as usual, let $X = X_0 \cup e$ and $M_0 = \mathrm{cl}[M - T]$, where T is the tubular neighbourhood of the stratum N corresponding to the cell e. First, notice that the restrictions ϕ_0 and ψ_0 of ϕ and ψ to M_0 both map M_0 into X_0. By induction, there is a homotopy

$$H_0 \colon M_0 \times [0,1] \longrightarrow X_0$$

such that $H_0(x,0) = \phi_0(x)$ and $H_0(x,1) = \psi_0(x)$. Since $\phi|_T$ is homotopic to $\psi|_T$ through maps which are at each time t trivialisations of the neighbourhood T of the closed stratum $N \subset T$, H_0 extends to a homotopy H from ϕ to ψ. ■

Let us now come back to the Morse-theoretic context. We have a block decomposition

$$S^u(a) = \bigcup_{a \succ z} S_{a,z}$$

induced by $\psi_a \colon S^u(a) \to Y(a)$, bundle maps $t_{a,z} \colon S_{a,z} \to D_z$ such that $h_z \circ t_{a,z} = \psi_a|_{S_{a,z}}$. We recall that h_z is the characteristic map for the cell associated to the critical point z. Moreover, the truncated strata are obviously given by the intersections $N(a,z) = M(a,z) \cap S_{a,z}$. We must study the behaviour of the Morse attaching map $\phi_a \colon S^u(a) \to Z^k$ with respect to the block decomposition induced by ψ_a.

Lemma 18 *One can construct the attaching map ϕ_a in such a way that for each critical point z, there is an open neighbourhood U_z of the truncated stratum $N(a,z)$ in $T_{a,z}$ and a subcell $E_z \subset e_z$ with the following properties:*

1. *$H_k \circ \phi_a|_{U_z} \colon U_z \to E_z \subset Y^k$ is a smooth fibration,*
2. *the fibre over $h_z(0)$ is $N(a,z)$ and*
3. *$H_k \circ \phi_a(S_{a,w}) \cap e_z = \varnothing$ if $\lambda_w < \lambda_z$.*

It is a standard fact that for any critical point z, there is an interval $[\alpha_z, \beta_z]$ about $f(z)$ and a retraction

$$r_z \colon f^{-1}\big((-\infty, \beta_z]\big) \longrightarrow f^{-1}\big((-\infty, \alpha_z]\big) \cup D^u(z).$$

Let $\delta > 0$. By construction, r_z has the following properties (see [7] or [5]):

1. If $x \in f^{-1}\big((-\infty, \beta_z]\big) - f^{-1}\big((-\infty, \alpha_z]\big)$ and is not within δ of $W^s(z)$, then $r_z(x)$ is the unique point on $f^{-1}(\alpha_z)$ on the same orbit as x.
2. On the other hand, if x is close to $W^s(z)$, then in Morse coordinates the retraction is given by the projection $(u,v) \mapsto (u,0)$.

For any critical point z with $a \succ z$, consider the manifold with corners $S(a,z) = \mathrm{cl}[S^u(a) - \bigcup_{b \in A} S_{a,b}]$, where A is the set of critical points b such that $f(b) \in \big(f(z), f(a)\big)$.

The proof of the lemma relies on the following fact: For any z, one can suppose that ϕ_a restricted to $S(a,z)$ has a factorization

$$S(a,z) \xrightarrow{R} f^{-1}\big((-\infty,\beta_z]\big) \xrightarrow{(g_z)\circ r_z} Z^{\lambda_z-1}\cup_{\phi_z} e_z \xrightarrow{i} Z^k.$$

Here, i is the obvious inclusion and $R(x)$ is by definition the unique point on $f^{-1}(\beta_z)$ which is on the same orbit as x. The existence of such a factorization follows easily from the construction of the retractions. ■

We now summarize the situation in the next lemma, where we use the notation introduced above.

Lemma 19 *Let ϕ and ψ be two maps from the manifold M to the transverse CW complex $X = \bigcup_{\lambda\in\Lambda} e_\lambda$. Let us assume that ψ is transverse and that for any $\lambda \in \Lambda$, there is an open neighbourhood U_λ of the truncated stratum N_λ (of $F(\psi, X)$) and a subcell $E_\lambda \subset e_\lambda$ such that*

1. *$\phi|_{U_\lambda} \colon U_\lambda \to E_\lambda \subset X$ is a smooth fibration,*
2. *the fibre over $h_\lambda(0)$ is N_λ and*
3. *$\phi(T_\gamma) \cap e_\lambda = \varnothing$ if $\dim[e_\gamma] < \dim[e_\lambda]$.*

Then, ϕ is homotopic to a transverse map $\phi' \colon M \to X$ whose block decomposition is comparable with the block decomposition induced by ψ.

Notice that in our situation, $M = S^u(a)$, $H_k \circ \phi_a = \phi$, $\psi_a = \psi$ and $X = Y^k$. As usual, we prove the lemma by induction on the number of cells in X. If $X = S^p$, with cellular structure $S^p = *\cup e$, this is obvious by uniqueness of tubular neighbourhood.

For X arbitrary, we write $X = X_0 \cup e$ and denote by ϕ_0 and ψ_0 the restrictions of ϕ and ψ to $M_0 = \mathrm{cl}[M - T]$. Here, T is the block of $F(\psi, X)$ corresponding to the top cell e. Clearly, ψ_0 and ϕ_0 map into X_0 (by 3). By induction, one can suppose that ϕ_0 is homotopic to a transverse map with the same block decomposition as ψ_0. It is not difficult to show that this homotopy can be extended to M and that one gets a transverse map which has the same block decomposition as ψ. ■

It follows that $H_k \circ \phi_a$ is homotopic to a map which has the same block decomposition as ψ_a. Moreover, the framing that this map induces on $N(a,c)$ is clearly the Morse framing, since it is given by the splitting of a Morse chart about the critical point c. Thus, it is the restriction of the framing of $W^s(c)$ in M to $N(a,c)$. By Lemma 17, $H_k \circ \phi_a$ is homotopic to ψ_a.

6 A Remark on the Cup Product

In [5], Franks pointed out that the way two connecting manifolds for a Morse-Smale flow (in our language, moduli spaces of flow lines) link in an unstable sphere $S^u(a) \subset W^u(a)$ should be reflected in the cohomology ring of the associated CW complex. We now describe a preliminary result in this direction.

Let X be a vector field on M which generates a Morse-Smale gradient flow. Let a, b, c be three rest points of X. Assume that $\lambda_a = \lambda_b + \lambda_c$ and that $a \succ_s b$, $a \succ_s c$. It follows that $M(a, b)$ and $M(a, c)$ are compact, disjoint submanifolds of $S^u(a)$.

Now, the Witten complex (see [10]) computes the homology and cohomology modules of M. If a determines a homology class ω_a and b, c cohomology classes κ_b and κ_c, then one has the following result.

Theorem 20 *If the above hypotheses hold then*

$$\langle \kappa_b \smile \kappa_c, \omega_a \rangle = \operatorname{lk}\big(M(a,b), M(a,c)\big),$$

where lk *is the linking number in* $S^u(a)$.

Let us sketch the proof. Using the flow, one can construct a retraction and by composing with a suitable collapsing map, we get

$$\phi\colon S^u(a) \longrightarrow S^{\lambda_b} \vee S^{\lambda_c}.$$

Now, we know that one can assume that, up to homotopy $\phi^{-1}(u) = M(a, b)$ and $\phi^{-1}(v) = M(a, c)$, for some regular values u, v in S^{λ_b}, S^{λ_c} respectively. Clearly, in the cohomology ring of the mapping cone $e_a \cup_\phi S^{\lambda_b} \vee S^{\lambda_c}$, one has the relation

$$\theta_b \smile \theta_c = n\theta_a,$$

where θ_a, θ_b and θ_c are the obvious generators in cohomology and n is an integer. It is not difficult to see that

$$n = \operatorname{lk}\big(M(a,b), M(a,c)\big).$$

The transfer of this result to the cohomology ring of the manifold M is a straightforward exercise in algebraic topology.

In [9], it is shown that a similar result holds under more general assumptions. There is obviously more to be said about the relations between the fine structure of Morse-Smale flows and cohomological invariants. We are currently investigating these questions.

7 Transversality

We now prove the transversality result (Section 2, Proposition 1). Thus, $f\colon M \to X$ is a map whose restrictions to the faces of M are transverse. Recall that, as stated at the beginning of Section 1, f is also compatible with the faces. It is not difficult to show that these conditions cause f to be transverse when restricted to a small collar neighbourhood of ∂M.

We now prove the result for $X = S^p = * \cup e$. Let us first approximate f by a smooth map, without perturbing it on ∂M (where it is already transverse). Let $h\colon D \to D/\partial D = S^p$ be a smooth characteristic map for the CW complex S^p. By

Sard's theorem, one can assume that $h(0)$ is a regular value of f. It follows that $N = f^{-1}\big(h(0)\big)$ is a smooth submanifold of M.

Now, it is necessary to prove that $f\colon M \to S^p$ can be identified with a projection map on the inverse image of a sufficently small neighbourhood of any regular value. When M is closed, it is a standard fact. Examining the proof, one sees that it immediately generalizes to maps from manifolds with faces to ordinary manifolds.

Consider an open subdisc $D(\epsilon) \subset D$ of radius ϵ. Let $e(\epsilon)$ be the image of $D(\epsilon)$ by the characteristic map $h\colon D \to S^p$. It follows from the preceding remark that if $\epsilon > 0$ is small enough, then f restricted to $f^{-1}\big(e(\epsilon)\big)$ is a bundle map. It can be factorized as

$$f^{-1}\big(e(\epsilon)\big) \xrightarrow{t_\epsilon} D(\epsilon) \xrightarrow{h|} S^p.$$

Consider a smooth homotopy $\phi_t\colon S^p \times [0,1] \to S^p$ of the identity of S^p such that ϕ_1 maps $h\big(D(\epsilon)\big)$ diffeomorphically onto $e \subset S^p$ and shrinks the complement to the 0-cell $*$. Such a homotopy always exists. Notice that in the standard Pontryagin-Thom construction, one simply considers the homotopy $\phi_t \circ f$ from f to $\phi_1 \circ f$ and one is finished. But recall that in our situation, we do not want to change the values of f on the boundary, since this is our basis for induction. By step 1, we know that f restricted to a suitable collar V of ∂M is transverse. In order to exploit this fact, choose an inner vector field on M (that is, on ∂M the field points toward the interior of M) and let γ_x be the integral curve of this vector field through x. Choose also a $\delta > 0$ such that γ_x maps the interval $[0,\delta]$ into V. For simplicity, assume that $\delta = 1$. Finally, consider a smooth function

$$\epsilon\colon \mathbf{R} \longrightarrow \mathbf{R}$$

satisfying $\epsilon(t) \geq 0$, $\epsilon'(t) \geq 0$, $\epsilon(t) = 0$ whenever $t \leq 0$ and $\epsilon(t) = 1$ whenever $t \geq 1$. Clearly, ϵ restricts to a map from $[0,1]$ to $[0,1]$.

Consider the homotopy

$$F(y,t) = \begin{cases} \phi_{\epsilon(s)t} \circ f(y) & \text{if } y = \gamma_x(s), x \in \partial M \text{ and } 0 \leq s \leq 1, \\ \phi_t \circ f(y) & \text{otherwise.} \end{cases}$$

The map $F(y,1)$ is transverse.

We can now proceed by induction on the number of cells in X. Consider a transverse CW complex $X = X_0 \cup e$, where X_0 is the subcomplex $X - e$ and e is a top-dimensional cell. Let $h\colon D \to X$ be a characteristic map for e and let $q\colon X \to X/X_0$ be the standard collapsing map. The characteristic map of the cell e gives an identification of X/X_0 with $D/\partial D$ and we choose an identification of $D/\partial D$ with S^p such that the collapsing map q is smooth on the top cell. With these choices, the composite

$$D \longrightarrow X \longrightarrow X/X_0 \longrightarrow D/\partial D \longrightarrow S^p$$

is smooth since it is the above collapsing map.

By step 1, we can assume that the map $f\colon M \to X$, restricted to some suitable neighbourhood V of ∂M, is transverse to X. There is therefore a factorization

$$f|_K\colon K \xrightarrow{t_V} D \xrightarrow{h} X$$

where $K = \mathrm{cl}[(f|_V)^{-1}(e)]$. Recall that t_V is smooth. It follows that $q \circ f|_K = q \circ (h \circ t_V) = (q \circ h) \circ t_V$ is smooth. It is also easy to see that $q \circ f$ is smooth on V. Notice that the complement of K in V goes to one point.

Let $W \subset V$ be some closed neighbourhood of ∂M. Since $q \circ f|_W$ is smooth, by standard differential topology, it extends to a smooth map $g\colon M \to S^p$ which is homotopic to $q \circ f$, and coincides with $q \circ f$ on the closed set W.

By Sard's theorem, one can suppose that $y = q\big(h(0)\big) \in S^p$ is a regular value of g. As before, let $D(\epsilon)$ be an open subdisc of D, of radius ϵ and denote by $e(\epsilon)$ the subset of S^p which is the image of $D(\epsilon)$ by the smooth map $q \circ h$. By the same argument as in step 2, if ϵ is small enough, then g is a projection when restricted to $T_\epsilon = g^{-1}\big(e(\epsilon)\big)$. Consider now the map $k\colon T_\epsilon \cup W \to X$ given by

$$k(x) = \begin{cases} q^{-1} \circ g(x) & \text{if } x \in T_\epsilon, \\ f(x) & \text{if } x \in W. \end{cases}$$

This map is well-defined since $g = q \circ f$ on W. It is not difficult to show that the map k is homotopic to f restricted to $T_\epsilon \cup W$, by a homotopy which does not perturb f on the boundary ∂M. Since $T_\epsilon \cup W$ is a retract of some open set of M, an application of the Homotopy Extension Property shows that k extends to a map $k\colon M \to X$, which is homotopic to f, coincides with f on ∂M and is smooth on $\mathrm{cl}[(q \circ k)^{-1} e(\epsilon)]$.

To summarize, one can suppose that f is smooth and trivial when restricted to the inverse image of a small subset of the top cell e. Consider such a small open set $e' \subset e \subset X$. There is an homotopy ϕ_t of the identity map of X which map e' diffeomorphically onto e, and is the identity on X_0. Using the same function $\epsilon\colon [0,1] \to [0,1]$ and the same inner vector field as in step 2, one gets a homotopy

$$F(y,t) = \begin{cases} \phi_{\epsilon(s)t} \circ f(y) & \text{if } y = \gamma_x(s) \text{ and } 0 \le s \le 1, \\ \phi_t \circ f(y) & \text{otherwise.} \end{cases}$$

Clearly, F is a homotopy rel ∂M of $F(y,0) = f(y)$. The map $F(y,1) = F_1$ has the property that

$$F_1\colon \ \mathrm{cl}[F_1^{-1}(e)] \xrightarrow{t} D \xrightarrow{h} X$$

and is thus transverse to the top cell e. Now, $M_0 = M - F_1^{-1}(e)$ is a manifold and the map F_1, restricted to the boundary of M_0 is transverse to X_0. Applying the induction hypothesis completes the proof.

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