Canad. Math. Bull. Vol. 60 (2), 2017 pp. 422–435
http://dx.doi.org/10.4153/CMB-2016-090-2

New Super-quadratic Conditions for Asymptotically Periodic Schrödinger Equations

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Abstract. We study the semilinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbf{R}^N, \\ u \in H^1(\mathbf{R}^N), \end{cases}$$

where f is a superlinear, subcritical nonlinearity. It focuses on the case where $V(x) = V_0(x)+V_1(x)$, $V_0 \in C(\mathbf{R}^N)$, $V_0(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, $\sup[\sigma(-\triangle + V_0) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\triangle + V_0) \cap (0,\infty)]$, $V_1 \in C(\mathbf{R}^N)$, and $\lim_{|x|\to\infty} V_1(x) = 0$. A new super-quadratic condition is obtained that is weaker than some well-known results.

1 Introduction

Consider the following semilinear Schrödinger equation

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $V\colon \mathbb{R}^N \to \mathbb{R}$ and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are asymptotically periodic in x and f is superlinear as $|u| \to \infty$.

The existence of a nontrivial solution for (1.1) has been widely investigated when $V(x)$ and $f(x,u)$ are periodic in x and satisfy the following basic assumptions:

(V) $V \in C(\mathbb{R}^N)$, $V(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and

$$\sup[\sigma(-\triangle + V) \cap (-\infty, 0)] < 0 < \overline{\Lambda} := \inf[\sigma(-\triangle + V) \cap (0,\infty)].$$

(F1) $f \in C(\mathbb{R}^N \times \mathbb{R})$, and there exist constants $p \in (2, 2^*)$ and $C_0 > 0$ such that

$$|f(x,t)| \le C_0(1 + |t|^{p-1}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

(F2) $f(x,t) = o(|t|)$ as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$, and $F(x,t) := \int_0^t f(x,s)ds \ge 0$.
(F3) $f(x,t)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$.

See [4,5,11,12,19,28,30] and the references therein. In those papers, (AR) is a classical existence condition that is due to Ambrosetti and Rabinowitz [2].

Received by the editors September 19, 2015; revised January 5, 2016.
Published electronically March 13, 2017.
This work was partially supported by the NNSF (No. 11171351) and the SRFDP (No. 20120162110021) of China.
AMS subject classification: **35J20**, 35J60.
Keywords: Schrödinger equation, superlinear, asymptotically periodic, ground state solutions of Nehari-Pankov type.

(AR) there exists a $\mu > 2$ such that

$$0 < \mu F(x,t) \le t f(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\}).$$

(AR) is a very convenient hypothesis, since it readily achieves mountain pass geometry as well as satisfaction of the Palais–Smale condition. However, it is a severe restriction, since it strictly controls the growth of $f(x,t)$ as $|t| \to \infty$. In recent years, there have been papers devoted to replacing (AR) with weaker conditions. For example, Liu and Wang [17] first introduced a more natural superquadratic condition.

(SQ) $\lim_{|t|\to\infty} \dfrac{|F(x,t)|}{|t|^2} = \infty$, uniformly in $x \in \mathbb{R}^N$.

Subsequently, (SQ) has been commonly used [6–8, 12, 13, 18, 21, 31]. However, it is not sufficient to guarantee that (1.1) has a nontrivial solution. Later, Ding and Lee [6] gave the following milder existence condition.

(DL) $\mathcal{F}(x,t) := \frac{1}{2} t f(x,t) - F(x,t) > 0$ if $t \ne 0$, and there exist $c_0 > 0$, $r_0 > 0$, and $\kappa > \max\{1, N/2\}$ such that $|f(x,t)|^\kappa \le c_0 \mathcal{F}(x,t)|t|^\kappa$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $|t| \ge r_0$.

Under Assumption (F1), Condition (DL) greatly weakens (AR). Soon after, (DL) was generalized to more general equations or systems (see *e.g.*, [3, 21, 23, 32, 34]).

Szulkin and Weth [20] developed an ingenious approach to find the ground state solutions for problem (1.1). They demonstrated that (SQ) together with the following Nehari type assumption (Ne) implies that (1.1) has a ground state solution.

(Ne) $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$.

Based on Szulkin and Weth [20], Liu [16] showed that (1.1) has a nontrivial solution by using the following weak version (WN) instead of (Ne).

(WN) $t \mapsto f(x,t)/|t|$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

Recently, Tang [22] introduced new super-quadratic conditions as follows.

(WS) $\lim_{|t|\to\infty} \dfrac{|F(x,t)|}{|t|^2} = \infty$, almost everywhere $x \in \mathbb{R}^N$;

(Ta) there exists a $\theta_0 \in (0,1)$ such that

$$\frac{1-\theta^2}{2} t f(x,t) \ge \int_{\theta t}^{t} f(x,s)\,\mathrm{d}s = F(x,t) - F(x,\theta t)$$

for all $\theta \in [0, \theta_0]$, $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Clearly, (WS) is slightly weaker than (SQ). Besides, (Ta) improves (AR), (WN), and the following weak version of (AR) (see [22]).

(WAR) there exists a $\mu > 2$ such that $0 \le \mu F(x,t) \le t f(x,t)$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Motivated by the aforementioned works, in the periodic case, we first weaken (DL) to the following condition, *i.e.*, $\mathcal{F}(x,t) > 0$, $t \ne 0$, to $\mathcal{F}(x,t) \ge 0$.

(F4) $\mathcal{F}(x,t) \ge 0$, and there exist $c_0 > 0$, $\delta_0 \in (0, \overline{\Lambda})$, and $\kappa > \max\{1, N/2\}$ such that

$$\frac{f(x,t)}{t} \ge \overline{\Lambda} - \delta_0 \quad \text{implies} \quad \Big[\frac{f(x,t)}{t}\Big]^\kappa \le c_0 \mathcal{F}(x,t).$$

https://doi.org/10.4153/CMB-2016-090-2 Published online by Cambridge University Press

Clearly, (WAR) and (DL) yield (F4). What we do notice, though, is that we cannot verify that (WN) implies (F4). However it is very difficult to find a function f that satisfies both (F2) and (WN,) but not (F4). Before presenting our first result, we give two nonlinear examples to illustrate Assumption (F4).

Example 1.1 Let $F(x,t) = t^2 \ln[1 + t^2 \sin^2(2\pi x_1)]$. Then

$$f(x,t) = 2t \ln[1 + t^2 \sin^2(2\pi x_1)] + \frac{2t^3 \sin^2(2\pi x_1)}{1 + t^2 \sin^2(2\pi x_1)},$$
$$\mathcal{F}(x,t) = \frac{t^4 \sin^2(2\pi x_1)}{1 + t^2 \sin^2(2\pi x_1)} \geq 0.$$

It is easy to see that f satisfies (WS) and (F4) with $\kappa > \max\{1, N/2\}$, but none of (AR), (SQ), (WAR), or (DL).

Example 1.2 Let $N \leq 4$ and $F(x,t) = a\Big(|t|^{13/4} - \frac{5}{2}|t|^{11/4} + \frac{45}{16}|t|^{9/4}\Big)$, $a > 0$. Then

$$f(x,t) = a\Big(\frac{13}{4}|t|^{5/4} - \frac{55}{8}|t|^{3/4} + \frac{405}{64}|t|^{1/4}\Big)t,$$
$$\mathcal{F}(x,t) = \frac{5}{8}a|t|^{9/4}\Big(\sqrt{|t|} - \frac{3}{4}\Big)^2 \geq 0.$$

Similarly, f satisfies (SQ) and (F4) with $\kappa = 12/5$ and $a \in (0, 64\overline{\Lambda}/405)$, but none of (AR), (WN), (WAR), (DL), or (Ta).

We are now in a position to state the first result of this paper.

Theorem 1.3 *Assume that V and f satisfy* (V), (F1), (F2), (F3), (F4), *and* (WS). *Then problem* (1.1) *has a nontrivial solution.*

When $V(x)$ is positive and asymptotically periodic, there are considerably fewer results [1, 14, 33]. In this case, the spectrum $\sigma(-\triangle + V) \subset (0, \infty)$. Comparing with appropriate solutions of a periodic problem associated with (1.1), a nontrivial solution can be found by using a version of the mountain pass theorem.

When $V(x)$ is periodic and sign-changing, while $f(x,u)$ is asymptotically periodic in x, there seems to be only one result [12]. Let Φ_0 and Φ denote the energy functionals associated with (1.1) with periodic and asymptotically periodic nonlinearity f, respectively. By using a generalized linking theorem for the strongly indefinite functionals and comparing with (C)c-sequences of Φ_0 and Φ, Li and Szulkin [12] proved that (1.1) has a nontrivial solution if V and f satisfy Assumptions (V) and (F1), (F2), (AR) and the following asymptotically periodic condition.

(F5) $f(x,t) = f_0(x,t) + f_1(x,t)$, $\partial_t f_0, f_1 \in C(\mathbb{R}^N \times \mathbb{R})$, $f_0(x,t)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$ and f_0 and f_1 satisfy the following:

$$0 < t f_0(x,t) < t^2 \partial_t f_0(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\}),$$
$$0 < F_0(x,t) := \int_0^t f_0(x,s)\mathrm{d}s \leq \frac{1}{\mu} t f_0(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\}),$$

$$F_1(x,t) := \int_0^t f_1(x,s)\mathrm{d}s > 0, \quad \forall (x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$$
$$|f_1(x,t)| \le a(x)\left(|t| + |t|^{p-1}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $\mu > 2$ is the same as in (AR), $a \in C(\mathbb{R}^N)$ with $\lim_{|x|\to\infty} a(x) = 0$.

We point out that the assumption in (F5) that $f_0(x,t)$ is differentiable in t and $0 < tf_0(x,t) < t^2\partial_t f_0(x,t)$ (which implies that $t \mapsto f_0(x,t)/|t|$ is strictly increasing on $(-\infty,0) \cup (0,\infty)$) is crucial in Li and Szulkin [12].

If $V(x)$ is both asymptotically periodic and sign-changing, the operator $-\triangle + V$ loses the $\mathbb{Z}^N$-translation invariance. For this reason, many effective methods for periodic problems cannot be applied to asymptotically periodic ones. To the best of our knowledge, there are no existence results for (1.1) when $V(x)$ is asymptotically periodic and sign-changing. Motivated by [6,12,14,24–27,33], we shall find new tricks to overcome the difficulties caused by dropping the periodicity of $V(x)$.

Before presenting our second theorem, we make the following assumptions instead of (V) and (F5), respectively.

(V1) $V(x) = V_0(x) + V_1(x)$, $V_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\sup[\sigma(-\triangle + V_0) \cap (-\infty,0)] < 0 < \overline{\Lambda} := \inf[\sigma(-\triangle + V_0) \cap (0,\infty)],$$

$V_1 \in C(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} V_1(x) = 0$;

(V2) $V_0(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and

$$0 \le -V_1(x) \le \sup_{\mathbb{R}^N}[-V_1(x)] < \overline{\Lambda}, \quad \forall x \in \mathbb{R}^N;$$

(F5′) $f(x,t) = f_0(x,t) + f_1(x,t)$, $f_0 \in C(\mathbb{R}^N \times \mathbb{R})$, $f_0(x,t)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, $f_0(x,t) = o(|t|)$ as $t \to 0$, uniformly in $x \in \mathbb{R}^N$; $t \mapsto f_0(x,t)/|t|$ is non-decreasing on $(-\infty,0) \cup (0,\infty)$; $\lim_{|t|\to\infty} |F_0(x,t)|/|t|^2 = \infty$, a.e. $x \in \mathbb{R}^N$; $f_1 \in C(\mathbb{R}^N \times \mathbb{R})$ satisfies the following:

$$-V_1(x)t^2 + F_1(x,t) > 0, \quad \forall\ (x,t) \in B_{1+\sqrt{N}}(0) \times (\mathbb{R} \setminus \{0\}),$$
$$F_1(x,t) \ge 0, \quad |f_1(x,t)| \le a(x)(|t| + |t|^{p-1}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $a \in C(\mathbb{R}^N)$ with $\lim_{|x|\to\infty} a(x) = 0$.

Remark 1.4 Comparing (F5) with (F5′), the condition that $f_0(x,t)$ is differentiable in t is dropped; the condition $0 < tf_0(x,t) < t^2\partial_t f_0(x,t)$ is weakened to the requirement that $t \mapsto f_0(x,t)/|t|$ is non-decreasing on $(-\infty,0) \cup (0,\infty)$; (AR) is also weakened to (WS) for f_0.

We are now in a position to state the second result in this paper.

Theorem 1.5 *Assume that* (V1), (V2), (F1), (F2), (F4) , *and* (F5′) *are satisfied. Then* (1.1) *has a nontrivial solution.*

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. The proofs of Theorems 1.3 and 1.5 are given in Section 3 and Section 4, respectively.

2 Preliminaries

Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. For a functional $\varphi \in C^1(X, \mathbb{R})$, φ is said to be weakly sequentially lower semi-continuous if for any $u_n \rightharpoonup u$ in X one has $\varphi(u) \le \liminf_{n\to\infty} \varphi(u_n)$, and φ' is said to be weakly sequentially continuous if $\lim_{n\to\infty} \langle \varphi'(u_n), v\rangle = \langle \varphi'(u), v\rangle$ for each $v \in X$.

Lemma 2.1 ([11, 12]) *Let $(X, \|\cdot\|)$ be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\varphi \in C^1(X, \mathbb{R})$ be of the form*

$$\varphi(u) = \frac{1}{2}(\|u^+\| - \|u^-\|) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(KS1) *$\psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*

(KS2) *ψ' is weakly sequentially continuous;*

(KS3) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that $\overline{\kappa} := \inf \varphi(S_\rho^+) > \sup \varphi(\partial Q)$, where*

$$S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}, \quad Q = \{v + se : v \in X^-, s \ge 0, \|v + se\| \le r\}.$$

Then there exist a constant $c \in [\overline{\kappa}, \sup \varphi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\varphi(u_n) \to c, \quad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0.$$

Let $\mathcal{A}_0 = -\triangle + V_0$. Then $\mathcal{A}_0$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}_0) = H^2(\mathbb{R}^N)$ (see [10, Theorem 4.26]). Let $\{\mathcal{E}(\lambda) : -\infty \le \lambda \le +\infty\}$ and $|\mathcal{A}_0|$ be the spectral family and the absolute value of $\mathcal{A}_0$, respectively, and let $|\mathcal{A}_0|^{1/2}$ be the square root of $|\mathcal{A}_0|$. Set $\mathcal{U} = \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}_0$, $|\mathcal{A}_0|$, and $|\mathcal{A}_0|^{1/2}$, and $\mathcal{A}_0 = \mathcal{U}|\mathcal{A}_0|$ is the polar decomposition of $\mathcal{A}_0$ (see [9, Theorem IV 3.3]). Let

$$E = \mathfrak{D}(|\mathcal{A}_0|^{1/2}), \quad E^- = \mathcal{E}(0)E, \quad E^+ = [\mathrm{id} - \mathcal{E}(0)]E.$$

For any $u \in E$, it is easy to see that $u = u^- + u^+$, where

$$u^- := \mathcal{E}(0)u \in E^-, \quad u^+ := [\mathrm{id} - \mathcal{E}(0)]u \in E^+$$

and

$$\mathcal{A}_0 u^- = -|\mathcal{A}_0|u^-, \quad \mathcal{A}_0 u^+ = |\mathcal{A}_0|u^+, \ \forall u \in E \cap \mathfrak{D}(\mathcal{A}_0). \tag{2.1}$$

Define an inner product $(u, v) = (|\mathcal{A}_0|^{1/2}u, |\mathcal{A}_0|^{1/2}v)_{L^2}$, for all $u, v \in E$ and the corresponding norm

$$\|u\| = \left\| |\mathcal{A}_0|^{1/2}u \right\|_2, \quad u \in E, \tag{2.2}$$

where $(\cdot, \cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^N)$ and $\|\cdot\|_s$ denotes the norm of $L^s(\mathbb{R}^N)$. By (V1), $E = H^1(\mathbb{R}^N)$ with equivalent norms. Therefore, E embeds continuously in $L^s(\mathbb{R}^N)$ for all $2 \le s \le 2^*$. In addition, one has the decomposition $E = E^- \oplus E^+$, orthogonal with respect to both $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$.

Under assumptions (V1), (F1), and (F2), the solutions of problem (1.1) are critical points of the functional

$$(2.3)\qquad \Phi(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x,\quad \forall u\in E,$$

Φ is of class $C^1(E,\mathbb{R})$, and

$$(2.4)\qquad \langle\Phi'(u),v\rangle=\int_{\mathbb{R}^N}(\nabla u\nabla v+V(x)uv)\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)v\mathrm{d}x,\quad \forall u,v\in E.$$

Let

$$(2.5)\qquad \Phi_0(u)=\frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2+V_0(x)u^2\right)\mathrm{d}x-\int_{\mathbb{R}^N}F_0(x,u)\mathrm{d}x,\quad \forall u\in E,$$

Then Φ_0 is also of class $C^1(E,\mathbb{R})$, and

$$\langle\Phi_0'(u),v\rangle=\int_{\mathbb{R}^N}(\nabla u\nabla v+V_0(x)uv)\mathrm{d}x-\int_{\mathbb{R}^N}f_0(x,u)v\mathrm{d}x,\quad \forall u,v\in E.$$

In view of (2.1) and (2.2), we have $\Phi_0(u)=\frac{1}{2}(\|u^+\|^2-\|u^-\|^2)-\int_{\mathbb{R}^N}F_0(x,u)\mathrm{d}x$ and

$$\langle\Phi_0'(u),u\rangle=\|u^+\|^2-\|u^-\|^2-\int_{\mathbb{R}^N}f_0(x,u)u\mathrm{d}x,\quad \forall u=u^-+u^+\in E.$$

We set $\Psi(u)=\int_{\mathbb{R}^N}[-V_1(x)u^2+F(x,u)]\mathrm{d}x$, for all $u\in E$.

Employing a standard argument, one can easily verify the following fact.

Lemma 2.2 *Suppose that* (V1), (V2), (F1)*, and* (F2) *are satisfied. Then Ψ is nonnegative, weakly sequentially lower semi-continuous, and Ψ' is weakly sequentially continuous.*

3 The Periodic Case

In this section, we assume that V and f are 1-periodic in each of $x_1,x_2,\dots,x_N$, *i.e.*, (V) and (F3) are satisfied. In this case, $V_0=V$, $V_1=0$, $f_0=f$, and $f_1=0$. Thus, $\Phi_0(u)=\Phi(u)$.

Lemma 3.1 ([25, Lemma 2.4]) *Suppose that* (V), (F1), (F2)*, and* (WN) *are satisfied. Then $\Phi(u)\geq\Phi(tu+w)+\frac{1}{2}\|w\|^2+\frac{1-t^2}{2}\langle\Phi'(u),u\rangle-t\langle\Phi'(u),w\rangle$, for all $u\in E$, $t\geq 0$, $w\in E^-$.*

Define $\mathcal{N}^-=\{u\in E\setminus E^-:\langle\Phi'(u),u\rangle=\langle\Phi'(u),v\rangle=0,\forall v\in E^-\}$. First introduced by Pankov [18], the set $\mathcal{N}^-$ is a subset of the Nehari manifold

$$\mathcal{N}=\{u\in E\setminus\{0\}:\langle\Phi'(u),u\rangle=0\}.$$

Corollary 3.2 *Suppose that* (V), (F1), (F2)*, and* (WN) *are satisfied. Then for $u\in\mathcal{N}^-$, $\Phi(u)\geq\Phi(tu+w)+\frac{1}{2}\|w\|^2$, for all $t\geq 0$, $w\in E^-$.*

Corollary 3.3 *Suppose that* (V), (F1), (F2)*, and* (WN) *are satisfied. Then*

$$\Phi(u) \geq \frac{t^2}{2}\|u\|^2 - \int_{\mathbb{R}^N} F(x, tu^+)\mathrm{d}x + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle + t^2\langle \Phi'(u), u^-\rangle,$$

for all $u \in E$, $t \geq 0$.

Analogous to the proof of [22, Lemma 3.3], it is easy to show the following lemma.

Lemma 3.4 *Suppose that* (V), (F1), (F2)*, and* (WS) *are satisfied. Then there exist a constant* $c > 0$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.1}$$

Lemma 3.5 *Suppose that* (V), (F1), (F2), (F3), (F4)*, and* (WS) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c \geq 0, \quad \langle \Phi'(u_n), u_n^\pm\rangle \to 0 \tag{3.2}$$

is bounded in E.

Proof In view of (3.2), there exists a constant $C_1 > 0$ such that

$$C_1 \geq \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \int_{\mathbb{R}^N} \mathcal{F}(x, u_n)\mathrm{d}x. \tag{3.3}$$

To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $1 = \|v_n\|^2$. If $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n^+|^2\mathrm{d}x = 0$, then by Lions' concentration compactness principle [15] or [29, Lemma 1.21], $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Set $\kappa' = \kappa/(\kappa - 1)$ and

$$\Omega_n := \Big\{ x \in \mathbb{R}^N : \frac{f(x, u_n)}{u_n} \leq \overline{\Lambda} - \delta_0 \Big\}. \tag{3.4}$$

Then using $\overline{\Lambda}\|v_n^+\|_2^2 \leq \|v_n^+\|^2$, one has

$$\int_{\Omega_n} \frac{f(x, u_n)}{u_n}(v_n^+)^2\mathrm{d}x \leq (\overline{\Lambda} - \delta_0)\|v_n^+\|_2^2 \leq 1 - \frac{\delta_0}{\overline{\Lambda}}. \tag{3.5}$$

On the other hand, by virtue of (F4), (3.3), and the Hölder inequality, one can get that

$$\begin{aligned} \int_{\mathbb{R}^N\setminus\Omega_n} \frac{f(x, u_n)}{u_n}(v_n^+)^2\mathrm{d}x &\leq \Big[\int_{\mathbb{R}^N\setminus\Omega_n} \Big|\frac{f(x, u_n)}{u_n}\Big|^\kappa \mathrm{d}x\Big]^{1/\kappa} \|v_n^+\|_{2\kappa'}^2 \\ &\leq C_2\Big(\int_{\mathbb{R}^N\setminus\Omega_n} \mathcal{F}(x, u_n)\mathrm{d}x\Big)^{1/\kappa} \|v_n^+\|_{2\kappa'}^2 \\ &\leq C_3\|v_n^+\|_{2\kappa'}^2 = o(1). \end{aligned} \tag{3.6}$$

So $\mathcal{F}(x,u) \geq 0$ implies that $uf(x,u) \geq 0$. Hence, combining (3.5) with (3.6) and making use of (2.4) and (3.2), we have

$$\begin{aligned}1+o(1) &= \frac{\|u_n\|^2 - \langle \Phi(u_n), u_n^+ - u_n^- \rangle}{\|u_n\|^2}\\ &= \int_{u_n \neq 0} \frac{f(x,u_n)}{u_n}[(v_n^+)^2 - (v_n^-)^2]\mathrm{d}x\\ &\leq \int_{\Omega_n} \frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x + \int_{\mathbb{R}^N \setminus \Omega_n} \frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x\\ &\leq 1 - \frac{\delta_0}{\overline{\Lambda}} + o(1).\end{aligned}$$

This contradiction shows that $\delta > 0$.

Going, if necessary, to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |v_n^+|^2\mathrm{d}x > \frac{\delta}{2}$. Let $w_n(x) = v_n(x + k_n)$. Since $V(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$, then $\|w_n\| = \|v_n\| = 1$, and

$$\int_{B_{1+\sqrt{N}}(0)} |w_n^+|^2\mathrm{d}x > \delta/2. \tag{3.7}$$

Passing to a subsequence, we have $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \leq s < 2^*$, $w_n \to w$ a.e. on $\mathbb{R}^N$. Obviously, (3.7) implies that $w \neq 0$.

Now we define $u_n^{k_n}(x) = u_n(x + k_n)$. Then $u_n^{k_n}/\|u_n\| = w_n \to w$ a.e. on $\mathbb{R}^N$, $w \neq 0$. For $x \in \{y \in \mathbb{R}^N : w(y) \neq 0\}$, we have $\lim_{n\to\infty} |u_n^{k_n}(x)| = \infty$. Hence, it follows from (2.3), (3.2), (F2), (F3), (WS), and Fatou's lemma that

$$\begin{aligned}0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\Phi(u_n)}{\|u_n\|^2} = \lim_{n\to\infty}\Big[\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) - \int_{\mathbb{R}^N} \frac{F(x,u_n^{k_n})}{(u_n^{k_n})^2} w_n^2\mathrm{d}x\Big]\\ &\leq \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n^{k_n})}{(u_n^{k_n})^2} w_n^2\mathrm{d}x \leq \frac{1}{2} - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n^{k_n})}{(u_n^{k_n})^2} w_n^2\mathrm{d}x = -\infty.\end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded. ■

Lemma 3.6 ([26, Theorem 1.2]) *Assume that* (V), (F1), (F2), (F3), (WN), *and* (WS) *are satisfied. Then problem* (1.1) *has a solution* $u_0 \in E$ *such that* $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi > 0$.

Proof of Theorem 1.3 Combining Lemma 3.4 with Lemma 3.5, we can get that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (3.1). Now the usual concentration-compactness argument suggests that $\Phi'(\overline{u}) = 0$ for some $\overline{u} \in E \setminus \{0\}$. ■

4 The Asymptotically Periodic Case

In this section, we always assume that V satisfies (V1) and (V2).

Lemma 4.1 *Suppose that* (V1), (V2), (F1), *and* (F2) *are satisfied. Then there exists* $\rho > 0$ *such that*

$$\hat{\kappa} := \inf\{\Phi(u) : u \in E^+, \|u\| = \rho\} > 0. \tag{4.1}$$

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Proof Set $\Theta_0 = \sup_{\mathbb{R}^N}[-V_1(x)]$. Let $\varepsilon_0 = (\overline{\Lambda} - \Theta_0)/3$. Then (F1) and (F2) imply that there exists a constant $C_{\varepsilon_0} > 0$ such that

$$F(x,t) \le \varepsilon_0 |t|^2 + C_{\varepsilon_0}|t|^p, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{4.2}$$

From (2.3), (4.2), and the Sobolev imbedding inequalities $\|u\|_p \le \gamma_p \|u\|$ and $\overline{\Lambda}\|u\|_2^2 \le \|u\|^2$ for $u \in E^+$, we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1(x)u^2 \mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \\ &\ge \frac{1}{2}\left[\|u\|^2 - (\Theta_0 + 2\varepsilon_0)\|u\|_2^2\right] - C_{\varepsilon_0}\|u\|_p^p \\ &\ge \frac{1}{2}\Big(1 - \frac{\Theta_0 + 2\varepsilon_0}{\overline{\Lambda}}\Big)\|u\|^2 - \gamma_p^p C_{\varepsilon_0}\|u\|^p, \quad \forall u \in E^+.\end{aligned}$$

This shows that there exists $\rho > 0$ such that (4.1) holds. ∎

Lemma 4.2 *Suppose that* (V1), (V2), (F1), (F2), *and* (WS) *are satisfied. Then for any $e \in E^+$, $\sup \Phi(E^- \oplus \mathbb{R}^+ e) < \infty$, and there is $R_e > 0$ such that $\Phi(u) \le 0$, for all $u \in E^- \oplus \mathbb{R}^+ e$, $\|u\| \ge R_e$.*

Proof Arguing indirectly, provided that for some sequence $\{w_n + s_n e\} \subset E^- \oplus \mathbb{R}e$ with $\|w_n + s_n e\| \to \infty$ such that $\Phi(w_n + s_n e) \ge 0$ for all $n \in \mathbb{N}$, set

$$v_n = (w_n + s_n e)/\|w_n + s_n e\| = v_n^- + t_n e.$$

Then $\|v_n^- + t_n e\| = 1$. Passing to a subsequence, we may assume that $t_n \to \bar{t}$, $v_n^- \rightharpoonup v^-$, and $v_n^- \to v^-$ a.e. on $\mathbb{R}^N$. Hence, it follows from (V2) and (2.3) that

$$\begin{aligned}0 &\le \frac{\Phi(w_n + s_n e)}{\|w_n + s_n u\|^2} \\ &= \frac{t_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1(x)(v_n^- + t_n e)^2 \mathrm{d}x - \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2}\mathrm{d}x \\ &\le \frac{t_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2}\mathrm{d}x.\end{aligned} \tag{4.3}$$

If $\bar{t} = 0$, then it follows from (4.3) that

$$0 \le \frac{1}{2}\|v_n^-\|^2 + \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2}\mathrm{d}x \le \frac{t_n^2}{2}\|e\|^2 \to 0,$$

which yields $\|v_n^-\| \to 0$, and so $1 = \|v_n^- + t_n e\|^2 \to 0$, a contradiction.

If $\bar{t} \neq 0$, then it follows from (4.3) and (WS) that

$$\begin{aligned}0 &\le \limsup_{n\to\infty}\Big[\frac{t_n^2}{2}\|e\|^2 - \frac{1}{2}\|v_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{\|w_n + s_n e\|^2}\mathrm{d}x\Big] \\ &\le \frac{\bar{t}^2}{2}\|e\|^2 - \liminf_{n\to\infty}\int_{\mathbb{R}^N} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2}(v_n^- + t_n e)^2 \mathrm{d}x \\ &\le \frac{\bar{t}^2}{2}\|e\|^2 - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x, w_n + s_n e)}{(w_n + s_n e)^2}(v_n^- + t_n e)^2 \mathrm{d}x = -\infty,\end{aligned}$$

a contradiction. ∎

Corollary 4.3 *Suppose that* (V1), (V2), (F1), (F2), *and* (WS) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there is an* $r > \rho$ *such that* $\sup \Phi(\partial Q) \le 0$, *where*

$$Q = \{w + se : w \in E^-, s \ge 0, \|w + se\| \le r\}.$$

Let $m_0 := \inf_{\mathcal{N}^0} \Phi_0$, where

$$\mathcal{N}^0 = \left\{ u \in E \setminus E^- : \langle \Phi_0'(u), u \rangle = \langle \Phi_0'(u), v \rangle = 0, \forall v \in E^- \right\}.$$

Lemma 4.4 *Suppose that* (V1), (V2), (F1), (F2), *and* (F5′) *are satisfied. Then there exist a constant* $c_* \in [\hat{\kappa}, m_0)$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0, \tag{4.4}$$

Proof Employing Lemma 3.6, there exists a $\overline{u} \in E$ such that $\overline{u} \not\equiv 0$ on $B_{1+\sqrt{N}}(0)$, $\Phi_0'(\overline{u}) = 0$, and $\Phi_0(\overline{u}) = m_0$. Set $\hat{E}(\overline{u}) = E^- \oplus \mathbb{R}^+ \overline{u}$ and $\zeta_0 := \sup\{\Phi(u) : u \in \hat{E}(\overline{u})\}$. Lemma 4.1 implies that $\zeta_0 \ge \hat{\kappa} > 0$. By (V2), (F5′), (2.3), (2.5), and Corollary 3.2, we have $\Phi(u) \le \Phi_0(u) \le m_0$, for all $u \in \hat{E}(\overline{u})$. Hence, $\zeta_0 \le m_0$. If $\zeta_0 = m_0$, then there is a sequence $\{u_n\}$ with $u_n = w_n + s_n \overline{u} \in \hat{E}(\overline{u})$ such that

$$m_0 - \frac{1}{n} < \Phi(u_n) = \Phi(w_n + s_n \overline{u}) \le m_0. \tag{4.5}$$

It follows from Lemma 4.2 and (4.5) that $\{s_n\} \subset \mathbb{R}$ and $\{w_n\} \subset E^-$ are bounded. Passing to a subsequence, we have $s_n \to \overline{s}$ and $w_n \rightharpoonup \overline{w}$ in E. It is easy to see that $\overline{s} > 0$. It follows from (2.3), (2.5), and Corollary 3.2 that

$$\begin{aligned} m_0 - \frac{1}{n} < \Phi(u_n) &= \Phi_0(u_n) + \frac{1}{2} \int_{\mathbb{R}^N} V_1(x) u_n^2 \mathrm{d}x - \int_{\mathbb{R}^N} F_1(x, u_n) \mathrm{d}x \\ &\le m_0 - \frac{1}{2}\|w_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} [-V_1(x) u_n^2 + 2F_1(x, u_n)] \mathrm{d}x, \end{aligned}$$

which yields that $\frac{1}{2}\|w_n\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} [-V_1(x)(w_n + s_n\overline{u})^2 + 2F_1(x, w_n + s_n\overline{u})] \mathrm{d}x \le \frac{1}{n}$. According to Fatou's Lemma and the weakly lower semi-continuity of the norm, one gets that $\frac{1}{2}\|\overline{w}\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} [-V_1(x)(\overline{w} + \overline{s}\overline{u})^2 + 2F_1(x, \overline{w} + \overline{s}\overline{u})] \mathrm{d}x = 0$. This, together with (F5′), implies that $\overline{w} = 0$ and $\overline{u} = 0$ on $B_{1+\sqrt{N}}(0)$, a contradiction. Therefore, $\zeta_0 \in [\hat{\kappa}, m_0)$. In view of Lemmas 2.1, 2.2, 4.1, and Corollary 4.3, there exist a constant $c_* \in [\hat{\kappa}, m_0)$ and a sequence $\{u_n\} \subset E$ satisfying (4.4). ∎

Lemma 4.5 *Suppose that* (V1), (V2), (F1), (F2), (F4), *and* (F5′) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying* (3.2) *is bounded in* E.

Proof Given the condition (3.2), there exists a constant $C_1 > 0$ such that (3.3) holds. To prove the boundedness of $\{u_n\}$, and arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $1 = \|v_n\|^2$. Passing to a subsequence, we have $v_n \rightharpoonup \overline{v}$ in E. There are two possible cases: (i) $\overline{v} = 0$ and (ii) $\overline{v} \not\equiv 0$.

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Case (i): $\overline{v} = 0$, *i.e.*, $v_n \rightharpoonup 0$ in E. Then $v_n^+ \to 0$ and $v_n^- \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $v_n^+ \to 0$ and $v_n^- \to 0$ a.e. on $\mathbb{R}^N$. From (V1), it is easy to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)(v_n^+)^2\mathrm{d}x = \lim_{n\to\infty}\int_{\mathbb{R}^N} V_1(x)(v_n^-)^2\mathrm{d}x = 0. \tag{4.6}$$

If $\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n^+|^2\mathrm{d}x = 0$, then by Lions' concentration compactness principle [15] or [29, Lemma 1.21], $v_n^+ \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Set Ω_n as (3.4). Then (3.5) and (3.6) hold also. Combining (3.5) with (3.6) and using (2.4), (3.2), and (4.6), we have

$$\begin{aligned}
1 + o(1) &= \frac{\|u_n\|^2 - \langle \Phi(u_n), u_n^+ - u_n^-\rangle}{\|u_n\|^2}\\
&= -\int_{\mathbb{R}^N} V_1(x)[(v_n^+)^2 - (v_n^-)^2]\mathrm{d}x + \int_{u_n\neq 0}\frac{f(x,u_n)}{u_n}[(v_n^+)^2 - (v_n^-)^2]\mathrm{d}x\\
&\le -\int_{\mathbb{R}^N} V_1(x)(v_n^+)^2\mathrm{d}x + \int_{\Omega_n}\frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x + \int_{\mathbb{R}^N\setminus\Omega_n}\frac{f(x,u_n)}{u_n}(v_n^+)^2\mathrm{d}x\\
&\le 1 - \frac{\delta_0}{\overline{\Lambda}} + o(1).
\end{aligned}$$

This contradiction shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)}|v_n^+|^2\mathrm{d}x > \frac{\delta}{2}$. Let $w_n(x) = v_n(x+k_n)$. Since $V_0(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, then

$$\int_{B_{1+\sqrt{N}}(0)}|w_n^+|^2\mathrm{d}x > \frac{\delta}{2}. \tag{4.7}$$

Now we define $\widetilde{u}_n(x) = u_n(x+k_n)$. Then $\widetilde{u}_n/\|u_n\| = w_n$ and $\|w_n\| = \|v_n\| = 1$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in E, $w_n \to w$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $w_n \to w$ a.e. on $\mathbb{R}^N$. Obviously, (4.7) implies that $w \neq 0$. Hence, it follows from (3.2), (F5′) and Fatou's Lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^2} = \lim_{n\to\infty}\frac{\Phi(u_n)}{\|u_n\|^2}\\
&= \lim_{n\to\infty}\Big[\frac{1}{2}(\|v_n^+\|^2 - \|v_n^-\|^2) + \frac{1}{2}\int_{\mathbb{R}^N} V_1(x)v_n^2\mathrm{d}x - \int_{\mathbb{R}^N}\frac{F(x+k_n,\widetilde{u}_n)}{\widetilde{u}_n^2}w_n^2dx\Big]\\
&\le \frac{1}{2} - \liminf_{n\to\infty}\int_{\mathbb{R}^N}\frac{F_0(x,\widetilde{u}_n)}{\widetilde{u}_n^2}w_n^2\,dx \le \frac{1}{2} - \int_{\mathbb{R}^N}\liminf_{n\to\infty}\frac{F_0(x,\widetilde{u}_n)}{\widetilde{u}_n^2}w_n^2\,dx = -\infty,
\end{aligned}$$

which is a contradiction.

Case (ii) $\overline{v} \neq 0$. In this case, we can also deduce a contradiction by a standard argument.

Cases (i) and (ii) both show that $\{u_n\}$ is bounded in E. ∎

Proof of Theorem 1.5 It is easy to see that (F5′) implies (WS). Applying Lemmas 4.4 and 4.5, we obtain that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (4.4). Passing to a subsequence, we have $u_n \rightharpoonup \overline{u}$ in E. Next we prove $\overline{u} \neq 0$.

Arguing by contradiction, suppose that $\bar{u} = 0$, *i.e.*, $u_n \rightharpoonup 0$ in E, and so $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $u_n \to 0$ a.e. on $\mathbb{R}^N$. By (V1) and (F5′), it is easy to show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1(x) u_n^2\, dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} V_1(x) u_n v dx = 0, \quad \forall\, v \in E \tag{4.8}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F_1(x, u_n) dx = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} f_1(x, u_n) v dx = 0, \quad \forall v \in E. \tag{4.9}$$

Note that

$$\Phi_0(u) = \Phi(u) - \frac{1}{2} \int_{\mathbb{R}^N} V_1(x) u^2 dx + \int_{\mathbb{R}^N} F_1(x, u) dx, \quad \forall u \in E \tag{4.10}$$

and

$$\langle \Phi_0'(u), v\rangle = \langle \Phi'(u), v\rangle - \int_{\mathbb{R}^N} V_1(x) u v dx + \int_{\mathbb{R}^N} f_1(x, u) v dx, \quad \forall u, v \in E. \tag{4.11}$$

From (4.4), (4.8)–(4.11), one can get that $\Phi_0(u_n) \to c_*$, $\|\Phi_0'(u_n)\|(1 + \|u_n\|) \to 0$.

A standard argument shows that $\{u_n\}$ is a non-vanishing sequence. Going, if necessary, to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that

$$\int_{B_{1+\sqrt{N}}(k_n)} |u_n|^2 \mathrm{d}x > \frac{\delta}{2}$$

for some $\delta > 0$. Let $v_n(x) = u_n(x + k_n)$. Then

$$\int_{B_{1+\sqrt{N}}(0)} |v_n|^2 \mathrm{d}x > \frac{\delta}{2}. \tag{4.12}$$

Since $V_0(x)$ and $f_0(x, u)$ are periodic on x, we have $\|v_n\| = \|u_n\|$ and

$$\Phi_0(v_n) \to c_*, \quad \|\Phi_0'(v_n)\|(1 + \|v_n\|) \to 0. \tag{4.13}$$

Passing to a subsequence, we have $v_n \rightharpoonup \bar{v}$ in E, $v_n \to \bar{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$ and $v_n \to \bar{v}$ a.e. on $\mathbb{R}^N$. Obviously, (4.12) and (4.13) imply that $\bar{v} \neq 0$ and $\Phi_0'(\bar{v}) = 0$. This shows that $\bar{v} \in \mathcal{N}^0$ and so $\Phi_0(\bar{v}) \ge m_0$. On the other hand, by using (4.13), (WN), and Fatou's Lemma, we have

$$\begin{aligned}
m_0 > c_* &= \lim_{n\to\infty} \Big[\Phi_0(v_n) - \frac{1}{2}\langle \Phi_0'(v_n), v_n\rangle \Big] \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^N} \Big[\frac{1}{2} f_0(x, v_n) v_n - F_0(x, v_n) \Big] \mathrm{d}x \\
&\ge \int_{\mathbb{R}^N} \lim_{n\to\infty} \Big[\frac{1}{2} f_0(x, v_n) v_n - F_0(x, v_n) \Big] \mathrm{d}x = \int_{\mathbb{R}^N} \Big[\frac{1}{2} f_0(x, \bar{v}) \bar{v} - F_0(x, \bar{v}) \Big] \mathrm{d}x \\
&= \Phi_0(\bar{v}) - \frac{1}{2} \langle \Phi_0'(\bar{v}), \bar{v}\rangle = \Phi_0(\bar{v}) \ge m_0.
\end{aligned}$$

This contradiction implies that $\bar{u} \neq 0$. It is obvious that $\bar{u} \in E$ is a nontrivial solution for problem (1.1). ∎

References

[1] S. Alama and Y. Y. Li, *On "multibump" bound states for certain semilinear elliptic equations*. Indiana J. Math. **41**(1992), no. 4, 983–1026. http://dx.doi.org/10.1512/iumj.1992.41.41052

[2] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Anal. **14**(1973), 349–381. http://dx.doi.org/10.1016/0022-1236(73)90051-7

[3] T. Bartsch and Y. H. Ding, *Solutions of nonlinear Dirac equations*. J. Differ. Equations **226**(2006), 210–249. http://dx.doi.org/10.1016/j.jde.2005.08.014

[4] T. Bartsch and Z.-Q. Wang, *Existence and multiplicity results for some superlinear elliptic problems on* $\mathbb{R}^N$. Comm. Partial Differential Equations **20**(1995), 1725–1741. http://dx.doi.org/10.1080/03605309508821149

[5] V. Coti Zelati and P. H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on* $\mathbb{R}^N$. Comm. Pure Appl. Math. **45**(1992), no. 10, 1217–1269. http://dx.doi.org/10.1002/cpa.3160451002

[6] Y. Ding and C. Lee, *Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms*. J. Differential Equations **222**(2006) 137–163. http://dx.doi.org/10.1016/j.jde.2005.03.011

[7] Y. Ding and S. X. Luan, *Multiple solutions for a class of nonlinear Schrödinger equations*. J. Differential Equations **207**(2004), 423–457. http://dx.doi.org/10.1016/j.jde.2004.07.030

[8] Y. Ding and A. Szulkin, *Bound states for semilinear Schrödinger equations with sign-changing potential*. Calc. Var. Partial Differential Equations **29**(2007), no. 3, 397–419. http://dx.doi.org/10.1007/s00526-006-0071-8

[9] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*. Clarendon Press, Oxford, 1987.

[10] Y. Egorov and V. Kondratiev, *On spectral theory of elliptic operators*. Operator Theory: Advances and Applications, 89. Birkhäuser Verlag, Basel, 1996. http://dx.doi.org/10.1007/978-3-0348-9029-8

[11] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to a semilinear Schrödinger equation*. Adv. Differ. Equations **3**(1998), 441–472.

[12] G. B. Li and A. Szulkin, *An asymptotically periodic Schrödinger equation with indefinite linear part*. Commun. Contemp. Math. **4**(2002). 763–776. http://dx.doi.org/10.1142/S0219199702000853

[13] Y. Q. Li, Z.-Q. Wang, and J. Zeng, *Ground states of nonlinear Schrödinger equations with potentials*. Ann. Inst. H. Poincaré Anal. Non Linéaire **23**(2006), 829–837. http://dx.doi.org/10.1016/j.anihpc.2006.01.003

[14] X. Lin and X. H. Tang, *Nehari-type ground state solutions for superlinear asymptotically periodic Schrödinger equation*. Abstr. Appl. Anal. (2014), ID 607078, 7.

[15] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II.* Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(1984), no. 4, 223–283.

[16] S. Liu, *On superlinear Schrödinger equations with periodic potential*. Calc. Var. Partial Differential Equations **45**(2012), 1–9. http://dx.doi.org/10.1007/s00526-011-0447-2

[17] Z. L. Liu, and Z.-Q. Wang, *On the Ambrosetti-Rabinowitz superlinear condition*. Adv. Nonlinear Stud. **4**(2004), no. 4, 561–572. http://dx.doi.org/10.1515/ans-2004-0411

[18] A. Pankov, *Periodic nonlinear Schrödinger equation with application to photonic crystals*. Milan J. Math. **73**(2005) 259–287. http://dx.doi.org/10.1007/s00032-005-0047-8

[19] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*. Z. Angew. Math. Phys. **43**(1992), 270–291. http://dx.doi.org/10.1007/BF00946631

[20] A. Szulkin and T. Weth, *Ground state solutions for some indefinite variational problems*. J. Funct. Anal. **257**(2009), 3802–3822. http://dx.doi.org/10.1016/j.jfa.2009.09.013

[21] X. H. Tang, *Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity*. J. Math. Anal. Appl. **401**(2013), 407–415. http://dx.doi.org/10.1016/j.jmaa.2012.12.035

[22] ———, *New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation*. Adv. Nonlinear Stud. **14**(2014), 361–373. http://dx.doi.org/10.1515/ans-2014-0208

[23] ———, *New conditions on nonlinearity for a periodic Schrödinger equation having zero as spectrum*. J. Math. Anal. Appl. **413**(2014), 392–410. http://dx.doi.org/10.1016/j.jmaa.2013.11.062

[24] ———, *Non-Nehari manifold method for superlinear Schrödinger equation*. Taiwanese J. Math. **18**(2014), 1957–1979.

[25] ———, *Non-Nehari manifold method for asymptotically linear Schrödinger equation*. J. Aust. Math. Soc. **98**(2015), 104–116. http://dx.doi.org/10.1017/S144678871400041X

[26] ———, *Non-Nehari manifold method for asymptotically periodic Schrödinger equations*. Sci. China Math. **58**(2015), 715–728. http://dx.doi.org/10.1007/s11425-014-4957-1

[27] X. H. Tang and B. T. Cheng, *Ground state sign-changing solutions for Kirchhoff type problems in bounded domains*. J. Differential Equations **261**(2016), 2384–2402 http://dx.doi.org/10.1016/j.jde.2016.04.032

[28] C. Troestler and M. Willem, *Nontrivial solution of a semilinear Schrödinger equation*. Commun. Partial Differ. Equ. **21**(1996), 1431–1449. http://dx.doi.org/10.1080/03605309608821233

[29] M. Willem, *Minimax theorems*. Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Boston, MA, 1996. http://dx.doi.org/10.1007/978-1-4612-4146-1

[30] M. Willem and W. M. Zou, *On a Schrödinger equation with periodic potential and spectrum point zero*. Indiana Univ. Math. J. **52**(2003), 109–132. http://dx.doi.org/10.1512/iumj.2003.52.2273

[31] M. Yang, *Ground state solutions for a periodic Schrödinger equation with superlinear nonlinearities*. Nonlinear Anal. **72**(2010), 2620–2627. http://dx.doi.org/10.1016/j.na.2009.11.009

[32] M. Yang, W. Chen, and Y. H. Ding, *Solution of a class of Hamiltonian elliptic systems in* $\mathbb{R}^N$. J. Math. Anal. Appl. **362**(2010), 338–349. http://dx.doi.org/10.1016/j.jmaa.2009.07.052

[33] H. Zhang, J. X. Xu, and F. B. Zhang, *Ground state solutions for asymptotically periodic Schrödinger equation with critical growth*. Electron. J. Differential Equations **2013**(2013), No.227, 16pp.

[34] R. M. Zhang, J. Chen, and F. K. Zhao, *Multiple solutions for superlinear elliptic systems of Hamiltonian type*. Discrete. Contin. Dyn. Syst. Ser. A **30**(2011), 1249–1262.

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