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RINGS ISOMORPHIC TO THEIR UNBOUNDED LEFT IDEALS

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A complete description is given of rings isomorphic to their unbounded left (right) ideals. The same problem for 2-sided ideals remains open.

Let R be a ring, and let R^+ denote the additive group of R. If R^+ is a bounded group then R is said to be bounded or have finite characteristic. In [1] Hill classified the rings that are isomorphic to each of their unbounded subrings. He also proved the following:

PROPOSITION 1. *Let R be a ring isomorphic to each of its unbounded ideals. Then R satisfies one of the following conditions:*

1) *R has finite characteristic;*
2) *R is the zeroring on $\mathbb{Z}(p^\infty)$, p a prime;*
3) *$R^2 = R$, R is a prime ring, and R^+ is a divisible torsion-free group;*
4) *R is the zeroring on $\mathbb{Z}$, with $\mathbb{Z}$ = the additive group of the ring of integers.*

PROOF: [1, Proposition 3.1, Lemma 3.3, Lemma 3.4 and Lemma 3.7.] ∎

The object of this note is to use Proposition 1 to prove:

THEOREM 2. *A ring R is isomorphic to each of its unbounded left (right) ideals if and only if R satisfies one of the following conditions:*

1) *R has finite characteristic;*
2) *R is the zeroring on $\mathbb{Z}(p^\infty)$, p a prime;*
3) *R is a division ring;*
4) *R is the zeroring on $\mathbb{Z}$.*

PROOF: Clearly if R satisfies one of conditions 1)–4) then R is isomorphic to each of its unbounded left (right) ideals. Conversely, suppose that R is isomorphic to each of its unbounded left ideals. By Proposition 1 it may be assumed that $R^2 = R$, R is prime, and that R^+ is a divisible torsion-free group.

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CLAIM 1. *The left annihilator, $\ell(R)$, and the right annihilator, $r(R)$, of R are trivial.*

PROOF: $\ell(R)$ is an ideal in R. If $\ell(R) \neq 0$, then $R \simeq \ell(R)$. Since $[\ell(R)]^2 = 0$ it follows that $R^2 = 0$, a contradiction. By the same argument, $r(R) = 0$. ∎

CLAIM 2. *Let $a, b \in R$. If $ab = 0$ then $ba = 0$.*

PROOF: It may be assumed that $a \neq 0$. By Claim 1 it follows that $R \simeq Ra$. Therefore the right annihilator of Ra in Ra is trivial. However, ba belongs to the right annihilator of Ra in Ra, and so $ba = 0$. ∎

CLAIM 3. *Let $a, b \in R$. If $ab = 0$ then $a = 0$ or $b = 0$.*

PROOF: $abR = 0$, so by Claim 2, $bRa = 0$. Since R is prime, either $a = 0$ or $b = 0$. ∎

CLAIM 4. *For all $a \in R$, $a \in Ra$.*

PROOF: It may be assumed that $a \neq 0$. Suppose that $a \notin Ra$. Since $(Ra)^+$ is divisible it follows that $na \notin Ra$ for every positive integer n. Let $A = (a) \oplus Ra$ with $(a) =$ the cyclic group generated by a. Then A is a left ideal in R, but A^+ is not divisible, a contradiction. ∎

Let $a \in R$, $a \neq 0$. By Claim 4 there exists $e \in R$ such that $ea = a$. Similarly, $e \in Re$, so there exists $e' \in R$ such that $e = e'e$. Now $e'a = e'ea = ea$, and so $(e' - e)a = 0$. By Claim 3 it follows that $e' = e$, and so $e^2 = e$. Therefore $e \in Re$ and is a right identity in Re. For $x \in R$ the fact that e is a right identity in Re yields that $(xe)e(xe) = (xe)^2$ and so $xe[e(xe) - (xe)] = 0$. Claim 3 yields that $e(xe) = xe$, that is, e is an identity in Re. Since $R \simeq Re$ it follows that R is a ring with identity 1. Since $Ra \simeq R$ there exists $c \in R$ such that ca is an identity in Ra. However $a \in Ra$ and so $a = aca = a \cdot 1$, that is, $a(ca - 1) = 0$. It follows from Claim 3 that $ca = 1$. Since every nonzero element in R has a left inverse, it is readily seen that R is a division ring.

The proof for right ideals follows similarly.

REFERENCES

[1] P. Hill, 'Some almost simple rings', *Canad. J. Math.* **25** (1973), 290–302.

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