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The C^*-algebras of Morse–Smale flows on two-manifolds

XIAOLU WANG†

Department of Mathematics, University of Maryland, College Park, MD 20742, *USA*

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Abstract. We give a classification of all the C^*-algebras of Morse–Smale flows on closed two-manifolds, and determine the relation between the invariants of dynamical systems and the topological invariants of the C^*-algebras.

0. *Introduction*

The C^*-algebras of smooth foliations, introduced by A. Connes [**C1**], [**C2**], have become one of the favourite objects studied in noncommutative differential geometry. Because of the inherited smooth structures, they are naturally noncommutative 'smooth manifolds'. Just as the classification of low-dimensional manifolds is a fundamental problem in topology, it is natural to wish to study and classify their 'noncommutative' counterparts. Such an investigation provides a new source of examples and as we shall see, adds new aspects to the well-known topological invariants of C^*-algebras by illuminating their close connection with some of the fundamental classical invariants which originate in the commutative world.

In [**W2**] we studied C^*-algebras of foliations of the plane and we showed that they naturally arise as noncommutative CW-complexes and can be characterized in terms of $\mathbb{R}$-trees. A foliation of the plane can be regarded as a singular foliation of S^2 induced by a flow with a singularity of arbitrary complexity. In this paper we study the C^*-algebras of flows on arbitrary closed two-manifolds. It is of course impossible to 'calculate' explicitly the C^*-algebras of all such flows. We shall concentrate on the most important class – the Morse–Smale flows. (On two-manifolds they are exactly the structurally stable flows, see the work of M. Peixoto and S. Smale.) Nevertheless, our method can be applied directly to more general types of flows on surfaces, for instance, the flows with wandering sets consisting of discrete critical elements for which the singularities are of finite type (e.g. saddles with n-prongs [**Lev1**]).

Our goal is to understand the relation of the topological and combinatorial invariants of the dynamical systems with those of the foliation C^*-algebras, especially their associated K-theoretical invariants. The main result is a complete classification of the C^*-algebras of all the Morse–Smale flows on closed two-manifolds in terms of *dual graphs* (Theorems 4.13–4.15). As a by-product of our

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investigation, it is shown that the *coloured dual graphs* (Definitions 4.5, 4.12) are exactly the complete combinatorial invariant of the topological conjugacy classes of the Morse–Smale flows on closed two-manifolds (Theorem 4.13). This classification is intrinsic and, therefore, much more effective and simpler than the classical result (again due to M. Peixoto [**P2**], see § 1). Peixoto's classification is in terms of the 'distinguished graphs', which are actually given by the flow diagram and the list exhausting all the 'distinguished sets' (i.e. the canonical regions) presented in the specific flow.

Perhaps the most interesting phenomenon is the analogue we observe in § 5 of the classification results of our 'noncommutative smooth manifolds' with that of the 'commutative' closed two-manifolds and simply connected four-manifolds. The *homeomorphism* classes of such *smooth* manifolds are classified by the equivalence classes of the intersection forms over the middle homology groups, with $\mathbb{Z}_2$-coefficients for surfaces and with $\mathbb{Z}$-coefficients of four-manifolds.

Recall that a graph G (a simplicial 1-complex) is determined by the (*symmetric*) *adjacency matrix*, whose ij-entry is the number of edges connecting the ith and jth vertices. A dual graph G is determined by a *nonsymmetric adjacency matrix*, whose ij-entry is 0 or ± 1 depending on whether or not the jth edge is incident to the ith vertex, and the grading of the edge. For the dual graphs classifying the C^*-algebras, the adjacency matrix A (or rather, the equivalence class) also determines a symmetric bilinear form over the K^1-group of a canonical ideal of the C^*-algebra, and we may view such bilinear forms as *the intersection forms* determined by the 'noncommutative manifolds'. A feature of 'noncommutativity' appears to be 'nonsymmetry'. These *noncommutative* manifolds are classified by the *nonsymmetric intersection matrices*, which are the nonsymmetric adjacency matrices of the dual graphs. It turns out that for each Morse flow $(M, \mathcal{F})$, there is a canonical decomposition of $C^*(M, \mathcal{F})$ defining a KK^1-element which can be identified with such a nonsymmetric intersection matrix. Here the KK^1-groups are viewed as the Ext-theory of Brown–Douglas–Fillmore and Kasparov. The generalized intersection form A is determined by the corresponding KK^1-element $A_{\mathcal{F}}$ through a simple formula (Theorem 5.12); the off-diagonal entries of A coincide with those of $A'_{\mathcal{F}} A_{\mathcal{F}}$.

For a Morse flow on a surface, the linking matrix of the stable manifolds of all the saddles gives the intersection form of the surface, and thus determines the surface. However, even for polar Morse flows, the symmetric adjacency matrix does not determine the C^*-algebra up to isomorphism for surfaces with genus $g \geq 8$ (Example 5.15). Thus in general no information contained in the nonsymmetric intersection matrices (equivalently, the dual graphs) is redundant for C^*-algebras.

For Morse flows with only index 2 saddles in a simply connected four-manifold, the nonsymmetric intersection form determined by the KK^1-element represented by the C^*-algebra plays a similar role. However, the situation is much more complicated because the complement of the stable and unstable manifolds (they are knotted S^2's, cf. [**H–W2**]) is connected. We plan to discuss this in a separate work.

Let $(M, \mathcal{F})$ be a Morse flow. Let $(\Sigma, \mathcal{F})$ be the foliated open two-manifold obtained by removing all the singularities from M. Then there is a canonical embedding of the $\mathbb{Z}_2$-graded dual graph $\hat{G}(\mathscr{F})$ into Σ inducing an isomorphism $\pi_1(\hat{G}(\mathscr{F})) \simeq \pi_1(\Sigma)$ (Theorem 2.9). An upshot of the embedding is a generalization of the results of **[Lev2]** and **[Mar]** (Remark 4.9). As the universal covering of Σ, the hyperbolic upper half plane $\mathbb{H}$ admits a unique covering foliation $\tilde{\mathscr{F}}$. The C^*-algebra $C^*(\mathbb{H}, \tilde{\mathscr{F}})$ is isomorphic to $C^*(T(\tilde{\mathscr{F}}))$ for some distinguished tree $T(\tilde{\mathscr{F}})$ (Theorem 20, **[W3]**). As one would expect, $T(\tilde{\mathscr{F}})$ is, up to similarity, actually the universal covering of the (embedded) dual graph $\hat{G}(\tilde{\mathscr{F}})$ and the C^*-algebra $C^*(\hat{G}(\tilde{\mathscr{F}}))$ is isomorphic to the crossed product $C^*(T(\tilde{\mathscr{F}}) \rtimes F(n)$, where $F(n) = \pi_1(\hat{G}(\mathscr{F}))$ (Theorem 3, **[W4]**).

M. Culler and K. Vogtmann showed that the moduli space X_n of (marked) graphs with fundamental group F_n has an Out (F_n)-equivariant deformation retraction to a simplicial 'spine' K_n, which is contractible and has dimension $2n-3$. From this triangulation, they showed that the virtual cohomological dimension of Out (F_n) is $2n-3$. Here we point out that each dual graph is a minimal marked graph without metric in the sense of **[C–V]**, and the parameter space of metrics on a dual graph (with fundamental group F_n) of total length 1 making it into an $\mathbb{R}$-graph (cf. Definition 2.5.1 **[W2]**) is an open simplex with dimension exactly equal to $2n-3$. It is an interesting problem to determine the contribution given by all the dual graphs, including those coming from structurally stable flows on unorientable surfaces. Roughly, a moduli space of graphs can be viewed sitting in the boundaries of the Teichmüller spaces for surfaces with appropriate numbers of punctures. The actions of Out (F_n) are the 'limits' of actions of the mapping class group.

For simplicity, in this paper we consider only orientable manifolds unless otherwise stated, although our method applies to unorientable manifolds.

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1. *Morse–Smale flows, Peixoto's work*

Let M be a closed, connected smooth two-manifold and let $\mathscr{X}(M)$ be all the C^∞-vector fields on M. A vector field $X \in \mathscr{X}(M)$ is called *Morse–Smale* if: (1) it has finitely many critical elements (singularities and closed orbits), all hyperbolic (they consist of sources, sinks, saddles, attracting and repelling closed orbits); (2) the α-limit and ω-limit sets of any orbit are critical elements; (3) there is no saddle connection (p. 122 **[P–M]**). The importance of Morse–Smale flows on surfaces can be seen from the following two remarkable theorems established by M. Peixoto **[P1]**, **[P2]** and Smale **[S]**: Morse–Smale vector fields form a dense open subset of $\mathscr{X}(M)$ and Morse–Smale flows are exactly the structurally stable flows on two-manifolds. The subset of Morse–Smale flows in $\mathscr{X}(M)$ is denoted by $\mathscr{X}_0(M)$ or just $\mathscr{X}_0$. We say two flows $(M_1, \mathscr{F}_1)$ and $(M_2, \mathscr{F}_2)$ are *conjugate* if there is a homeomorphism from M_1 to M_2 taking the orbits of $\mathscr{F}_1$ onto the orbits of $\mathscr{F}_2$, preserving the flow orientation. A flow is *gradient-like* if it is conjugate to a gradient flow. A gradient-like flow is *polar* if it has only one source and one sink. For simplicity we shall call a gradient-like Morse–Smale flow a *Morse flow*.

Given a graph $G=(V, E)$ (cf. p. 13, [**Ser**]), a *direction* on G is defined by a fixed choice between each edge e and its inverse $\bar{e}$. A *digraph* is a directed graph. A pair $\tilde{e}=\{e, \bar{e}\}$ corresponds to a 1-simplex in the geometrical realization of a graph G, and will be called a *geometric edge* of G. Sometimes we may identify G with its realization, and simply call a geometric edge an 'edge' if no confusion will arise.

Given a gradient-like flow X with finitely many singularities, one defines the *phase diagram* Γ of X to be the digraph constructed as follows: the vertices are the critical elements and there is a directed edge from σ_1 to σ_2 if the intersection $W^u(\sigma_1)\cap W^s(\sigma_2)$ of the unstable manifold of σ_1 and the stable manifold of σ_2 is nonempty (p. 123, [**M–P**], cf. also p. 7, [**S**]), one edge for each component in the intersection. Sources, sinks and saddles will be denoted by α, ω and σ, respectively.

Using the phase diagrams with some other structures (the *distinguished sets*, as he called them), M. Peixoto gave a classification of all Morse–Smale flows on closed two-manifolds in [**P2**]. For flows on orientable surfaces without closed orbits, there are three types of distinguished sets (figure 1.1) which are denoted by $[i, j; k, l]$, $[i; j, k]$ and $[l, m; n]$ and called the sets of type 1, type 2 and type 3, respectively. A pair of edges of a distinguished set, such that one edge enters a saddle point and the other leaves it, is said to be *associated*.

Definition 1.1 (Definition 4.2, 4.5 [**P2**].) A *Peixoto graph* G^* is a digraph Λ, which is either the one-edged graph $\overrightarrow{\alpha\omega}$, corresponding to the polar flow on S^2, or else a digraph with vertices in three levels, α, σ and ω. All edges must be oriented from α to σ and from σ to ω, in such a way that to each σ there are associated two incoming and two outgoing edges. Besides, one assigns a certain number of subsets of Λ as distinguished sets, three types of them defined as above (figure 1.1) satisfying the following axioms:

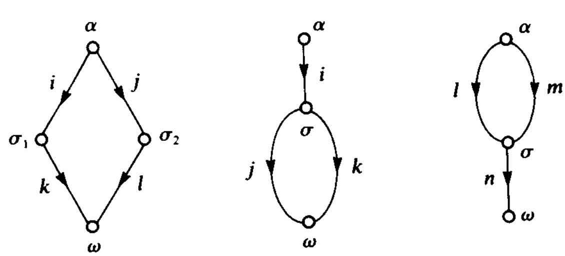

FIGURE 1.1

(1.1) every edge of G belongs to exactly two distinguished sets except the ones which are the first edge of a distinguished set of type 2 or the third edge of a distinguished set of type 3, and these belong to only one distinguished set;

(1.2) when the first edge of a distinguished set of type 2 is incident on α then no other edge of G is incident on α, and similarly for the third edge of a distinguished set of type 3;

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(1.3) if two distinguished sets have in common a pair of associated edges, then these distinguished sets coincide;

(1.4) when more than one distinguished set is incident at a given vertex then they can be written in cyclic order, i.e. in the form $D_1, D_2, \ldots, D_p$, where D_k has one edge in common with D_{k-1} and another with D_{k+1}, $k = 1, \ldots, p$, $D_{p+1} = D_1$, these two edges being incident at the vertex and adjacent to each other as edges of D_k;

(1.5) the distinguished sets can be coherently oriented, i.e. an orientation can be assigned to each one such that at every vertex, as in (1.4), the orientation of D_k induces on its pair of adjacent edges incident at the vertex a sense of rotation which is the same for all $k = 1, \ldots, p$.

Let p, q, r be the number of α, σ and ω in G^*. Then $p - q + r = \chi(G^*)$ is called the Euler characteristic of G^*.

Let Σ be the open two-manifold obtained by removing all the singularities of the flow X from M. For a Morse–Smale flow X (or the gradient flow of a Morse function), the *separatrices* (cf. [**H–W1**]) of X are the closed orbits along with the stable and the unstable manifolds of saddles, and the *canonical regions* of X are the connected components of $\Sigma\backslash\{\text{separatrices}\}$.

Clearly the distinguished sets correspond to the canonical regions. Therefore, if $X, Y \in \mathscr{X}_0(M)$ then X, Y are topologically equivalent if and only if the Peixoto graphs $G^*(X)$ and $G^*(Y)$ are equivalent in the obvious sense. The following is the main theorem in [**P2**].

THEOREM 1.2 [**P2**]. *An abstract distinguished graph* $\Lambda \cong G^*(X)$ *for some* $X \in \mathscr{X}_0(M)$ *if and only if* Λ *and* M *have the same Euler characteristic.*

For a general Morse–Smale flow $\mathscr{F}$ on an orientable surface M with closed orbits, the nonwandering set has form

$$\Omega = \{\alpha_1, \ldots, \alpha_p, \alpha_1^1, \ldots, \alpha_m^1; \sigma_1, \ldots, \sigma_r; \omega_1, \ldots, \omega_q; \omega_1^1, \ldots, \omega_n^1\}$$

where the α^1 and ω^1 are the one-dimensional attractors and repellers (closed orbits). Peixoto defines the digraph $G(\mathscr{F})$ as before (p. 416, [**P1**]), with Ω as vertices: the α and α^1 on the first line, the σ on the second, and the ω and ω^1 on the third. The directed edges are assigned as before. Now in figure 1.1 the α can be replaced by α^1 and the ω can be replaced by ω^1, they may also reduce to a single edge, and we have a total of 16 types of different canonical regions.

The axioms (1.1)–(1.5) need to be modified. There are ten axioms to regulate how the 16 types of distinguished sets can fit together. We omit the list of these axioms as we shall not need them later one. Thus a distinguished graph consists of a flow diagram, and a full list enumerating all the distinguished sets (of up to 16 types).

In § 4 we shall prove a much simpler classification theorem (Theorems 4.12, 4.13), for general Morse–Smale flows, in terms of a single $\mathbb{Z}_2$-graded digraph.

2. *C^*-algebras and dual graphs of Morse flows*

A flow $\mathscr{F}$ on a closed manifold M is a one-parameter transformation group. Associated with it, there is the C^*-algebra $C(M) \rtimes \mathbb{R}$ [**E–H**]. The flow $\mathscr{F}$ induces a foliation on the open manifold $M\backslash\{\text{singularities}\}$. By a slight abuse of notation, we simply write $C^*(M, \mathcal{F})$ for the C^*-algebra of this foliation **[C2]** and refer to it as the C^*-algebra of the flow $(M, \mathcal{F})$. For a Morse–Smale flow $\mathcal{F}$ on a closed surface, every closed orbit has holonomy cover the real line. So the holonomy groupoid of the foliation can be identified with the transformation groupoid (the singularities deleted) and the C^*-algebra $C^*(M, \mathcal{F})$ is canonically isomorphic to the C^*-algebra $C_0(M\backslash\{\text{singularities}\}) \rtimes \mathbb{R}$ (Proposition 1.11, **[W1]**). Let n be the number of all singularities of $(M, \mathcal{F})$. We have an exact sequence (Proposition 1.12, **[W1]**)

$$0 \to C^*(M, \mathcal{F}) \to C(M) \rtimes \mathbb{R} \to C_0(\mathbb{R})^n \to 0.$$

Since each singularity gives rise to a one-parameter family of one-dimensional representations of the transformation group C^*-algebra, and all one-dimensional representations arise this way, the ideal $C^*(M, \mathcal{F})$ is exactly the commutator ideal of $C(M) \rtimes \mathbb{R}$. The local structure of $C(M) \rtimes \mathbb{R}$ around singularities is explicitly described in **[W1]**. In this paper we only treat $C^*(M, \mathcal{F})$. However, our method applies to the full algebra $C(M) \rtimes \mathbb{R}$ using **[W1]**. In many situations C^*-algebras of foliations can be profitably thought of as noncommutative simplicial complexes, which are in turn the global fibered products of graphs of C^*-algebras (Definition 2.1). A procedure dealing with the more general noncommutative CW complexes is described in § 4, **[W2]**, and **[W3]**. To every vertex x and every geometrical edge e, we associate C^*-algebras A_x with a surjective homomorphism $\pi_{e,x}$ from A_x onto A_e if e is incident on x. The C^*-algebras A_x and A_e are the 'cells' and the $\pi_{e,x}$ are the gluing maps.

Definition 2.1. Let (G, A) be a finite graph of C^*-algebras. Then the associated *global fibered product* is

$$C^*(G, A) = \left\{ (a_x) \in \prod_{x \in V} A_x \,\middle|\, \pi_{e,x}(a_x) = \pi_{e,x'}(a_{x'}),\ e \in E \right\},$$

where x, x' are the two ends of e.

For flows with wandering sets consisting of infinitely many critical elements, we need $\mathbb{R}$-*graphs* (Definition 2.5.3, **[W2]**). In this case, we also need to introduce the condition of vanishing at infinity and the global gluing condition for the global fibered product. (See Definitions 4.2.7, 4.2.9, 4.2.11 of **[W2]**.) In the rest of this section we consider only gradient-like flows with finitely many singularities, for which it is enough to consider finite graphs.

Let $\mathfrak{G}(M, \mathcal{F})$ be the holonomy groupoid of a foliation $(M, \mathcal{F})$. For a closed saturated submanifold N of the foliated manifold $(M, \mathcal{F})$, we define $\mathfrak{G}(N, \mathcal{F}) = \{\gamma \in \mathfrak{G}(M, \mathcal{F}) \mid \gamma \in N\}$. Note that in general $\mathfrak{G}(N, \mathcal{F}) / \mathfrak{G}(N, \mathcal{F} \mid N)$. Let $C^*(\mathfrak{G}(N, \mathcal{F}))$ be the C^*-algebra of the groupoid $\mathfrak{G}(N, \mathcal{F})$.

The proof of the following lemma is straightforward and we omit it.

LEMMA 2.2. *Let $(M, \mathcal{F})$ be a foliation of a connected manifold M which can be either closed or open, maybe with boundary. Let $M = \bigcup_{i=1}^n M_i$ be a decomposition of $(M, \mathcal{F})$ into finitely many saturated* closed *submanifolds with boundaries. Let $G = (V, E)$ be*

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the associated graph of the partition with vertices $V = \{M_i\}$ and edges $E = \{M_{ij}\}$, where $M_{ij} = M_i \cap M_j$. Let $A_x = C^*(\mathfrak{G}(M_i, \mathcal{F}))$ if $x = M_i$, and $M_e = C^*(\mathfrak{G}(M_{ij}, \mathcal{F}))$ if $e = M_{ij}$. Let $\pi_{e,x}$ be the restriction maps. Then the global fibered product $C^*(G, A)$ is isomorphic to $C^*(M, \mathcal{F})$.

It turns out that quite often these cells take very simple forms. For gradient-like flows, the C^*-algebra A_e is isomorphic to $\mathcal{K}$, the compact operators in a Hilbert space, for all the edges e, while the C^*-algebras A_v associated with all the vertices v are given by the following

THEOREM 2.3. *Let $M = ([0, 1] \times \mathbb{R} \backslash \{(0, 0), (1, 0)\}$. Let $(M, \mathcal{F})$ be the foliation with vertical constant leaves. Then the C^*-algebra $C^*(M, \mathcal{F})$ is canonically isomorphic to the C^*-algebra*

$$A_v = \left\{ f \in C([0, 1], M_2(\mathcal{K})) \,|\, f(0) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, f(1) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\},$$

where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space.

Proof. It follows from Theorem 4.1.2, [**W2**]. □

In order to 'calculate' the C^*-algebras for a given flow by Lemma 2.2, we need only specify two more data: (1) the graph $G(\mathcal{F})$ associated with the flow, and (2) the gluing maps $\pi_{x,e}$ for $x \in V$, and $e \in E$. If $(M, \mathcal{F})$ is the north–south polar flow on S^2, then we define the *dual graph* $\hat{G}(\mathcal{F})$ to be just 1 point. In the rest of the paper, we rule out this trivial case unless otherwise specified.

Definition 2.4. Let $(M, \mathcal{F})$ be a Morse flow on a closed orientable manifold M. The *dual graph* $\hat{G}(\mathcal{F}) = (V, E)$ of the flow $\mathcal{F}$ is a graph defined by the following (2.4.1) and (2.4.2), with a partition on E satisfying (2.4.3) below.

(2.4.1) The vertices are in 1–1 correspondence with the canonical regions;

(2.4.2) The geometric edges connecting two vertices are in 1–1 correspondence with the separatrices in the common boundary of the closures of two canonical regions.

Clearly we have

PROPOSITION 2.5. *A vertex corresponds to a distinguished set of Peixoto (figure* 1.1). *The incidence number* Inc $(x) = 4$ *for each* $x \in V$.

(2.4.3) For each $x \in V$, the four edges in E_x are divided evenly into two pairs, and each pair corresponds to a pair of stable and unstable trajectories of a saddle.

At each vertex we use two small arcs to identify the two pairs of edges, as in the illustration (figure 2.1(b)).

Sometimes we shall represent the partition of edges E by a $\mathbb{Z}_2$-grading on E, so for each $x \in V$ one pair of edges in E_x has degree 0 and the other has degree 1. Of course, switching the degrees of the two pairs gives an equivalent representation. For a geometric edge $\tilde{e} = (e, \bar{e})$, it may well happen that deg $e \neq$ deg $\bar{e}$.

For a gradient flow $\mathcal{F}$ of a Morse function, we define its *dual graph* $\hat{G}(\mathcal{F})$ as before by (2.4.1), (2.4.2) but (2.4.3) is replaced by

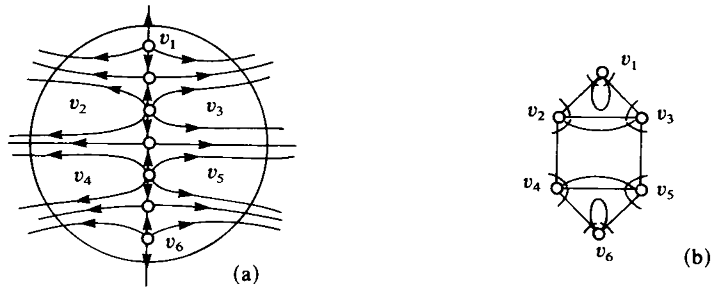

FIGURE 2.1

(2.4.3)′ For each $x \in V$, the edges in E_x are $\mathbb{Z}_2$-graded so that any two edges e and e' in E_x have the same grading if and only if the corresponding separatrices L and L' sit in the closure of a Cauchy sequence $\{L_1, L_2, \ldots\}$ of leaves contained in the same canonical region in the coarse leaf topology.

For separatrices and coarse leaf topology, see the paragraph preceding Theorem 1.2 and the beginning of § 3. Dual graphs can be defined for more general flows on surfaces.

Example 2.6. In general $G(\mathcal{F})$ is not a combinatorial graph, namely, it may have multiple geometric edges and loops. In figure 2.1(a) a Morse–Smale flow on S^2 (there is only one sink on its back) is shown; its dual graph is (b).

Let $\hat{G}(\mathcal{F}) = (V, E)$ be the dual graph of a Morse flow $(M, \mathcal{F})$. Recall that A_v is given in Theorem 2.3 and $A_e = \mathcal{K}$ for any $v \in V$ and $e \in E$. We fix a labelling $E_v = \{e_j^i(v)\}_{i,j=0,1}$ for each $v \in V$, such that $\{e_1^i, e_2^i\}$ is an associated pair, $i = 0, 1$. Then the gluing maps $\pi_{v,e_1^i} \oplus \pi_{v,e_2^i} : A_v \to A_{e_1^i} \oplus A_{e_2^i}$ are given by $f \mapsto f(i)$, for all $f \in A_v$, $i = 0, 1$. If the labellings of the two pairs of edges in E_v are switched, then the $f(0)$ and $f(1)$ are all switched for $f \in A_v$ and one gets an isomorphic C^*-algebra. So the ambiguity in the labelling is harmless. Let $C^*(\hat{G}(\mathcal{F}))$ be the corresponding fibered product (Definition 2.1). We say two dual graphs are *isomorphic* if there is an abstract graph isomorphism from one to the other preserving the partition of the edges. Now the structure of $C^*(M, \mathcal{F})$ follows from Lemma 2.2 and Theorem 2.3.

THEOREM 2.7. *Let $(M, \mathcal{F})$ be a Morse flow on a closed orientable two-manifold. Let $\hat{G}(\mathcal{F})$ be the dual graph* (2.4). *Then the C^*-algebra $C^*(M, \mathcal{F})$ of the flow is isomorphic to the C^*-algebra $C^*(\hat{G}(\mathcal{F}))$ of the graph. Moreover, C^*-algebras of two such Morse flows are isomorphic if and only if the two dual graphs of the flows are isomorphic.*

Example 2.8. Theorem 2.7 and (2.4.3) tell us that the C^*-algebra of a gradient-like stable flow is always constructed with '1-cells' with exactly two 'branches' at each 'end'. Therefore, stability of flows can be detected by the structure of C^*-algebras. Figure 2.2 illustrates the gradient flow of a Morse function (the upright height function) on a torus, and the dual graph. The flow is not stable (there is a saddle connection) and the C^*-algebra is built with 1-cells with three 'branches' on each 'end'. A small perturbation changes the flow to a stable one which is illustrated in figure 2.3.

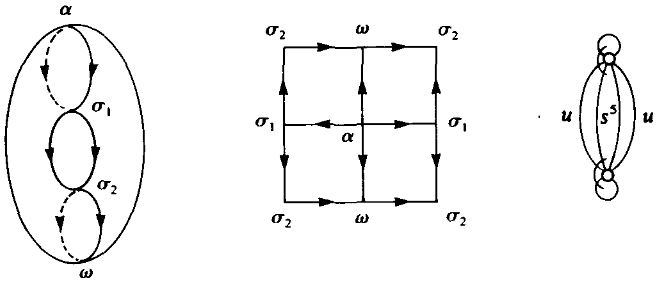

FIGURE 2.2

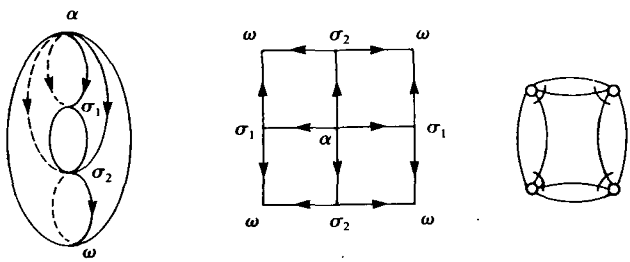

FIGURE 2.3

THEOREM 2.9. *Let $(M, \mathscr{F})$ be a Morse flow on a closed orientable two-manifold. There is an embedding ι of the geometrical realization of $\hat{G}(\mathscr{F})$ into Σ which induces a natural isomorphism $\pi_1(\hat{G}(\mathscr{F}), *) \simeq \pi_1(\Sigma, \iota(*))$.*

Proof. Encircle every source and sink by a small transversal on Σ. The union C of all these disjoint circles forms a faithful transversal of $(M, \mathscr{F})$ (i.e. every leaf intersects C), because $(M, \mathscr{F})$ has no saddle connections. Deform isotopically the two transverse arcs in each canonical region until they become tangent at a *single* point v_x in the interior of the canonical region, while remaining transversal (as shown in figure 2.4).

Now we check that if we identify a vertex x in $\hat{G}(\mathscr{F})$ with the 'tangential point' v_x in the canonical region represented by x, and identify every geometric edge with an appropriate transversal arc, then we obtain a desired embedding ι. By an isotopy, we may choose the basepoint $*$ to be a vertex in $\hat{G}$. The induced map $\iota_*: \pi_1(\hat{G}, *) \to \pi_1(\Sigma, \iota(*))$ is clearly injective: no loop in $\hat{G}$ can bound a disc in Σ, because otherwise the transversality of the flow along the loop would lead to a contradiction to the Poincaré–Hopf theorem.

Next we show that ι_* is also surjective. Let p be the quotient map from Σ to the leaf space $\Sigma/\mathscr{F}$. Let $f:(S^1, *) \to (\Sigma, *)$ be a loop in Σ. Then $\gamma(f) = p \circ f$ defines a loop in $(\Sigma/\mathscr{F}, *)$. Although the $\mathbb{R}$-action is non-proper and consequently $\Sigma/\mathscr{F}$ is not Hausdorff, the $\mathbb{R}$-action is free, so γ induces a canonical isomorphism γ_* from $\pi_1(\Sigma, *)$ onto $\pi_1(\Sigma/\mathscr{F}, *)$. It is enough to show that $[f]$ is in the image of ι_* if $p \circ f$ has no backtracking.

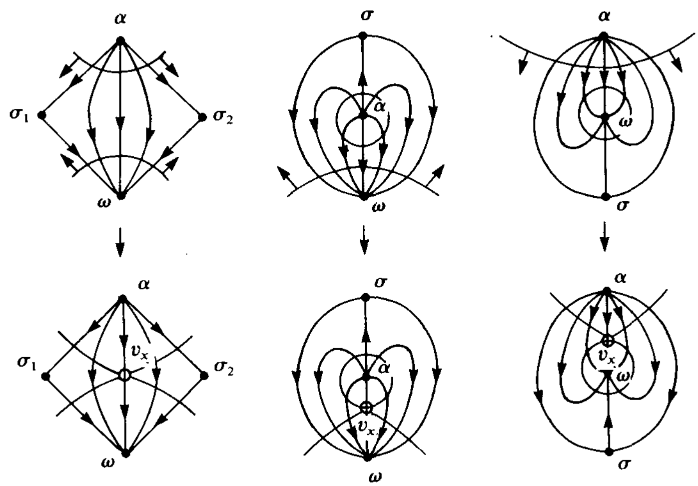

FIGURE 2.4

(a) Im $(p\circ f)$ contains no separatrices. In this case Im (f) is contained in the canonical region with $*$. Thus $[f]=id$ is in Im (σ_*).

(b) Im $(p\circ f)$ has separatrices. The loop $p\circ f$ defines a cyclic order among all the separatrices in Im $(p\circ f)$: $L_1, L_2, \ldots, L_n$ and then L_1 again. There are no separatrices between L_i and L_{i+1}, for $i=1,2,\ldots, n$, in Im $(p\circ f)$. Thus L_i and L_{i+1} are connected by all the leaves in a single canonical region. In other words, the two directed edges e_i, e_{i+1} in $\hat{G}(\mathcal{F})$ corresponding to two separatrices L_i, L_{i+1} with the orientation given by $p\circ f$ incident to the same vertex. Therefore, $\{e_1, e_2, \ldots, e_n\}$ define an oriented loop $\hat{f}$ in $\hat{G}(\mathcal{F})$. (When $n=1$, the loop f encircles either a source or a sink, and the loop $\hat{f}$ has both ends incident to the same vertex.) Clearly $\iota_*[\hat{f}]=[f]$. Thus ι_* is an isomorphism. □

Denote the numbers of sources, sinks and saddles by $\#\alpha$, $\#\omega$, and $\#\sigma$. Let g be the genus of M. Clearly $\pi_1(\Sigma)$ is a free group on $(2g+\#\alpha+\#\omega+\#\sigma-1)$ generators. The graph $\hat{G}$ has $4\#\sigma$ geometric edges and $2\#\sigma$ vertices. So a maximal spanning tree of $\hat{G}$ has $2\#\sigma-1$ geometric edges and $\hat{G}$ is homotopy equivalent to a bouquet of $(4\#\sigma-(2\#\sigma-1))=2\#\sigma+1$ circles. Thus $\pi_1(\hat{G})$ is a free group with $(2\#\sigma+1)$ generators. Theorem 2.9 shows that

$$2g+\#\alpha+\#\omega+\#\sigma-1=2\#\sigma+1.$$

Thus we have reproved the celebrated Euler–Poincaré formula that $\#\alpha+\#\omega-\#\sigma=2(1-g)$.

The proof of Theorem 2.9 has other consequences (Remark 4.9).

It is well known that for any free discrete subgroup Γ of SL$(2,\mathbb{R})$ with finitely many generators, the quotient space $\Sigma=H/\Gamma$ is an open two-manifold with finitely many ends. Moreover, every open two-manifold of finite homology type arises in this way. A natural question is, can one relate the number of ends of Σ to rank Γ?

We list the following simple facts from the proof above.

PROPOSITION 2.10. *Let Γ be a free discrete subgroup of* $SL(2,\mathbb{R})$ *with n generators. Let m be the number of ends of the open two manifold $\Sigma = H/\Gamma$. Then $m \le n+1$. Conversely for each $g = 0, 1, 2, \ldots, [n/2]$, there is an embedding $i_g : F_n \to SL(2, \mathbb{R})$ such that $\mathbb{H}/i_g(F_n)$ has exactly $(n+1-2g)$ ends. If the $(n+1-2g)$ ends are exactly the singularities of a Morse–Smale flow on M_g, then n is odd and $g-r+k-1$ must be one of $0, 1, \ldots, [(n-1)/4]$, where r is the number of closed orbits and k is the number of the connected components when the closed orbits are removed from M_g.*

Proof. One observes that $n = m + 2g - 1$, so $m = n + 1 - 2g$. If the m ends are exactly the singularities of a Morse flow on M_g, then $n = 2\#\sigma + 1$ and

$$\#\alpha + \#\omega = \frac{n-1}{2} + 2 - 2g$$

hence $g \le [(n-1)/4]$.

Assume one has a Morse-Smale flow $(M, \mathscr{F})$ with r closed orbits. Remove all the r closed orbits. Replace a repeller (attractor) by a pair of sources (sinks). This surgery will produce Morse flows $(M_1, \mathscr{F}), \ldots, (M_k, \mathscr{F})$, where M_i are connected surfaces. Assume that $(M_i, \mathscr{F})$ has genus g_i, fundamental group of rank n_i, a total of m_i ends and l_i saddles. Then we have

$$\sum_{i=1}^{k} g_i = g - r + k - 1, \quad \sum_{i=1}^{k} n_i = n + k - 1$$

(since n_i are odd, so n is also odd), $\sum_{i=1}^{k} m_i = n + 1 - 2(g - r)$, and $\sum_{i=1}^{k} l_i = \#\sigma$ unchanged.

Applying the result for Morse flows, we get

$$g - r + k - 1 = \sum_{i=1}^{k} g_i \le \sum_{i=1}^{k} \left[\frac{n_i - 1}{4}\right] \le \left[\frac{1}{4}\sum_{i=1}^{k} (n_i - 1)\right] = \left[\frac{n-1}{4}\right]. \qquad \square$$

COROLLARY 2.11. *Let $\mathscr{F}$ be a Morse flow. Let $\# V$ be the number of vertices in $\hat{G}(\mathscr{F})$ and $n = \operatorname{rank} \pi_1(\Sigma)$, then $n = \# V + 1$ and $\# V = 2\#\sigma$.*

Proof. It follows from the proof of Theorem 2.9 and the discussion following it.

COROLLARY 2.12. *Let $(M, \mathscr{F})$ and $(M', \mathscr{F}')$ be two Morse flows. Then the corresponding punctured surfaces Σ and Σ' are homotopy equivalent if and only if $\# V = \# V'$, equivalently if and only if $\#\sigma = \#\sigma'$.*

Proof. It is a standard fact in topology that the two open two-manifolds Σ and Σ' are homotopy equivalent if and only if $\pi_1(\Sigma) \simeq \pi_1(\Sigma')$, as both of the groups are free and with finitely many generators. So the conclusion follows from Corollary 2.11.

$\square$

COROLLARY 2.13. *If $C^*(M, \mathscr{F}) \simeq C^*(M', \mathscr{F}')$ then Σ and Σ' are homotopy equivalent.*

Example 2.14. It may very well happen that $C^*(M_g, \mathscr{F}) \simeq C^*(M_{g'}, \mathscr{F}')$, but $(M_g, \mathscr{F})$ and $(M_{g'}, \mathscr{F}')$ are not conjugate, in fact neither M_g and $M_{g'}$ nor Σ_g and $\Sigma_{g'}$ are homeomorphic. Consider the 'Monkey seat' shown in figure 2.5(a). The gradient flow (b) of its height function has the same flow graph (c) as that of the stable flow of the torus shown in figure 2.3. If the rank of the fundamental group $n = 5$, the

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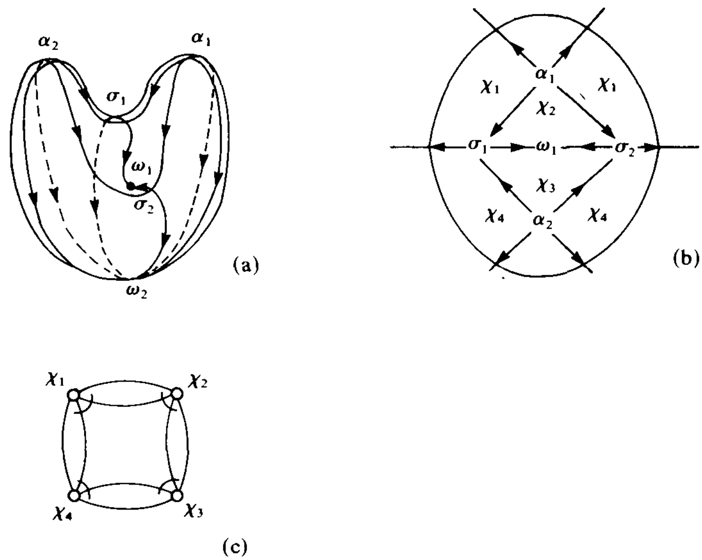


FIGURE 2.5

only possible numbers of ends are 6 (on S^2) and 4 (on a torus) by Proposition 2.10. In other words, these are the only two Morse flows with two saddles. By Proposition 2.10, one can check that a Morse flow with three saddles also can only occur on the sphere and the torus. It is a fun exercise to conceive their shapes with such Morse flows.

3. *Crossed products of C^*-algebras by free groups*

Let $(M, \mathcal{F})$ be a Morse–Smale flow on a closed orientable surface and $(\Sigma, \mathcal{F})$ be the foliated open manifold obtained by removing all the singularities of $\mathcal{F}$ from M. Let $(\mathbb{R}^2, \tilde{\mathcal{F}})$ be the universal covering foliation of $(\Sigma, \mathcal{F})$. For the definitions of separatrices and the coarse leaf topology of the plane, see 1.5.1 [**W2**] or [**H–W**]. One observes first

LEMMA 3.1. *The separatrices of the foliation* $(\mathbb{R}^2, \tilde{\mathcal{F}})$ *are precisely the pre-images of all the closed orbits and the stable and unstable manifolds of the saddles of* $(M, \mathcal{F})$.

COROLLARY 3.2. *The foliation* $(\mathbb{R}^2, \tilde{\mathcal{F}})$ *has* T_2*-separatrices* (*Definition* 1.5.12, [**W2**]).

Proof. The stable and unstable manifolds of saddles form a finite subset in $\Sigma/\mathcal{F}$. So its preimage in $\mathbb{R}^2/\tilde{\mathcal{F}}$ is discrete, in particular Hausdorff. A limit separatrix (1.5.1, [**W2**]) is precisely a leaf in the preimage of a limit cycle of a stable or unstable trajectory of a saddle. Again there are only finitely many closed orbits, so the limit separatrices are discrete. As a general fact for any foliation of the plane, the limit of a convergent sequence of non-limit separatrices is a unique leaf. So the separatrices of $(\mathbb{R}^2, \tilde{\mathcal{F}})$ are Hausdorff in the quotient topology. □

We recall briefly some results about C^*-algebras of foliations of the plane (see [**W2**] and [**W3**] for details). An $\mathbb{R}$-*tree* is a separable metric space such that any pair of points are joined by a unique path which is the isometric image of an interval [**M–S**]. A *regular* $\mathbb{R}$*-tree* (Definition 2.3.3, [**W2**]) is an $\mathbb{R}$-tree with vertices and edges. The C^*-algebras of foliations with T_2-separatrices are classified by *distinguished trees* (§ 3, [**W2**]), which are a special class of regular $\mathbb{R}$-trees characterized in Definition 3.3.1 of [**W2**]. There is a canonical *tree* of C^**-algebras* associated with a distinguished tree such that the *global fibered product* (Definition 4.2.11, [**W2**]) is isomorphic to the C^*-algebra of the foliation defining the distinguished tree (Theorem 4.3.1, [**W2**]). It follows from Corollary 3.2. that there is a distinguished tree $T(\tilde{\mathscr{F}})$ associated to $(\mathbb{R}^2, \tilde{\mathscr{F}})$ and we have

COROLLARY 3.3. $C^*(T(\tilde{\mathscr{F}})) \simeq C^*(\mathbb{R}^2, \tilde{\mathscr{F}})$.

The following proposition follows from Theorem 4 of [**W4**].

PROPOSITION 3.4. *Let M be a manifold without boundary. Let G be a connected Lie group acting locally freely on M such that the isotropy group bundle can be identified to the holonomy group bundle of the induced foliation* [**W1**]. *Let $\tilde{M}$ be a regular covering of M with covering group Γ. Suppose that G also acts on $\tilde{M}$ and induces the covering foliation $(\tilde{M}, \tilde{\mathscr{F}})$. Then $C^*(M, \mathscr{F}) \simeq C^*(\tilde{M}, \tilde{\mathscr{F}}) \rtimes_\alpha \Gamma$, where the Γ-action α is determined by the deck transformations of Γ on the holonomy groupoid $G(\tilde{M}, \tilde{\mathscr{F}})$ $(\simeq \tilde{M} \times G)$.*

Note that the situation treated in Proposition 3.4 is similar to the situation 10 in [**R1**]. However Green and Rieffel's theorem does not apply here, because the action of G may be neither free nor wandering. Now we consider the particular situation where M is the open two-manifold Σ, the group $G = \mathbb{R}$, the foliation $(M, \mathscr{F})$ is a Morse–Smale flow, and $\Gamma = \pi_1(\Sigma)$ is a Fuchsian group. We have

COROLLARY 3.5. $C^*(M, \mathscr{F}) \simeq C^*(\mathbb{R}^2, \mathscr{F}) \rtimes_{\tilde{\alpha}} \Gamma$.

When $(M, \mathscr{F})$ is a Morse flow, the C^*-algebra $C^*(\Sigma, \mathscr{F})$ is isomorphic to $C^*(\hat{G}(\mathscr{F}))$ (Theorem 2.7) and $C^*(\tilde{\Sigma}, \tilde{\mathscr{F}}) \simeq C^*(T(\tilde{\mathscr{F}}))$ (Corollary 3.3). By Theorem 2.9, there is an embedding ι of the graph $\hat{G}(\mathscr{F})$ into the surface Σ inducing an isomorphism $\iota_* : \pi_1(\hat{G}(\mathscr{F})) \to \pi_1(\Sigma)$. The distinguished tree $T = T(\tilde{\mathscr{F}})$ is a simplicial tree and can be regarded as the universal covering of $\hat{G}(\mathscr{F})$. The lift $\tilde{\sigma}$ of σ is an embedding of T into $\tilde{\Sigma}$, and the deck transformations $\tilde{\alpha}$ of Γ on $\tilde{\Sigma}$ naturally induce an action on T, which induces naturally a Γ-action, again denoted by $\tilde{\alpha}$, on the global product $C^*(T)$ of C^*-algebras. There follows

COROLLARY 3.6. *When $(M, \mathscr{F})$ is a Morse flow*

$$C^*(\hat{G}(\mathscr{F})) \simeq C^*(T(\tilde{\mathscr{F}})) \rtimes_{\tilde{\alpha}} \Gamma.$$

When $(M, \mathscr{F})$ is a general Morse–Smale flow, the closed orbits cut M into a finite union of surfaces $\bigcup M_i$, and each M_i has a Morse flow $\mathscr{F}_i$ from the restriction of $\mathscr{F}$. The dual graph $\hat{G}(\mathscr{F}_i)$ has a universal covering $T(\tilde{\mathscr{F}}_i)$. If we identify $\hat{G}(\mathscr{F}_i)$ with an embedded image (Theorem 2.9) in M, then $T(\tilde{\mathscr{F}}_i)$ can be identified with a component of the preimage of $\hat{G}(\mathscr{F}_i)$ in $T(\tilde{\mathscr{F}})$ under the universal covering map

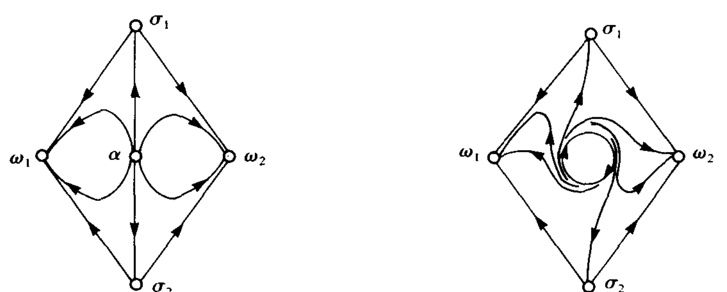
$\mathbb{R}^2 \to \Sigma$. The union of all the preimages of $\hat{G}(\mathscr{F}_i)$'s is a simplicial 'forest' with infinitely many trees.

Although any Morse–Smale flow $(M, \mathscr{F})$ has only finitely many critical points, the tree associated to the universal covering foliation $(\mathbb{R}^2, \tilde{\mathscr{F}})$ is in general an $\mathbb{R}$-tree. In fact, $(\mathbb{R}^2, \tilde{\mathscr{F}})$ has limit separatrices if and only if $(M, \mathscr{F})$ has some saddle σ, whose α-limit (or ω-limit) is a repeller (or an attractor). This is illustrated by a simple example.

Example 3.7. We 'blow up' a source and get a repeller (figure 3.1). The universal covering of *this region* (not of the whole punctured surface) is shown in figure 3.2(a).

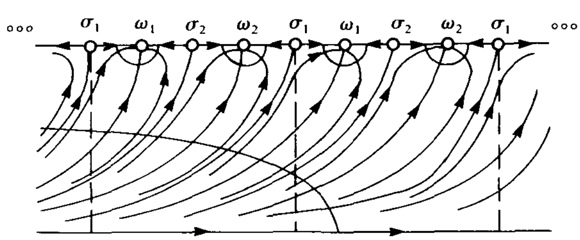
We may associate an $\mathbb{R}$-tree (figure 3.2(b)) with the foliated region. To each vertex in (b) we attach a C^*-algebra

$$A_1 = \{f \in C([0,1], M_2(\mathscr{K})) \mid f(0)\} = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad f(1) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

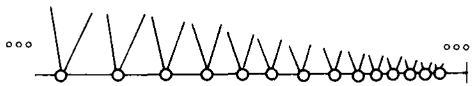


(a) (b)

FIGURE 3.1



(a)



(b)

FIGURE 3.2

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Then the global fibered product $C^*(T)$ of the tree of C^*-algebras is the C^*-algebra of the foliated region (a), which consists of all $(f_n) \in \prod_{-\infty}^{\infty} A_1$, satisfying

(i) the local gluing condition

$$f_n(0) = \begin{pmatrix} a_n^1 & 0 \\ 0 & a_n^0 \end{pmatrix}, \quad f_n(1) = \begin{pmatrix} a_n^2 & 0 \\ 0 & a_{n+1}^0 \end{pmatrix}, \quad a_n^i \in \mathcal{K};$$

(ii) the global gluing condition

$$\sup_{t\in[0,1]} \|f_n(t) - a_\infty \otimes e_{2,2}\| \to 0, \quad \text{as } n \to \infty$$

for some $a_\infty \in \mathcal{K}$ (depending on (f_n)).

(iii) the vanishing-at-∞ condition

$$\sup_{t\in[0,1]} \|f_n(t)\| \to 0 \quad \text{as } n \to -\infty.$$

Then by Theorem 3.6 the C^*-algebra of the foliated region (figure 3.1) is isomorphic to $C^*(T) \rtimes \mathbb{Z}$ with the $\mathbb{Z}$-action translating along the tree by two steps each time. Ignoring the limit point of the tree, we have the quotient graph given by figure 3.3, which is just the portion associated with (b) (or (a)), figure 3.1 of the dual graph. Note the tree (b) of figure 3.2 is a component of the preimage of the portion of dual graph in the tree associated with the universal cover foliation $(\mathbb{R}^2, \tilde{\mathcal{F}})$.

Example 3.8. It is not hard to visualize a slightly more complicated region involving simultaneously repellers and attractors (figure 3.4(a)), and get more complicated examples of $\mathbb{R}$-trees. In fact, the numbers of limit cycles can be arbitrary (figure 3.5).

Again the C^*-algebra of such a foliated region is the crossed product of the global product of certain trees of C^*-algebras with F_n's. These $\mathbb{R}$-trees are universal coverings of the corresponding *portions* of dual graphs and the F_n are free groups on n generators, n being the number of closed orbits, acting on these $\mathbb{R}$-trees freely

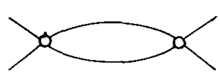
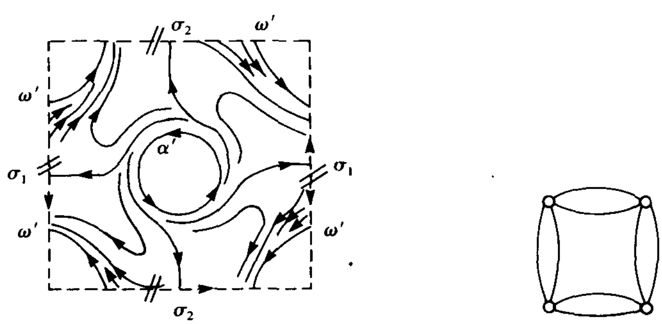


FIGURE 3.3



On $(T^2 - 2 \text{ discs})$

(a) (b)

FIGURE 3.4

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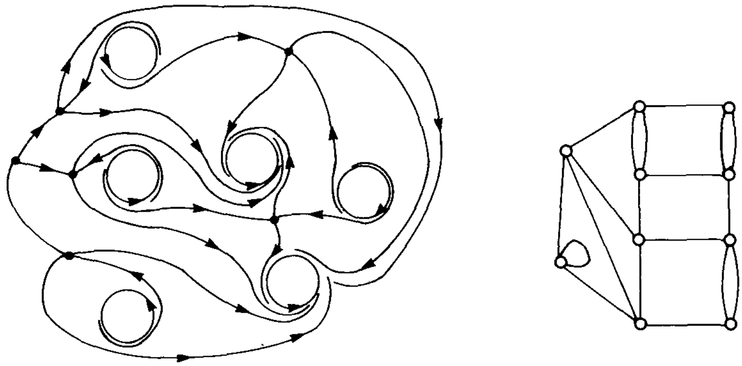


(a) (b)

FIGURE 3.5

and properly discontinuously everywhere except on the limit sets of the vertices. If we remove these limit sets, then the quotient graphs are exactly the portions of dual graphs as shown in (b) of both figures 3.4 and 3.5.

4. *The characterization of dual graphs of Morse-Smale flows*

By a *simple n-cycle*, or just an *n-cycle*, in a graph G, we mean a sequence of distinct edges of the form $\{(e_1, \bar{e}_1), (e_2, \bar{e}_2), \ldots, (e_n, \bar{e}_n)\}$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \ldots, n$ and $e_{n+1} = e_1$. An *orientation* on an n-cycle is an assignment of one of the two subsequences $\{e_1, \ldots, e_n\}$ or $\{\bar{e}_{n+1}, \ldots, \bar{e}_1\}$.

The verification of the following lemma is straightforward:

LEMMA 4.1. *Let $\hat{G} = (V, E)$ be a graph, with* Inc $(E_v) = 4$ *and a partition of E_v into two associated pairs for every $v \in V$. Then this decomposes E into a disjoint union of cycles of the form $\{(e_1, \bar{e}_1), \ldots, (e_n, \bar{e}_n)\}$ where the inverse $\bar{e}_i$ of e_i is in an associated pair with e_{i+1}, for $i = 1, \ldots, n$, and $e_{n+1} = e_1$.*

The dual graphs of Morse flows are characterized by

THEOREM 4.2. *Let $\hat{G} = (V, E)$ be a graph, with* Inc $(E_v) = 4$ *and a partition of E_v into two pairs for each $v \in V$. Then $\hat{G}$ is the dual graph of a Morse flow on a closed surface of genus g if and only if the following conditions* (1), (2) *and* (3) *hold*:

(1) *The partition of E decomposes* (*Lemma* 4.1) *E into a disjoint union of* 4*-cycles.*

Before stating condition (2), we note that if we colour each 4-cycle by 'u' and 's' such that the opposite edges have the same colour, then there is a new partition of E_v by the two colours, and again by Lemma 4.1, E is the disjoint union of s-cycles and u-cycles.

(2) There is such a colouring and a direction on G which induces an orientation on each 4-cycle, each u-cycle, and each s-cycle.

(3) Let p, q, r be the numbers of the u-cycles, the s-cycles, and the 4-cycles. Then $p + q - r = 2 - 2g$.

Proof. Suppose that $\hat{G} = (V, E)$ is the dual graph of a Morse flow $\mathcal{F}$ on an orientable closed surface of genus g. We fix an embedding ι of $\hat{G}$ into Σ by Theorem 2.9. The four trajectories connecting each saddle σ of $\mathcal{F}$ correspond to a 4-cycle $Q(\sigma)$ in $\hat{G}$. Since there are no saddle connections, this gives a canonical decomposition of E. The embedded 4-cycle $\iota Q(\sigma)$ bounds a disc containing σ. An orientation on M determines an orientation at each saddle, thus an orientation on the 4-cycle $Q(\sigma)$. Mark each geometric edge of $\hat{G}(\mathcal{F})$ by s or u according to whether the corresponding trajectory is a stable or unstable manifold of σ. We need to show that the directions on the 4-cycles induce a consistent orientation on each u-cycle and s-cycle. All the edges corresponding to the trajectories connecting to a given source α form a u-cycle $Q(\alpha)$, and $\iota Q(\alpha)$ also bounds a disc containing α (cf. the proof of Theorem 2.9). Similarly there is also a 1–1 correspondence between the sinks and the s-cycles. The orientation on M induces an orientation on each α and ω, which assigns a direction on each u-edge and s-edge. It is easy to see that for each s-edge a (u-edge b) this direction *always* conflicts with the direction assigned by the 4-cycle containing a (respectively b) (figure 4.1).

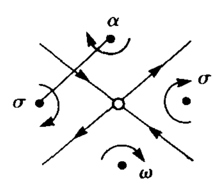


FIGURE 4.1

Thus the orientations on all the 4-cycles always specify a coherent orientation on each s-cycle and u-cycle. Note that there is a unique s-trajectory connecting the α and σ, intersecting $\iota(a)$. Condition (3) now follows from the Euler–Poincaré theorem.

Conversely, let $\hat{G}$ be such a dual graph satisfying (1), (2) and (3). Let $\hat{G}^*$ be $\hat{G}$ with such a fixed colouring and an orientation. We show that $\hat{G} = \hat{G}(\mathcal{F})$ for a Morse flow $\mathcal{F}$ on a closed surface M of genus g. Reversing our construction of $\hat{G}(\mathcal{F})$ from $(M, \mathcal{F})$ yields a direct proof. The idea is that to each 4-cycle, s-cycle and u-cycle, we associate a 2-cell with the cycle as its boundary. Then we glue these 2-cells together 'along' $\hat{G}$, namely, we construct a cellular 2-complex CM with $\hat{G}$ as its 1-skeleton. It is easy to verify that the space M of CM is an orientable closed surface. There is an obvious way to 'assign' a Morse flow $\mathcal{F}$ on M, i.e. put an $\alpha(\omega, \sigma)$ in each 2-cell bounded by an s-cycle (u-cycle, 4-cycle) and connecting them by s-trajectories (u-trajectories) crossing the s-edges (u-edges) of $\hat{G}$. Finally we check the axioms of Morse–Smale flows (§ 1).

However, here we give another detailed proof in order to exhibit the connection of this classification with Peixoto's work. We shall show how to construct a Peixoto graph from a dual graph and see how the rather artificial-sounding axioms (§ 1) are easily satisfied.

A flow diagram G is constructed from $\hat{G}$ as above, i.e. for every s-cycle, 4-cycle, or u-cycle, we associate respectively an α, σ or ω. There is a directed edge $\check{e}$ from α (resp. σ) to σ (resp. ω) for each geometric edge e in the intersection of the corresponding s-cycle (u-cycle) with the 4-cycle. For convenience we shall denote by $\hat{e}$ the corresponding edge in the coloured dual graph $\hat{G}^*$ for each (geometric) edge e in the Peixoto graph G^*.

We associate a distinguished set with each vertex v in $\hat{G}$. Let $a = s$-in, $b = s$-out, $c = u$-out, and $d = u$-in be the four edges incident to V (figure 4.2). There are three cases. (a) There are four vertices connected to v. The distinguished set is $[\check{a}, \check{b}; \check{c}, \check{d}]$ (in the bracket we identify an edge with its inverse). (b) There is an s-loop at v. Then $a = \bar{b}$, and the distinguished set is $[\check{a}; \check{c}, \check{d}]$. (c) There is a u-loop at v. Then $c = \bar{d}$ and the distinguished set is $[\check{a}, \check{b}; \check{c}]$. We note that two edges e_1, e_2 in G^* are associated if and only if $\hat{e}_1$ and $\hat{e}_2$ are consecutive edges in $\hat{G}^*$ with *distinct* colours.

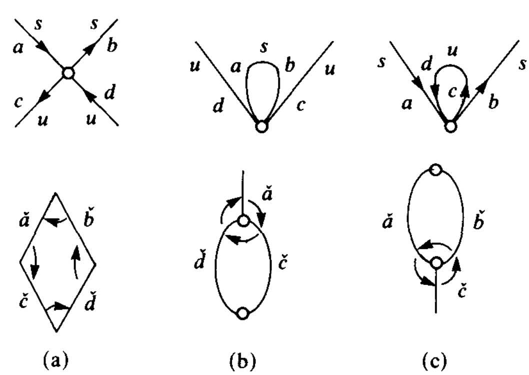


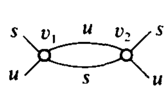
FIGURE 4.2

Now we see that the axioms (1.1)–(1.5) of a Peixoto graph in § 1 translate to the natural properties of the dual graph $\hat{G}^*$, and the verification of these axioms is straightforward:

(1.1). An edge e in G^* belongs to exactly two distinguished sets if and only if $\hat{e}$ connects distinct vertices.

(1.2) e is the first (or third) edge of type 2 (or type 3) distinguished set if and only if $\hat{e}$ is an s-loop (or u-loop).

(1.3) With a moment of reflection one checks that



cannot be included in a 4-cycle in G^*.

(1.4) A vertex in G^* corresponds to either an s-cycle, a u-cycle or a 4-cycle.

(1.5) For each pair of edges e, e' in the same distinguished set, the fixed orientation in $\hat{G}^*$ specifies a rotation from e to e' or e' to e, depending on whether

$\xrightarrow{\hat{e}}\xrightarrow{e'}$ or $\xrightarrow{\hat{e}'}\xrightarrow{\hat{e}}$ in $\hat{G}^*$. Now if e, e' have the same colour (u or s), we define the rotation in G^* by reversing the rotation, but if e, e' have different colours, then we take the sense of rotation defined as above. One checks easily that for every type of distinguished set (see figure 4.2), such a rotation defines a coherent orientation of the distinguished set, independent of which pair of edges e, e' was chosen. Therefore, the axiom (1.5) *follows from the condition* (2). □

The following definition is motivated by the proof of Theorem 4.2.

Definition 4.3. Let $(M, \mathscr{F})$ be a Morse flow. The *coloured dual graph* $\hat{G}^*(\mathscr{F})$ of $(M, \mathscr{F})$ is the dual graph $\hat{G}(\mathscr{F})$ (Definition 2.4) with a colouring (on the edges) and an orientation. Here an edge is 's' ('u') if and only if the corresponding orbit in $\mathscr{F}$ is contained in the stable (unstable) manifold of the saddle. The orientation on $\hat{G}(\mathscr{F})$ is given by the orientations assigned to all the 4-cycles given by an orientation on M.

Since there are exactly two possible orientations on M, there are also two on $\hat{G}(\mathscr{F})$. Two coloured dual graphs are *isomorphic* if there is a graph isomorphism from one to the other preserving the colouring and either preserving or reversing all the orientations on the edges. It follows from the proof of Theorem 4.2 that we have

THEOREM 4.4. *Two Morse flows* $(M, \mathscr{F})$ *and* $(M, \mathscr{F}')$ *are* topologically conjugate *if and only if their coloured dual graphs are isomorphic.*

Definition 4.5. A (*connected*) *coloured dual graph* is either a single vertex, or a (connected) digraph $\hat{G}^* = (E, V)$, $\text{Inc}(x) = 4$ for all $x \in V$, with two colours u and s on the edges E such that

(1) E is a disjoint union of 4-cycles. In each 4-cycle the opposite edges have the same colour.

(2) The direction on $\hat{G}^*$ induces an orientation on each 4-cycle, u-cycle and s-cycle.

The colouring and orientation induce a partition of E_x into two pairs, for $x \in V$, such that each pair of edges have the opposite colours but coherent orientation. A (connected) *dual graph* is a (connected) coloured dual graph stripped of the colouring and orientation, but retaining the induced partition of the edges.

Let p, q, r be the numbers of the u-cycles, the s-cycles, and the 4-cycles, respectively. Then $\chi(\hat{G}^*) = p + q - r$ is called the *Euler number* of $\hat{G}^*$. Recall that the dual graph of the north–south flow on S^2 consists of a single vertex. We define $\chi(\text{point}) = 2$.

THEOREM 4.6. *A coloured dual graph* $\hat{G}^*$ *has form* $\hat{G}^*(\mathscr{F})$ *for a Morse flow* $(M, \mathscr{F})$ *if and only if* $\chi(\hat{G}^*) = \chi(M)$.

Theorems 4.4 and 4.6 provide a simple classification of the conjugacy classes of Morse flows. Let x be a vertex in a dual graph. Let a, b, c, d be the four edges in E_x where a, c and b, d are the two associated pairs (figure 4.3(a), figure 4.2). Then around x there are exactly eight possibilities for colouring and orientations. Figure 4.3(b) shows four of them. The other four have the same colouring but the opposite orientation.

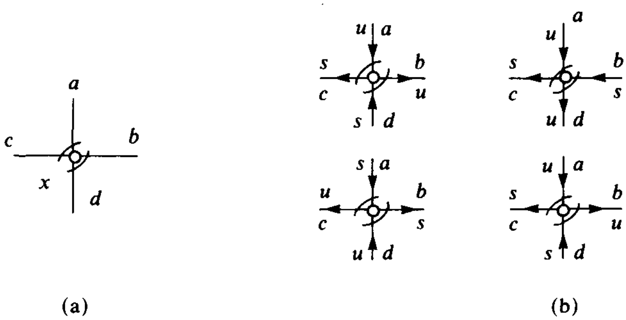


FIGURE 4.3

Example 4.7. The two Morse flows illustrated in figures 2.4 and 2.6 are not conjugate so their coloured dual graphs (figure 4.4) are not isomorphic. But their dual graphs are the same.

It is easy to construct two nonisomorphic dual graphs, both having one u-cycle, one s-cycle and $4g$ vertices, for $g \geq 2$. Thus for a surface with genus $g \geq 2$, there are polar flows with nonisomorphic C^*-algebras. Of course, then the polar flows are nonconjugate.

It is not so easy to find nonconjugate polar flows on the same surface but with isomorphic C^*-algebras. In fact one can show that for two coloured dual graphs with one u-cycle, one s-cycle and fewer than 12 vertices, they are isomorphic as dual graphs if and only if they are isomorphic as coloured dual graphs. A 'minimal' counterexample is given below.

Example 4.8. Figures 4.5 and 4.6 show two non-isomorphic coloured dual graphs which are isomorphic as dual graphs. In (a) of both figures, circles are the s-cycles, while in (b) the circles are the u-cycles. By Proposition 2.10 and Corollary 2.11, for a polar Morse flow, we have

$$g = \tfrac{1}{4}(n-1) = \tfrac{1}{4}\# V.$$

So the flows are on a closed surface with genus 4.

(1) The C^*-algebras of two flows are isomorphic. One checks that a cyclic bijection of vertices as shown from figure 4.5(a) to figure 4.6(a) induces an isomorphism of the dual graphs. A graph isomorphism preserves the partition on the edges, if and only if the isomorphism takes any 4-cycle in the first partition to a 4-cycle in the second. These 4-cycles in both dual graphs (a) are marked by c_i, $i = 1, \ldots, 8$, under the one-to-one correspondence induced by the graph isomorphism.

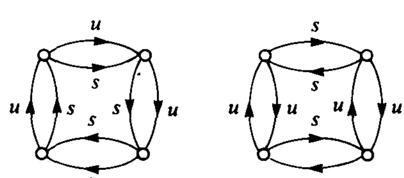


FIGURE 4.4

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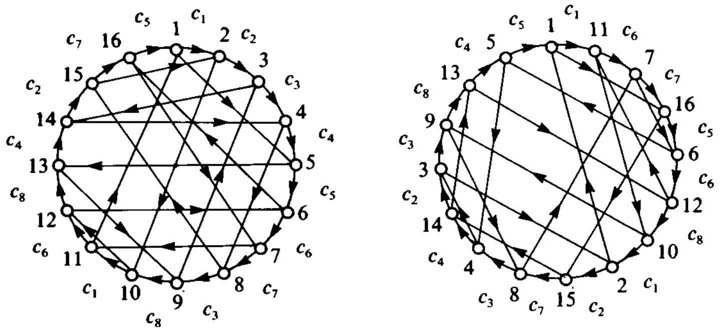
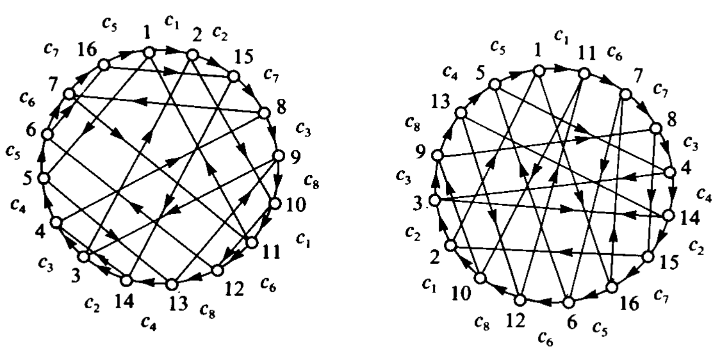


(a) (b)

FIGURE 4.5



(a) (b)

FIGURE 4.6

(2) The two foliations induced by the two Morse flows are *not* conjugate. Take the s-cycle of figure 4.6, i.e., the circle of (a), which cannot be mapped to either the s-cycle (the circle in (a)) or the u-cycle (the circle in (b)) of figure 4.5, under any graph isomorphism.

When we draw each 4-cycle C_i with a distinct colour, then in figures 4.5 and 4.6, both parts (a) (both parts (b)) give the Heegard diagram at the source (the sink) (see § 5). Thus the decomposition into 4-cycles of a dual graph generalizes the Heegard splitting of two-manifolds (cf, [**F**]) from polar flows (i.e., with exactly one source and one sink) to arbitrary Morse flows.

Remark 4.9. By the argument in the proof of Theorem 2.9 and the basic facts of coloured dual graphs, one can generalize the result in [**Lev2**] and [**Mar**], about 'pairs of pants decompositions', to the more general situation, where sources (sinks) are present. Thus one gets 'caps and pants decompositions'. We do not elaborate, but indicate the procedure:

(1) Cut the surface along the small circles around all sources and sinks, and take out the 'caps'.
(2) In the embedded coloured dual graph constructed in the proof of Theorem 2.9, delete the edges *alternately* in both colour and orientation.

(3) Then the remainder of the graph is a collection of circles. Together with the small circles in (1), they form the transversal which provides the 'caps and pants decompositions'.

We illustrate it, again by the simplest example (figures 2.3 and 4.7).

Now we define and characterize the dual graphs for general Morse-Smale flows $(M, \mathcal{F})$ with closed orbits. Following the notation of Peixoto, we denote an attracting (repelling) periodic orbit of $\mathcal{F}$ by α_i^1 (ω_i^1), which is a one-dimensional source (sink) of $\mathcal{F}$. Removing all the closed orbits from $(M, \mathcal{F})$, we get a disjoint union $\bigcup_i (M_i, \mathcal{F}_i)$ of Morse flows. Let $\hat{G}^*(\mathcal{F}_i)$ be the coloured dual graph of $(M_i, \mathcal{F}_i)$ defined as in § 2. Every attractor α_j^1 (repeller ω_j^1) corresponds to a *pair* of sources (sinks) in $\bigcup (M_i, \mathcal{F}_i)$. Let their coloured dual graphs be $\hat{G}^*(\mathcal{F}_i)$, defined as in § 2. Recall that a source (sink) in $(M_i, \mathcal{F}_i)$ corresponds to an s-cycle (u-cycle) in $\hat{G}^*(\mathcal{F}_i)$. So the closed orbits in $(M, \mathcal{F})$ establish a *pairing P* between some s-cycles, and between some u-cycles of $\bigcup \hat{G}^*(\mathcal{F}_i)$.

Definition 4.10. The *coloured dual graph* $\hat{G}^*(\mathcal{F})$ of a Morse-Smale flow $(M, \mathcal{F})$ is the pair $(\bigcup_i \hat{G}^*(\mathcal{F}_i), P)$, namely, the disjoint union of the $\hat{G}^*(\mathcal{F}_i)$, and the pairing P. Similarly the *dual graph* $\hat{G}(\mathcal{F})$ is $(\bigcup \hat{G}(\mathcal{F}_i), P)$, the disjoint union of the dual graphs of $(M_i, \mathcal{F}_i)$ with the same pairing P among some cycles in $\hat{G}(\mathcal{F}_i)$.

Here we recall that the dual graph $\hat{G}(\mathcal{F})$ of a Morse flow $(M, \mathcal{F})$ can be obtained from the coloured dual graph $\hat{G}^*(\mathcal{F})$ by ignoring the colouring and orientation, but retaining the partition on E_x (determined by the colouring and orientation) for each vertex x, namely, the partition of E_x into two pairs of edges (figure 4.3). In the illustration of $\hat{G}^*(\mathcal{F})$, we use a two-way arrow connecting a pair of two s-cycles (u-cycles) to represent a pair in P.

Example 4.11. We denote the (coloured) dual graph of the north-south polar flow on S^2 by a point. Then the Morse-Smale flow on a torus with two ω, two σ, one α^1 (figure 4.5(a)) has the coloured dual graph figure 4.8(b).

Definition 4.12. A (*general*) *coloured dual graph* $\hat{G}^*$ consists of a collection $\bigcup_{i=1}^n \hat{G}_i^*$ of connected coloured dual graphs and some pairings $\mathcal{P}$ between some of the u-cycles, the s-cycles, and the one-point graphs (no pairing between a u-cycle and an s-cycle), such that there are some pairings $P_i \in \mathcal{P}$ involving cycles in $\hat{G}_i^*$ and $\hat{G}_{i+1}^*$, for each $i = 1, \ldots, n$. The *Euler number* $\chi(\hat{G}^*)$ is $\sum_{i=1}^n \chi(\hat{G}_i^*) - 2|\mathcal{P}|$ where $|\mathcal{P}|$ is the cardinality of $\mathcal{P}$.

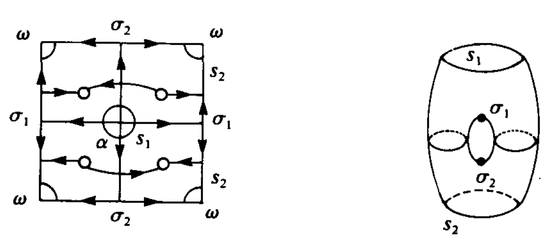


FIGURE 4.7

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(a) (b)

FIGURE 4.8

A *(general) dual graph* $(\hat{G}, \mathscr{P})$ is a (general) coloured dual graph $(\hat{G}^*, \mathscr{P})$ stripped of the colouring and orientation, but retaining the partition of edges and the pairing $\mathscr{P}$.

THEOREM 4.13. *A general coloured dual graph $\hat{G}^*$ has form $\hat{G}^*(\mathscr{F})$ for a Morse–Smale flow $(M, \mathscr{F})$ if and only if $\chi(\hat{G}^*) = \chi(M)$.*

We say two general coloured dual graphs $\hat{G}_j^* = (\bigcup_{i=1}^{n_j} \hat{G}_{j,i}^*, \mathscr{P}_j), j = 1, 2$, are *isomorphic* if and only if $n_1 = n_2$ and after renumbering, there is a family $(\phi_i)_{i=1,\dots,n}$ of isomorphisms of $\hat{G}_{1,i}^*$ with $\hat{G}_{2,i}^*$ identifying the pairing $\mathscr{P}_1$ with $\mathscr{P}_2$.

THEOREM 4.14. *Two general coloured dual graphs are isomorphic if and only if the two corresponding Morse–Smale flows are topologically conjugate.*

If we forget about hte 'colour', we obtain the complete invariants for the C^*-algebras.

THEOREM 4.15. *Two general dual graphs are isomorphic if and only if the two corresponding C^*-algebras of Morse–Smale flows are isomorphic.*

Proof. Suppose the two C^*-algebras $A_i = C^*(M_i, \mathscr{F}_i)$, $i = 1, 2$ are isomorphic. Then the two spectra $\hat{A}_i$ are homeomorphic. We claim that every closed orbit in $\mathscr{F}_i$ corresponds to a closed subset in $\hat{A}_i$ homeomorphic to a circle, which is contained in the closure of any single point in two other 'nearby' circles. To see this, notice that the union of the unstable manifold of a repeller (or the stable manifold of an attractor) with the closed orbit is always a foliated open annulus as shown in figure 4.9 (cf. figure 3.1). Its C^*-algebra A_a is isomorphic to $C_0(-1, 1) \times_\alpha \mathbb{Z}$, where the $\mathbb{Z}$-action is given by a homeomorphism of $(-1, 1)$ having 0 as the unique fixed point (of either expansion or contraction). Clearly any such homeomorphism gives rise to a C^*-algebra unique up to isomorphism. The spectrum $\hat{A}_a$ is described as earlier (figure 4.10).

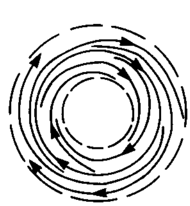

FIGURE 4.9

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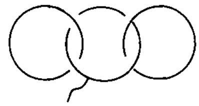

FIGURE 4.10

Eliminating these limit cycles from the spectrum $\hat{A}_i$, then one gets a T_1-space. Thus a homeomorphism between $\hat{A}_1$ and $\hat{A}_2$ establishes a bijection between $\mathcal{P}_1$ and $\mathcal{P}_2$. After removing all the closed orbits in $\mathcal{P}_1$ and $\mathcal{P}_2$, we get foliated open submanifolds $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$. Their C^*-algebras are two ideals I_1 and I_2 of A_1 and A_2 and the restriction to I_1 of an isomorphism ϕ from A_1 to A_2 must have image I_2, due to the consideration of the spectra as above. Therefore $\phi|I_1$ induces a family of isomorphisms from the dual graphs of the Morse flows which are the components of $\mathcal{F}_1$ to those of $\mathcal{F}_2$.

Since the general dual graph $\hat{G} = (\bigcup \hat{G}_j, \mathcal{P})$ provides all the information needed to construct the C^*-algebra of a Morse-Smale flow (which has $\hat{G}$ as its dual graph), the 'only if' part is clear. □

The above theorem is interesting because the underlying manifolds of the flows are ignored as a whole. In fact, we know that the dual graph $\hat{G}(\mathcal{F})$ of flow $(M, \mathcal{F})$ does not even determine the homeomorphism type of M (Example 4.8).

The C^*-algebra of a Morse-Smale flow is always GCR. In general the length of a composition series with continuous trace factors is 3.

COROLLARY 4.16. *A Morse-Smale flow $(M, \mathcal{F})$ has no closed orbits if and only if the C^*-algebra $C^*(M, \mathcal{F})$ is CCR.*

Proof. From the proof of Theorem 4.14, the spectrum of $C^*(M, \mathcal{F})$ is T_1 if and only if $\mathcal{F}$ has no closed orbits. The C^*-algebra $C^*(M, \mathcal{F})$ is isomorphic to the C^*-algebra of 1-parameter transformation group (Proposition 1.11 **[W1]**). By **[Wil]** or **[Gt]**, the corollary follows. □

5. *Intersection forms given by the KK-intersection product*

We have obtained a complete classification of the C^*-algebras of all the structurally stable Morse-Smale flows on closed two-manifolds in terms of dual graphs (Theorems 4.2, 4.13, 4.15). Recall that the homeomorphism types of closed two-manifolds and simply-connected four-manifolds are classified by symmetric intersection forms over homology. In this section we interpret the combinatorial invariants in our classification results in terms of the familiar KK-invariant, developed by Brown-Douglas-Fillmore and Kasparov. Then we point out an analogy between the classification result for C^*-algebras and that for low-dimensional manifolds: the C^*-algebra of a Morse flow naturally defines a symmetric intersection form over K-homology (Theorem 5.14). Moreover, the isometry equivalence class of the nonsymmetric intersection matrices (Definition 5.8) classify the isomorphism class of the C^*-algebra (Theorem 5.11).

It is however an intriguing question how far the concept of dimension of manifolds can be carried over to C^*-algebras. The topological stable rank introduced by M.

Rieffel can be regarded as such a generalization [**R2**]. For a compact orientable differentiable manifold, its dimension equals its cohomological dimension. For general C^*-algebras, such a satisfactory concept of dimension cannot be found if it is to be invariant under strong Morita equivalence, because it has been known for quite a while that there is no nontrivial $\mathbb{Z}$-graded Morita-invariant cohomology theory on all separable C^*-algebras which is not related to K-theory. In fact, such a functor which is half exact, stable and homotopy-invariant, has to satisfy Bott periodicity [§ 4, **Cun**]. Therefore, it seems not just a sheer coincidence that any *stable* C^*-algebra has topological stable rank either 1 or 2 (Theorem 6.4 of [**R2**]).

Hopefully, however, for sufficiently general noncommutative *smooth* manifolds, the concept of 'dimensions' may be (and should be) eventually defined and tied up with the associated 'total' smooth structures. It seems that again C^*-algebras of smooth foliations serve as good candidates for such study.

The intersection form of a surface is given by the linking matrix associated to a polar flow. A *polar flow* is the gradient flow of a 'perfect' Morse function. Let M be a closed two-manifold with genus g. Then a polar flow on M has $2g$ saddles. For each saddle σ, there is a stable cycle (unstable cycle) consisting of the union of the stable (unstable) manifold of σ, the source α (the sink ω), and σ. The $2g$ stable (unstable) cycles generate $\pi_1(M)$. Let $C_s(C_u)$ be a small circle around the source (the sink) transversal to the flow. The intersection of C_s with the $2g$ stable (unstable) cycles consists of $2g$ pairs of points. The circle $C_s(C_u)$ along with the $2g$ pairs of coloured points will be called the (dual) *Heegard diagram of the flow* $\mathcal{F}$ (Fleitas [**F**], p. 172, whose analogue for four-dimensional manifolds is known as a *Heegard splitting*), again denoted by $C_s(C_u)$. Two polar Morse–Smale flows are isomorphic if and only if the two Heegard diagrams are isomorphic in the obvious sense.

Let $e_1, \ldots, e_{2g}$ be the stable cycles. For $i \neq j$ the linking number $\langle e_i, e_j\rangle \in \mathbb{Z}_2$, is defined as follows.

Let $\langle e_i, e_j\rangle = 1$ if the corresponding S^0 (two pairs of points) are linked and let $\langle e_i, e_j\rangle = 0$ if the two S^0 are unlinked (figures 5.1). For simplicity, we also denote by e_i both the S^0 and the element in $H_1(M, \mathbb{Z}_2)$ represented by the cycle e_i for all i.

Let $\langle e_i, e_i\rangle = 0$, for all i. The $2g \times 2g$ (skew) symmetric matrix $L_s = (\langle e_i, e_j\rangle)$ is the *linking matrix* of (e_i). Because of Poincaré duality, L_s is nondegenerate. Note that $\langle e_i, e_j\rangle$ is the same as the intersection number of the two closed curves e_i and e_j (see p. 1, [**L**]), and the linking matrix represents the intersection form on $H_1(M, \mathbb{Z}_2)$.

Example 5.1. The 'stable' and 'unstable' Heegard diagrams of the polar flow (a) are shown in (b), (c) of figure 5.2.

Notice that both of the Heegard diagrams have the same linking matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

However, this is only a coincidence. First an elementary observation.

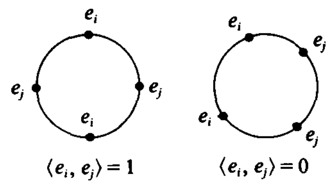

FIGURE 5.1

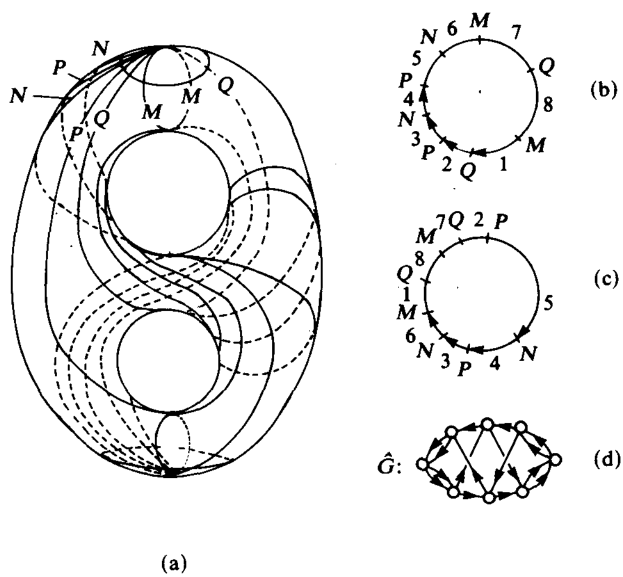

FIGURE 5.2

LEMMA 5.2. *Let M be a vector space over a field R. Let* $\langle\ ,\ \rangle$ *be a nondegenerate symmetric bilinear form defined on M. Let* $\{e_1, \ldots, e_n\}$ *be a basis of M, with matrix* $L_e = (\langle e_i, e_j\rangle)$. *Suppose* $\{f_1, \ldots, f_n\}$ *is another basis of M given by* $e_j = \Sigma\langle e_i, e_j\rangle f_i$. *Then the associated matrix* $L_f = (\langle f_i, f_j\rangle)$ *is the inverse of* L_e.

Proof. Exercise.

COROLLARY 5.3. *Let M be a closed orientable 2n-manifold, admitting a polar flow* $\mathcal{F}$ *on M with only saddles of Morse index n. Let* $e_1, \ldots, e_m$ *(respectively,* $f_1, \ldots, f_m$*) be the basis of* $H_n(M, \mathbb{Z})$ *represented by the stable (respectively, unstable) cocycles. Then the linking matrix* $L_u = (\langle f_i, f_j\rangle)$ *is the inverse of the linking matrix* $L_s = (\langle e_i, e_j\rangle)$.

Example 5.4. Figure 5.3 illustrates another polar Morse flow (a) with the Heegard diagrams at the source and the sink (b), and their linking matrices (c).

For closed surfaces, the symmetric bilinear forms are all even and the diagonals of L_s and L_u are zero. Corollary 5.3 will be used on four-manifolds [**H–W**], where the links in S^3 have self-linking numbers, which are the entries on the diagonals of linking matrices L_u and L_s. They are usually nonzero.

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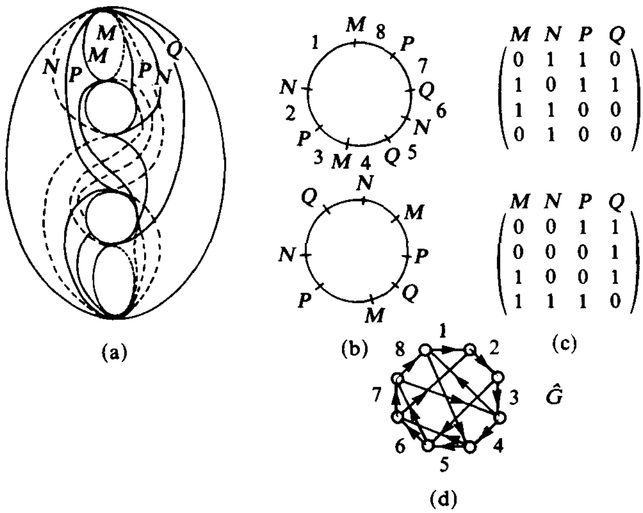

FIGURE 5.3

Example 5.5. The linking matrix L_s does not determine the Morse–Smale flow up to conjugacy for two-manifolds with genus $g \geq 2$.

Figure 5.4 is a Heegard diagram of a Morse–Smale flow. Its linking matrix is

$$L_s = \begin{bmatrix} 0 & 1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{bmatrix}.$$

By switching one point of e_4 (as the arrow indicates), we can get a nonisomorphic Heegard diagram with the same linking matrix.

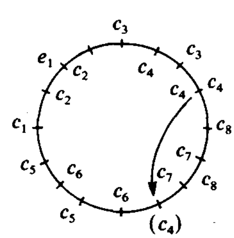

FIGURE 5.4

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The equivalence classes of linking matrices, that is, the equivalence classes of intersection forms over $H_1(M, \mathbb{Z}_2)$, classify the two-manifolds. Before discussing the analogy for C^*-algebras, we record some facts about the dual graphs.

LEMMA 5.6. *If two edges e_1, e_2 in a dual graph have the same origin $x = o(e_1) = o(e_2)$ and the same terminus $t(e_1) = t(e_2)$, then the two edges have opposite grading in E_x. (In particular, if $x = o(e) = t(e)$, then e and $\bar{e}$ have opposite grading.)*

Proof. Assume that $o(e_1)$ and $t(e_1)$ are distinct vertices. Suppose e_1 and e_2 have the same grading. Then the two geometric edges $|e_1|$ and $|e_2|$ have distinct colours (figure 4.3). However, we already know that such a loop with two distinct coloured directed edges cannot be included in a 4-cycle in a dual graph (cf. (1.3) in the proof of Theorem 4.2).

Assume that $o(e) = t(e)$. Then with the same colour, the two edges e and $\bar{e}$ have different grading. □

Recall that given a graph G, a *symmetric adjacency matrix* is the matrix whose ij entry is the number of geometric edges connecting the ith and jth vertices, for a fixed labelling of vertices. Let $\mathbb{P}_n$ be the group of matrices generated by elementary $n \times n$ matrices E_{ij}, which are obtained by permuting the ith row and the jth row of the identity matrix. Let $\mathbb{P} = \bigcup_n \mathbb{P}_n$. Further, we set $\bar{\mathbb{P}}_n$ to be the group of $n \times n$ matrices generated by the elementary matrices E_{ij}, and $E_i = \operatorname{diag}(1, \ldots, -1, \ldots, 1)$ and $\bar{\mathbb{P}} = \bigcup_n \bar{\mathbb{P}}_n$. The groups $\mathbb{P}_n$ and $\bar{\mathbb{P}}_n$ are isomorphic to the Weyl groups A_{n-1} and B_n $(=C_n)$ respectively, for $n = 2, 3, 4, \ldots$. The group $\bar{\mathbb{P}}_n$ is isomorphic to $O(n, \mathbb{Z})$ $(= O(n) \cap GL(N, \mathbb{Z}))$.

For a graph G with n vertices and m edges, a *nonsymmetric adjacency matrix* $N(G)$ is the $n \times 2m$ integer matrix whose ijth entry n_{ij} is the number of ends of the jth geometric edge incident to the ith vertex. For a dual graph $\hat{G}$, a (*signed*) *nonsymmetric adjacency matrix* $N(\hat{G})$ is the $n \times 2n$ integer matrix whose ij entry $\hat{n}_{ij}$ is 0 if $n_{ij} = 0$ or 2, otherwise the entry $\hat{n}_{ij}$ is $(-1)^{\deg e_j}$, where $\deg e_j$ is the degree of the end of the jth geometric edge incident to the ith vertex.

Two symmetric adjacency matrices M_1 and M_2 are *equivalent* if there is some $P \in \mathbb{P}$ such that $M_1 = PM_2P^{-1}$. Two nonsymmetric adjacency matrices N_1 and N_2 are *equivalent* if there is some $\bar{P}_1 \in \bar{\mathbb{P}}$ and $P_2 \in \mathbb{P}$ such that $N_1 = \bar{P}_1 N_2 P_2$.

Let (G, A) be a graph of C^*-algebras (Definition 2.1). We set

$$I(G) = \{(a_x) \in C^*(G, A) \mid \pi_{e,x}(a_x)\} = 0, \quad \forall e \in E$$

and $Q(G) = \prod_{\tilde{e} \subset \tilde{E}} A_{\tilde{e}}$, where $\tilde{E}$ is the set of geometric edges of G. Then we have a short exact sequence

$$0 \to I(G) \to C^*(G, A) \to Q(G) \to 0,$$

defining canonically an element $\alpha_G \in \operatorname{Ext}(Q(G), I(G))$, which is isomorphic to $KK^1(Q(G), I(G))$, since the algebras are separable and nuclear. When G is a dual graph, then $G = \hat{G}(\mathcal{F})$ for some Morse flow $(M, \mathcal{F})$ and we write $\alpha_{\mathcal{F}} = \alpha_G$. It is easy to see that $I(G) \simeq C_0(0, 1) \otimes \mathcal{K}^n$ and $Q(G) \simeq \mathcal{K}^{2n}$, where $n = \# V$. By the universal coefficient theorem [**R–S**] we have

$$KK^1(Q(G), I(G)) \simeq \operatorname{Hom}(K_0(Q(G)), K_1(I(G)))$$

where the isomorphism is given by the KK-intersection product. Thus we can represent $\alpha_{\mathcal{F}}$ by a matrix $A_{\mathcal{F}}$ in $\mathrm{Hom}(\mathbb{Z}^{2n}, \mathbb{Z}^n)$ for a chosen canonical basis of $K_1(C_0(0,1)\otimes\mathcal{K}^n)\simeq\mathbb{Z}^n$ and $K_0(\mathcal{K}^{2n})\simeq\mathbb{Z}^{2n}$.

LEMMA 5.7. *For each nonsymmetric adjacency matrix $A_{\mathcal{F}}$ of a dual graph $\hat{G}(\mathcal{F})$, there is a canonical basis of $K_0(Q(G))$ and $K_1(I(G))$, such that the matrix $A_{\mathcal{F}}$ in* $\mathrm{Hom}(\mathbb{Z}^{2n}, \mathbb{Z}^n)$ *represents the KK^1-element associated with $C^*(M, \mathcal{F})$.*

Proof. The point is to show that it is possible to choose an orientation of the basis so that the matrix comes out with the given signs. It is most convenient to consider the C^*-algebra $C^*(\hat{G}(\mathcal{F}))$ of the graph as $C^*(M, \mathcal{F})$. Since we are considering only Morse flows, there is only one 'cell' A_v given by Theorem 2.3. Fix some 'compatible' basis for $K_1(C_0(0,1)\otimes\mathcal{K})$ and $K_0(\mathcal{K}^4)$ so that the exponential map takes both 'positive' generators at $K_0(\mathcal{K})$ at the end '1' on $(0,1)$ to the positive generator of $K_1(C_0(0,1)\otimes\mathcal{K})$ while doing the opposite at the end '0'. Thus the extension associated to A_v yields the matrix $(-1,-1,1,1)$ for the KK^1-element, by the argument in Example 4.6 of [**W1**].

Fix a nonsymmetric adjacency matrix $A=(a_{ij})$. Choose the basis of $K_1(Q(G))$ and $K_1(I(G))$ as follows. Their orders are consistent with the labeling of $A_{\mathcal{F}}$. Their positive generators are the images of $(\tau_x)_*$, $x\in V$, of the positive generators of the K-theory associated with A_v given above. Here τ_x is the embedding of A_v into $C^*(\hat{G}(\mathcal{F}))$ such that the '0' end is mapped to deg 0 edges while the '1' end to deg 1 edges.

Now assume $\alpha_{\mathcal{F}}[(v_i])=a'_{ij}[u_i]$, where u_i, v_j correspond to the ith vertex x_i and the jth geometric edge $\tilde{e}_j$. So if x_i is not incident on $\tilde{e}_j$, then $a'_{ij}=0=a_{ij}$. Let $\tilde{e}_j=(e_j, \bar{e}_j)$. If only $e_j=E_{x_i}$, then $a'_{ij}=(-1)^{\deg e_j}=a_{ij}$. Similarly $a_{ij}=a'_{ij}$ if only $\bar{e}_j\in E_{x_i}$. Finally if both $e_j, \bar{e}_j\in E_{x_i}$, then by Lemma 5.6, $\deg e_j \deg \bar{e}_j=-1$ thus

$$a_{ij}=(-1)^{\deg e_j}+(-1)^{\deg \bar{e}_j}=0.$$

So $a_{ij}=a'_{ij}$ for all i, j and the map $\alpha_{\mathcal{F}}$ is represented by the matrix $A_{\mathcal{F}}$. □

There is some subtlety in both the statement and the proof of Lemma 5.7. In general a matrix $A_{\mathcal{F}}\in\mathrm{Hom}(\mathbb{Z}^{2n}, \mathbb{Z}^n)$ representing the KK^1-element $\alpha_{\mathcal{F}}$ is *not* a nonsymmetric adjacency matrix of the dual graph. In the proof, the possibility of orienting the bases to attain the given signs in $A_{\mathcal{F}}$ is not automatic. One cannot change the signs of arbitrary specific entries only by multiplying by matrices in $O(n, \mathbb{Z})$ in general!

Fix a canonical basis of $K^1(I(\hat{G}))$ as above. The canonical inner product $\langle\ ,\ \rangle$ on the lattice $\mathbb{Z}^n$ given by $\langle (n_i), (m_i)\rangle=\Sigma n_i m_i$ defines an inner product on $K^1(I(\hat{G}))$, denoted again by $\langle\ ,\ \rangle$.

Definition 5.8. For elements $u, v\in K^1(I(\hat{G}))$, the nonnegative symmetric bilinear form $f(u,v)=\langle u, A_{\mathcal{F}}A'_{\mathcal{F}}v\rangle$ is called a (*symmetric*) *intersection form* associated with $C^*(M, \mathcal{F})$. A nonsymmetric adjacency matrix $A_{\mathcal{F}}$ will be called the (*nonsymmetric*) *intersection matrix* associated with $C^*(M, \mathcal{F})$.

By Lemma 5.7, $A_{\mathcal{F}}$ is given by the KK-intersection product with the KK^1 element associated with $C^*(M, \mathcal{F})$.

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Definition 5.9. Let f_i, $i=1,2$, be two symmetric bilinear forms over $\mathbb{Z}^n$. Assume $f_i=\langle u, B_i v\rangle$ with respect to a canonical basis of $\mathbb{Z}^n$. We say the two forms f_1, f_2 are *isometrically equivalent* if the two matrices B_1 and B_2 are *isometrically congruent*, that is, $B_1=OB_2O^{-1}$ for some orthogonal matrix O in $O(n,\mathbb{Z})$.

Let A_1, A_2 be any (nonsymmetric) integer matrices. We say A_1 and A_2 are (*nonsymmetrically*) *isometrically equivalent*, if $A_1=OA_2P$, for some $O\in\bar{\mathbb{P}}$ and $P\in\mathbb{P}$. (Clearly then the two positive bilinear forms $\langle u, A_1A'v\rangle$ and $\langle u, A_2A'v\rangle$ are isometrically equivalent.)

From Lemma 5.7, Theorem 4.14 can be restated as

THEOREM 5.10. *Two C^*-algebras of Morse flows are isomorphic if and only if their KK^1-elements are the same, that is, their intersection matrices* (*Definition* 5.8) *are nonsymmetrically isometrically equivalent* (*Definition* 5.9).

There exist nonisomrophic C^*-algebras of Morse flows $\mathscr{F}_1$ and $\mathscr{F}_2$, with intersection matrices $A_{\mathscr{F}_1}$ and $A_{\mathscr{F}_2}$ such that $A_{\mathscr{F}_1}=A_{\mathscr{F}_2}O$, for some $O\in\bar{\mathscr{P}}$ (Example 5.16).

LEMMA 5.11. *The nonnegative symmetric bilinear form $f_{\mathscr{F}}$ over $\mathbb{Z}^n$ is up to isometric equivalence uniquely determined by the C^*-algebra $C^*(M,\mathscr{F})$ of the foliation with respect to a canonical basis of $K_1(I(G))$.*

Proof. A relabelling or reorientation of the basis of $K_1(I(G))$ and $K_0(Q(G))$ produces another nonsymmetric intersection matrix $A'=\bar{P}_1A\bar{P}_2$, where $\bar{P}_1, \bar{P}_2\in\bar{\mathbb{P}}$. The associated bilinear form

$$f'\langle u, v\rangle=\langle u, \bar{P}_1A_{\mathscr{F}}A'_{\mathscr{F}}\bar{P}_1v\rangle$$

is isometrically equivalent to $f_{\mathscr{F}}$ with respect to the canonical basis of $K_1(I(G))$. □

The converse question is how a bilinear form $\langle\ ,\ \rangle$, or rather its isometric class, determines the C^*-algebra $C^*(M,\mathscr{F})$. Recall that $C^*(M,\mathscr{F})\simeq C^*(\hat{G}(\mathscr{F}))$, where $\hat{G}(\mathscr{F})$ is the dual graph of the flow. We denote by $\bar{G}(\mathscr{F})$ the underlying graph of $\hat{G}(\mathscr{F})$, forgetting the grading. Then $\bar{G}(\mathscr{F})$ is determined by the isometric equivalence class of its adjacency matrices. Thus Proposition 5.12 asserts that the form $f_{\mathscr{F}}$ determines $\bar{G}(\mathscr{F})$, therefore 'almost' determines the C^*-algebra $C^*(M,\mathscr{F})$.

THEOREM 5.12. *Let $A_{\mathscr{F}}\in\mathrm{End}\,(\mathbb{Z}^{2n},\mathbb{Z}^n)$ represent the KK^1-element as above. Then*

$$M:=|A_{\mathscr{F}}A'_{\mathscr{F}}-\tfrac{3}{2}\,\mathrm{diag}\,(A_{\mathscr{F}}A'_{\mathscr{F}})+2I|$$

is an adjacency matrix of $\hat{G}(\mathscr{F})$. Here for an arbitrary matrix $B=(b_{ij})$, the matrix diag B *is* diag $(b_{11},\ldots,b_{nn})$ *and $|B|$ is the matrix whose i, j entries are the $|b_{ij}|$.*

Proof. Let $A_{\mathscr{F}}=(a_{ij})$ be as in the proof of Proposition 5.6 and $M=(m_{ij})$ be the adjacency matrix corresponding to the basis of $K_1(I(G))$. For $i\neq j$, the lth geometric edge connecting the vertices x_i and x_j if and only if $|a_{il}a_{jl}|=1$. There are at most two edges connecting any pair x_i and x_j. Assume that they are the lth and l'th geometric edges, then $a_{il}a_{il'}=a_{jl}a_{jl'}$ by Lemma 5.6. Therefore $|\sum a_{il}a_{jl}|$ is in any case the number of edges connecting x_i and x_j, so $|m_{ij}|=|\sum a_{il}a_{jl}|$.

A diagonal entry m_{ii} is the number of geometric edges with both ends being e_i. Clearly $m_{ii}=0$ or 1. By the discussion above, Inc $E_{x_i}=2m_{ii}+\sum_l a_{il}^2$. Since Inc $E_{x_i}=4$

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(Proposition 2.5),

$$m_{ii} = 2 - \tfrac{1}{2}\sum_l a_{il}^2. \qquad \square$$

THEOREM 5.13. *Let G be an underlying graph of a dual graph $\hat{G}$. Let $\mathscr{F}$ be a Morse flow that $\hat{G} = \hat{G}(\mathscr{F})$. Then the nonnegative symmetric form $f_{\mathscr{F}}\langle\cdot,\cdot\rangle$ uniquely determines the graph $\bar{G}$ and conversely the graph $\bar{G}$ uniquely determines the form $f_{\mathscr{F}}\langle\cdot,\cdot\rangle$.*

Proof. The first assertion is immediate from Theorem 5.12. Conversely let $M = (m_{ij})$ be an adjacency matrix of the graph G. Let $B = (b_{ij})$ be given by $b_{ij} = m_{ij}$ if $i \neq j$, but $b_{ii} = 4 - 2m_{ii}$. Then from the proof of Theorem 5.14, there is *a* matrix $A_{\mathscr{F}} \in \mathrm{Hom}(\mathbb{Z}^{2n}, \mathbb{Z}^n)$, such that $A_{\mathscr{F}}$ represents the KK^1-element given by $C^*(\hat{G}(\mathscr{F}))$ and $B = A'_{\mathscr{F}} A_{\mathscr{F}}$ defines the nonnegative symmetric form $f_{\mathscr{F}}\langle\cdot,\cdot\rangle$ over $K_1(I(\hat{G}))$. The basis $K_1(I(G))$ is chosen to correspond to the labelling of vertices given by M. $\square$

Example 5.14. For planar coloured dual graphs with either only one s-cycle or only one u-cycle (figure 5.5), one can show by induction on the number of vertices that the underlying graphs determine the dual graphs. Thus the symmetric forms $f_{\mathscr{F}}\langle\cdot,\cdot\rangle$ determine the C^*-algebras up to isomorphism.

The symmetric form $f_{\mathscr{F}}\langle\cdot,\cdot\rangle$ does not determine the grading of the graph $\hat{G}$. For polar flows on a closed surface of genus less than 7, one may check that two dual graphs are isomorphic if and only if the two underlying graphs are isomorphic.

Example 5.15. Figures 5.6 and 5.7 are two coloured dual graphs on the closed surface with genus $g = 8$. The two underlying *graphs* are isomorphic. However, the two underlying *dual graphs* are not isomorphic. One can show that the automorphism group of the graph is $\mathbb{Z}_2$, where the nontrivial element interchanges the two vertices P_1 and P_2, and does not carry the grading of one dual graph to the other.

Question. Do there exist nonisomorphic underlying graphs G_1, G_2 of some dual graphs $\hat{G}_1$ and $\hat{G}_2$, and some $A \in GL(2n, \mathbb{Z})$ such that the adjacency matrices are congruent: $M(G_1) = AM(G_2)A'$?

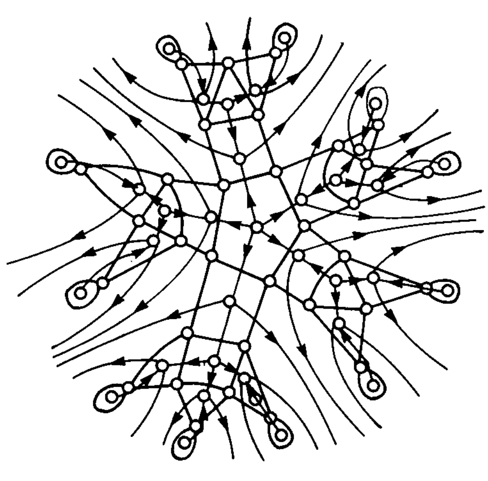

FIGURE 5.5

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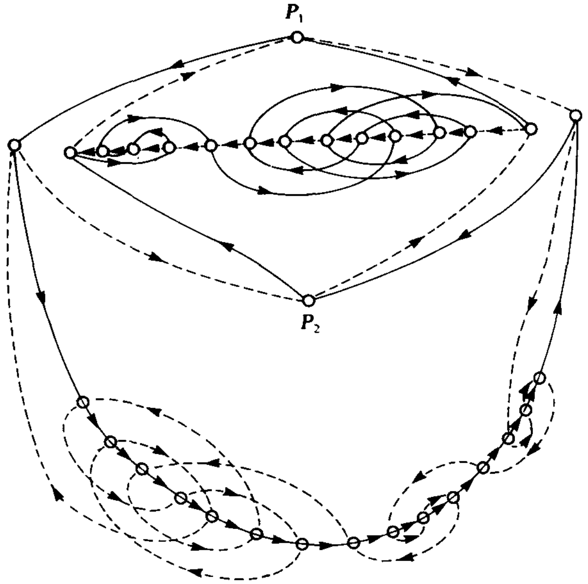

FIGURE 5.6

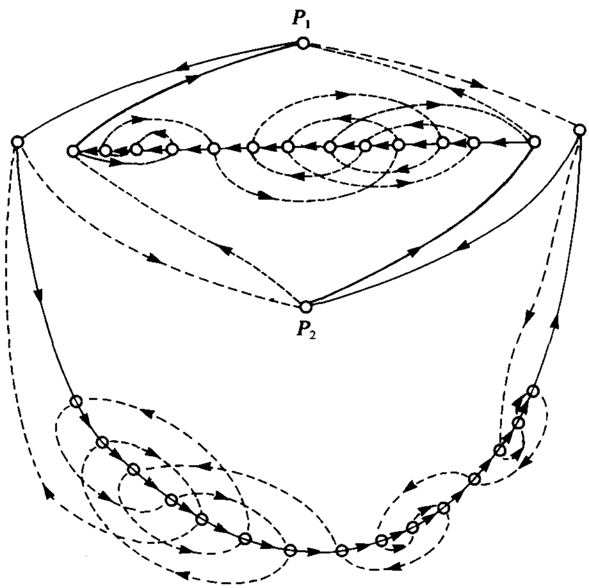

FIGURE 5.7

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