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RATIONAL HOMOTOPY TYPES WITH THE RATIONAL COHOMOLOGY ALGEBRA OF STUNTED COMPLEX PROJECTIVE SPACE

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ABSTRACT. We consider the number of spaces, up to rational homotopy equivalence, which have rational cohomology algebra isomorphic to that of stunted complex projective space $\mathbb{C}P^n/\mathbb{C}P^k$. Using a classification theory due to Schlessinger and Stasheff, we determine the number of rational homotopy types with rational comology algebra isomorphic to $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$, for any given n and k. The necessary computations make use of a spectral sequence introduced by the second named author.

1. **Introduction and notation.** A fundamental problem in homotopy theory is to classify all homotopy types that realize a given commutative graded algebra H, *i.e.*, that have cohomology algebra isomorphic to H. A general solution to this problem is unknown; indeed, it is difficult in general to decide whether or not there is any space that realizes H. In rational homotopy theory, however, the situation is more straightforward. Quillen showed, among other things, that rational homotopy types of simply connected spaces are in bijective correspondence with homotopy types of connected, rational, differential graded Lie algebras [Qu]. It follows from this that every one-connected, finite-type, commutative graded algebra over the rationals is realized by some simply connected space [Qu, p. 206]. Subsequently, Sullivan's concept of a minimal model was extended into the differential graded Lie algebra setting by a number of authors [B-L], [Ne]. A rational homotopy equivalence between two spaces corresponds to an isomorphism between the corresponding minimal models, and this makes the classification of rational homotopy types a more tractable problem. Solutions to the problem of classifying rational homotopy types that realize a given algebra H, using minimal models, have been given by Félix [Fe], Halperin and Stasheff [H-S], Lemaire and Sigrist [L-S] and Schlessinger and Stasheff [S-S]. We describe the Schlessinger-Stasheff classification below (Theorem 1.2).

Each of these solutions to the rational classification problem is theoretical, in the sense that direct application is limited by the elaborate calculations required. However, Umble [Um] introduced a spectral sequence that helps perform some of the calculations necessary for the Schlessinger and Stasheff approach. The basic facts concerning this spectral sequence are reviewed in Section 2.

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In this paper we use the Schlessinger and Stasheff classification, with the spectral sequence introduced in [Um] and analyze the family of algebras $H = H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$; for each n and k we determine the number of rational homotopy types that realize H. The spaces $\mathbb{C}P^n/\mathbb{C}P^k$, known as stunted complex projective spaces, are discussed in Section 2. Our main result is the following:

THEOREM 1.1. *$H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is realized by*

(a) one rational homotopy type if

(i) $k = 1$,

(ii) $n = \infty$,

(iii) $n \leq 4k+8$, *or*

(iv) $k = 2$ *and* $n = 21, 22, 23$ *or* 30;

(b) two rational homotopy types if

(i) $n = 4k+9$,

(ii) $k = 3$ *and* $n = 29$, *or*

(iii) $k = 2$ *and* $n = 20$;

(c) a countably infinite number of rational homotopy types otherwise.

This result is the composite of Theorems 2.3, 2.6, 2.7, 2.9, 4.1 and 4.2 below.

The paper is organized as follows: In Section 2 we determine certain values n and k for which $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is realized uniquely. Here we prove part (a) of Theorem 1.1, and in particular obtain a new proof of the fact that $H^*(\mathbb{C}P^\infty/\mathbb{C}P^k;\mathbb{Q})$ is realized by a unique rational homotopy type for all k—a result due to Tanré [Ta$_1$]. Other values for n and k are analyzed in Sections 3 and 4; in these cases we show that $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is realized by multiple rational homotopy types.

We assume familiarity with basic rational homotopy theory, and in particular 'perturbation' techniques relevant to differential graded Lie algebra minimal models. A brief review of these ideas is given below, following some notation. References for basic rational homotopy theory are [B-G], [D-G-M-S], [G-M], [Ha], [Ne], [Qu] and [Su]. References for differential graded Lie algebra minimal models and perturbations are [B-L], [Mi], [N-M], [St]. A treatment of rational homotopy theory from both the differential graded commutative algebra and differential graded Lie algebra points of view is included in [Ta$_2$], which is a useful reference.

For basic terminology and notation, see [B-L]. In particular, we adopt the following notation: The prefix DG means differential graded; *algebra* means one-connected, commutative, finite type graded algebra over $\mathbb{Q}$; *Lie algebra* means connected graded Lie algebra over $\mathbb{Q}$. $\mathbb{L}(V)$ denotes the free graded Lie algebra on the graded vector space V, and $\langle V\rangle$ denotes the abelian Lie algebra on V. If x is an element of a graded vector space, $|x|$ denotes the degree of x. A linear map θ of degree p on a graded Lie algebra L, is a *derivation of degree p* if for all $x, y \in L$, $\theta([x,y]) = [\theta(x), y] + (-1)^{p|x|}[x, \theta(y)]$. We will frequently write $\mathrm{ad}(x)(y)$ for the Lie bracket $[x,y]$, or more generally $ad^r(x)(y)$ for the bracket $\Big[x, \big[x, \ldots, [x,y]\cdots\big]\Big]$ with x occurring r times. For an algebra H, the dual coalgebra will be denoted by H_*.

Every algebra H is realized by some rational homotopy type; in particular, H is realized by a unique formal rational homotopy type [H-S]. This formal rational homotopy type should be thought of as the canonical choice of rational homotopy type that realizes H. According to Quillen [Qu], every rational homotopy type corresponds to some DG Lie algebra; so for the purposes of rational homotopy theory, spaces can be thought of as DG Lie algebras and vice-versa. Let $s^{-1}H_*$ denote the graded vector space obtained from H_* by setting $(s^{-1}H_*)_0 = 0$ and $(s^{-1}H_*)_n = (H_*)_{n+1}$, for $n \geq 1$. Here and in the sequel, the symbol s^{-1} denotes desuspension. Then the multiplication in H induces a quadratic differential d of degree -1 on $\mathbb{L}(s^{-1}H_*)$ in a standard way [Qu, p. 287]. The free DG Lie algebra $\big(\mathbb{L}(s^{-1}H_*), d\big)$ is called the *Quillen model* of H; we denote it by $\mathbb{L}(s^{-1}H_*, d)$. Under the bijection between homotopy types of DG Lie algebras and rational homotopy types of spaces referred to above, the Quillen model of H corresponds to the unique formal rational homotopy type that realizes H. Furthermore, if X is a formal space that realizes H, then the Quillen model of H satisfies $H\big(\mathbb{L}(s^{-1}H_*, d)\big) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$, the *rational homotopy Lie algebra* of X.

Let H be an algebra, with Quillen model $\mathbb{L}(s^{-1}H_*, d)$. A *perturbation* of d is a degree -1 derivation P on $\mathbb{L}(s^{-1}H_*)$ such that P extends bracket length by at least two, and $d+P$ is a differential [N-M]. Given a perturbation P of d, the DG Lie algebra $\mathbb{L}(s^{-1}H_*, d+P)$ represents some rational homotopy type that realizes H [Ne, p. 437]. Conversely, every rational homotopy type that realizes H can be represented by $\mathbb{L}(s^{-1}H_*, d + P)$, for some P [Mi], [B-L]. Whereas many perturbations can represent the same rational homotopy type, the problem of classification up to rational homotopy type corresponds to identifying isomorphism classes of perturbations.

Occupying a central place in the Schlessinger and Stasheff classification program is a certain DG Lie algebra of derivations that arises as follows: Given an algebra H, construct $\mathbb{L}(s^{-1}H_*, d)$ and re-grade $\mathbb{L}(s^{-1}H_*)$ so that elements in degree n are now in degree $-n$. Denote the re-graded DG Lie algebra by (L_H, d); L_H is negatively graded and d is a degree +1 differential. This step, while unnecessary, is consistent with [S-S] and aligns our statements with theirs. Consider the DG Lie algebra $(\operatorname{Der} L_H, \delta)$, where $\operatorname{Der} L_H$ is the graded Lie algebra of graded derivations on L_H and $\delta = \operatorname{ad}(d)$ is given by $\delta(\theta) = d\theta - (-1)^{|\theta|}\theta d$ for $\theta \in \operatorname{Der} L_H$; δ is a degree +1 differential. If $x \in L_H$ is homogeneous with respect to bracket length; define the *weight* of x by $\operatorname{wt}(x) = |x| - \operatorname{length}(x)$. If x is zero, define its weight to be $-\infty$. Furthermore, since L_H is free, every element of $\operatorname{Der} L_H$ can be written as the sum of its bi-homogeneous components, *i.e.*, degree and change in bracket length; so for a bi-homogeneous derivation θ, say that θ is *weight-decreasing* if $\operatorname{wt}\big(\theta(x)\big) < \operatorname{wt}(x)$, for all non-zero $x \in L_H$. Since d is quadratic of degree +1, $\operatorname{wt}\big(\delta\theta(x)\big) \leq \operatorname{wt}\big(\theta(x)\big)$ for $\theta \in \operatorname{Der} L_H$. Thus, if $\mathcal{L}_H$ denotes the sub-Lie algebra of $\operatorname{Der} L_H$ generated by weight-decreasing derivations, then δ restricts to $\mathcal{L}_H$ giving a DG Lie algebra $\big(\mathcal{L}_H, \delta\big)$, which is the one that appears in the classification theory.

There is a bijection between rational homotopy types that realize H and isomorphism classes of DG Lie algebras $\mathbb{L}(s^{-1}H_*, d + P)$, where P is a perturbation of d [Mi]. The perturbation P is thought of as representing the corresponding rational homotopy type.

Now a group of Lie algebra automorphisms of $\mathbb{L}(s^{-1}H_*)$ induces a group action on the set of perturbations, *i.e.*, the set $V = \{P \in \mathcal{L}_H^1 \mid (d+P)^2 = 0\}$. Thus, perturbations P and Q represent the same rational homotopy type that realize H if P and Q lie in the same orbit under such an induced action. There are two such actions, namely, an 'exponential action of $\mathcal{L}_H^0$' and an action induced by Aut H—the algebra automorphisms of H. We refer the reader to [S-S]for details and state the following fundamental results for reference:

THEOREM 1.2 ([S-S]). *Let H be an algebra and let $V = \{P \in \mathcal{L}_H^1 \mid (d+P)^2 = 0\}$. There is a bijection of sets:*

$$\left\{ \begin{array}{c} \textit{Rational Homotopy Types} \\ \textit{of Spaces That Realize } H \end{array} \right\} \longleftrightarrow \left\{ \frac{V}{\exp(\operatorname{ad} \mathcal{L}_H^0)} \Big/ \operatorname{Aut} H \right\}.$$

If an algebra is realized by only one rational homotopy type, the algebra is called *intrinsically formal*. If $H^1(\mathcal{L}_H, \delta) = 0$, then it can be shown that $V\big/\exp(\operatorname{ad} \mathcal{L}_H^0) = \{0\}$. This proves:

THEOREM 1.3 ([S-S, 8.0]). *If $H^1(\mathcal{L}_H, \delta) = 0$, then H is intrinsically formal.*

All intrinsic formality results in this paper are obtained by applying Theorem 1.3.

2. **Intrinsic formality and the algebras $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$.** In this section we prove:

THEOREM 2.1. *$H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$ is intrinsically formal if*

(i) $k = 1$,

(ii) $n = \infty$,

(iii) $n \leq 4k + 8$ *or*

(iv) $k = 2$ *and* $n = 21, 22, 23$ *or* 30.

The proof of Theorem 2.1 is a composite of Theorems 2.3, 2.6, 2.7 and 2.9 below. In Sections 3 and 4 we show that Theorem 2.1 is a sharp result. Although the case $n = \infty$, which is Theorem 2.6 below, was obtained by Tanré [Ta$_1$], our proof is of independent interest for two reasons: First, because we use the Schlessinger and Stasheff approach and second, because the calculations are used subsequently.

Theorem 1.3 asserts that $H^1(\mathcal{L}_H, \delta) = 0$ is a sufficient condition for intrinsic formality. In general, however, direct computation of $H^1(\mathcal{L}_H, \delta)$ is a formidable task. One helpful tool is a spectral sequence, introduced by Umble [Um], which was invented for the purpose of calculating $H^*(\mathcal{L}_H, \delta)$. Since our calculations use the spectral sequence, we review the basic facts here and refer the reader to [Um] for details.

Given an algebra H with Quillen model $\mathbb{L}(s^{-1}H_*, d)$, let L_H be the graded Lie algebra obtained by negatively re-grading $\mathbb{L}(s^{-1}H_*)$, as in the introduction. Let $\{\bar{x}_j\}_{j\in J}$ be an additive basis for $s^{-1}H_*$, and let $\{x_j\}$ be the corresponding free Lie algebra basis for L_H; *i.e.*, $|x_j| = -|\bar{x}_j|$. The basis $\{x_j\}$ is said to have *cellular* indexing if $i < j$ implies $|\bar{x}_i| \leq |\bar{x}_j|$ or equivalently $|x_i| \geq |x_j|$. It is always possible to choose a basis with cellular indexing, so assume that $\{x_j\}$ is such a basis.

Let $\alpha, x_i \in L_H$ with x_i a generator, and let $\alpha \cdot \partial x_i$ denote that derivation sending x_i to α, and other generators of L_H to zero. Similarly, if W is a sub-vector space of L_H, then $W \cdot \partial x_p$ denotes the vector space of derivations consisting of all derivations $w \cdot \partial x_p$ for $w \in W$. Since any element of $\operatorname{Der} L_H$ can be written as a sum of these so-called basic derivations, the cellular indexing of the indecomposables induces a decreasing filtration of $\operatorname{Der} L_H$ with $\mathcal{F}^p = \oplus_{k \geq p}\{\alpha \cdot \partial x_k \mid \alpha \in L_H\}$. Now $\operatorname{Der} L_H$ is bigraded with respect to cellular index p and derivation degree q; $\operatorname{Der}^{p,q} L_H$ denotes the (p,q)-component. Furthermore there are compatible maps $\delta_i \colon \operatorname{Der}^{p,q} L_H \longrightarrow \operatorname{Der}^{p+i,q+1} L_H$, $i = 0, 1, 2, \ldots$, such that $\delta = \operatorname{ad}(d) = \sum_{i \geq 0} \delta_i$ with δ_i defined as follows: Write the differential in L_H in terms of basic derivations, $d = \sum_{j \in J} \beta_j \cdot \partial x_j$, and consider an element $\alpha \cdot \partial x_p \in \operatorname{Der}^{p,q} L_H$. Then $\delta_0(\alpha \cdot \partial x_p) = (d\alpha) \cdot \partial x_p$ and for $i > 0$ we have $\delta_i(\alpha \cdot \partial x_p) = -(-1)^q\big(\alpha \cdot \partial x_p(\beta_{p+i})\big) \cdot \partial x_{p+i}$. Thus the bigraded vector space $\operatorname{Der}^{*,*} L_H$, with maps δ_i, is a filtered multi-complex that gives rise to a spectral sequence $\{E_r^{*,*}, \bar{\delta}_r\}$, with $E_0^{p,q} = \operatorname{Der}^{p,q} L_H$ and $\bar{\delta}_0 = \delta_0$. The differential $\bar{\delta}_r$ on E_r is induced by the maps δ_i for $0 \leq i \leq r$. In particular, the E_1-term satisfies $E_1^{p,q} \cong \big(\pi_{|\bar{x}_p|-q}(\Omega X) \otimes \mathbb{Q}\big) \cdot \partial x_p$, where $\bar{x}_p$ has positive degree and X is the formal space that realizes H; this latter follows from the fact that $H\big(\mathbb{L}(s^{-1}H_*, d)\big) \cong \pi_*(\Omega X) \otimes \mathbb{Q}$. The sequence $\{E_r^{*,*}, \bar{\delta}_r\}$ is a spectral sequence of vector spaces; in general, the multiplicative structure is lost in the limit.

The spectral sequence restricts to the sub-DG Lie algebra $\mathcal{L}_H \subset \operatorname{Der} L_H$. Here, the E_1-term satisfies $E_1^{p,q} \cong \oplus_{s \geq q+2}\big(\pi_{|\bar{x}_p|-q,s}(\Omega X) \otimes \mathbb{Q}\big) \cdot \partial x_p$ where $\pi_{*,s}(\Omega X) \otimes \mathbb{Q}$ denotes the homology generated by cycles of bracket length s in the Quillen model of H. This makes sense because $\mathbb{L}(s^{-1}H_*, d)$ has quadratic differential. The nature of the E_1-term suggests that the spectral sequence will be most useful when there is a good description of the rational homotopy Lie algebra of the formal space that realizes H. We now give such a description for the algebras $H = H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$.

Let $\mathbb{C}P^m$ denote m-dimensional complex projective space. For $1 \leq k < n \leq \infty$, define $\mathbb{C}P^n/\mathbb{C}P^k$ as the cofibre of the natural inclusion map $i \colon \mathbb{C}P^k \longrightarrow \mathbb{C}P^n$. The map i is a so-called formalisable map and it is well-known that the cofibre of such a map is a formal space [F-T]. Thus $\mathbb{C}P^n/\mathbb{C}P^k$ is formal, and hence is the formal space that realizes $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$. Following Tanré [Ta$_1$], we give the following useful descriptions of the multiplicative structure of $H^*(\mathbb{C}P^\infty/\mathbb{C}P^k; \mathbb{Q})$, and the rational homotopy Lie algebra of $\mathbb{C}P^\infty/\mathbb{C}P^k$.

PROPOSITION 2.2 ([TA$_1$, PROPOSITION 1]).

$$H^*(\mathbb{C}P^\infty/\mathbb{C}P^k; \mathbb{Q}) \cong \mathbb{Q}[y_{k+1}, \ldots, y_{2k+1}]/\mathcal{R},$$

where $|y_i| = 2i$ *and* $\mathcal{R}$ *is the ideal generated by* $\{y_{k+1+i}y_{k+1+j} - y_{k+1}y_{k+1+i+j}\}$ *for* $0 \leq i+j \leq k$ *and* $\{y_{k+1+i}y_{k+1+j} - y_{k+1}^2 y_{i+j}\}$ *for* $k+1 \leq i+j \leq 2k$. ∎

For any $i \geq k+1$, write $i = p(k+1) + q$ with $0 \leq q \leq k$, $1 \leq p$; then the elements $y_i = y_{k+1}^{p-1} y_{k+1+q}$ form a basis for the vector space $H^*(\mathbb{C}P^\infty/\mathbb{C}P^k; \mathbb{Q})$.

THEOREM 2.3. *If $n \leq \infty$, then $H^*(\mathbb{C}P^n/\mathbb{C}P^1;\mathbb{Q})$ is intrinsically formal.*

PROOF. If $n < \infty$, write $n+1 = 2p+q$ with $q = 0$ or 1, so that $H^*(\mathbb{C}P^n/\mathbb{C}P^1;\mathbb{Q}) \cong \mathbb{Q}[y_2, y_3]/(y_3^2 - y_2^3, y_2^{p-1}y_{2+q})$. Also, $H^*(\mathbb{C}P^\infty/\mathbb{C}P^1;\mathbb{Q}) \cong \mathbb{Q}[y_2,y_3]/(y_3^2 - y_2^3)$. In either case, the algebras are the quotient of a free algebra by a regular sequence of relations; and such algebras are well-known to be intrinsically formal ([Su, p. 317]). ∎

We include Theorem 2.3 for completeness only; indeed, it is implicit in the work of Tanré. Henceforth, for the space $\mathbb{C}P^n/\mathbb{C}P^k$, it is assumed that $k \geq 2$.

Continuing our description of $\pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$, given a basis element $y_i \in H^*(\mathbb{C}P^\infty/\mathbb{C}P^k;\mathbb{Q})$, let y_i also denote the corresponding dual basis element in the homology coalgebra $H_*(\mathbb{C}P^\infty/\mathbb{C}P^k;\mathbb{Q})$. Let $\bar{x}_i = s^{-1}y_i$ for $i \geq k+1$. Then the Quillen model of $H^*(\mathbb{C}P^\infty/\mathbb{C}P^k;\mathbb{Q})$ is given by $\mathbb{L}(\bar{x}_{k+1}, \bar{x}_{k+2}, \ldots; d)$, with

$$d(\bar{x}_m) = -\frac{1}{2}\sum_{i=k+1}^{m-(k+1)} [\bar{x}_i, \bar{x}_{m-i}]$$

for $m \geq 2k+2$, and $d(\bar{x}_m) = 0$ otherwise. In particular, for $k+1 \leq i \leq 2k+1$, the d-cycles $\bar{x}_i$ each represent distinct classes in $H\big(\mathbb{L}(\bar{x}_{k+1}, \bar{x}_{k+2}, \ldots; d)\big)$. The full Lie algebra structure of $H\big(\mathbb{L}(\bar{x}_{k+1}, \bar{x}_{k+2}, \ldots; d)\big) \cong \pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$ is described by:

PROPOSITION 2.4 ([TA$_1$, PROPOSITION 2]). *There is a short exact sequence of Lie algebras*

$$0 \longrightarrow \mathbb{L}(\bar{x}_{k+2}, \bar{x}_{k+3}, \ldots, \bar{x}_{2k+1}) \longrightarrow \pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big) \otimes \mathbb{Q} \longrightarrow \langle \bar{x}_{k+1} \rangle \longrightarrow 0$$

with the remaining brackets in $\pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$ given by $[\bar{x}_{k+1}, \bar{x}_{k+2}] = 0$, and $[\bar{x}_{k+1}, \bar{x}_{k+r}] = -\frac{1}{2}\sum_{i=2}^{r-1}[\bar{x}_{k+i}, \bar{x}_{k+r+1-i}]$ for $3 \leq r \leq k+1$. ∎

NOTATION 2.5. Let H be an algebra and let $(\mathcal{L}_H, \delta)$ be the DG Lie algebra of weight decreasing derivations on L_H as constructed above. If $H = H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$, then we denote $\mathcal{L}_H$ by $\mathcal{L}_{H(n/k)}$.

We now prove part (ii) of Theorem 2.1. We remark once more that this is a new proof of a theorem of Tanré's. We include all details since the calculations in this proof form a starting point for the calculations in our subsequent results.

THEOREM 2.6 ([TA$_1$, 3.2]). *$H^*(\mathbb{C}P^\infty/\mathbb{C}P^k;\mathbb{Q})$ is intrinsically formal.*

PROOF. Consider the E_1-term in the spectral sequence described above:

$$E_1^{p,q} \cong \bigoplus_{s \geq q+2}\Big(\pi_{|\bar{x}_p|-q,s}\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big) \otimes \mathbb{Q}\Big) \cdot \partial x_p,$$

and the differential $\bar{\delta}_1 \colon E_1^{p,q} \longrightarrow E_1^{p+1,q+1}$ induced by $\delta_1 \colon \mathcal{L}^{p,q}_{H(\infty/k)} \longrightarrow \mathcal{L}^{p+1,q+1}_{H(\infty/k)}$. If $\alpha \cdot \partial x_p \in \mathcal{L}^{p,q}_{H(\infty/k)}$ and d is written $d = \sum_j \beta_j \cdot \partial x_j$, then δ_1 is in turn given by $\delta_1(\alpha \cdot \partial x_p) = \big(\alpha \cdot \partial x_p(\beta_{p+1})\big) \cdot \partial x_{p+1}$. Now in the Quillen model, $d\bar{x}_{p+1} = -\frac{1}{2}\sum_{i=k+1}^{p-k}[\bar{x}_i, \bar{x}_{p+1-i}] = \bar{\beta}_{p+1}$, so $\alpha \cdot \partial x_p(\beta_{p+1}) = 0$ since β_{p+1} contains no bracket with an entry in x_p. Thus δ_1, and hence $\bar{\delta}_1 \colon E_1^{p,q} \longrightarrow E_1^{p+1,q+1}$ is zero. Similarly, $\bar{\delta}_r \colon E_r^{p,q+1} \longrightarrow E_r^{p+r,q+1}$ is zero for all $r = 1, \ldots, k$;

thus $E_1^{p,q} \cong E_{k+1}^{p,q}$ and it is necessary to understand $\bar{\delta}_{k+1}\colon E_{k+1}^{p,q} \longrightarrow E_{k+1}^{p+k+1,q+1}$. As a vector space, $E_{k+1}^{p,*} \cong \langle x_{k+1}\rangle \oplus \mathbb{L}(x_{k+2},\ldots,x_{2k+1})$. For an element $\alpha \cdot \partial x_p \in \mathcal{L}^{p,q}_{H(\infty/k)}$,

$$\begin{aligned}\delta_{k+1}(\alpha\cdot\partial x_p) &= \big(\alpha\cdot\partial x_p(\beta_{p+k+1})\big)\cdot\partial x_{p+k+1}\\ &= -\frac{1}{2}\Big(\alpha\cdot\partial x_p\Big(\sum_{i=k+1}^{p}[x_i,x_{p+k+1-i}]\Big)\Big)\cdot\partial x_{p+k+1}\\ &= -\big(\alpha\cdot\partial x_p([x_{k+1},x_p])\big)\cdot\partial x_{p+k+1},\end{aligned}$$

where this last step follows from the fact that $\alpha\cdot\partial x_p$ is zero on all other brackets of β_{p+k+1}. Hence the differential $\bar{\delta}_{k+1}\colon E_{k+1}^{p,q} \longrightarrow E_{k+1}^{p+k+1,q+1}$ is induced by

$$\delta_{k+1}(\alpha\cdot\partial x_p) = -(-1)^q[x_{k+1},\alpha]\cdot\partial x_{p+k+1}$$

and if $p = k+1$, then a $\frac{1}{2}$ should be placed before the bracket. Thus $\bar{\delta}_{k+1}$ can be understood in terms of the action of $\mathrm{ad}(x_{k+1})$ on the vector space $\langle x_{k+1}\rangle \oplus \mathbb{L}(x_{k+2},\ldots,x_{2k+1})$, and this is described by Proposition 2.4. For brackets $\alpha \in \mathbb{L}(x_{k+2},\ldots,x_{2k+1})$, $\bar{\delta}_{k+1}(\alpha\cdot\partial x_p) = \theta(\alpha)\cdot\partial x_{p+k+1}$, where θ is a derivation that satisfies

$$\begin{aligned}\theta(x_{k+2}) &= 0\\ \theta(x_{k+3}) &= -\frac{1}{2}[x_{k+2},x_{k+2}]\\ &\ \vdots\\ \theta(x_{2k+1}) &= -\frac{1}{2}\sum_{i=0}^{k-2}[x_{k+2+i},x_{2k-i}].\end{aligned}$$

CLAIM. $E_{k+2}^{*,1} = 0$.

PROOF OF CLAIM. Recall that $H^*(\mathbb{C}P^k;\mathbb{Q})$ has Quillen model $\mathbb{L}\big(s^{-1}H_*(\mathbb{C}P^k;\mathbb{Q}),d_\theta\big)$, where if $\{\bar{z}_i\}_{i=1,\ldots,k}$ is a choice of basis for $s^{-1}H_*(\mathbb{C}P^k;\mathbb{Q})$, then $d_\theta(\bar{z}_p) = -\frac{1}{2}\sum_{i=1}^{p-1}[\bar{z}_i,\bar{z}_{p-i}]$ for $2 \le p \le k$, and $d_\theta(\bar{z}_1) = 0$. It is well-known that this Quillen model has homology $H\big(\mathbb{L}(s^{-1}H_*(\mathbb{C}P^k;\mathbb{Q}),d_\theta)\big) \cong \langle \bar{z}_1,\eta\rangle$; where $\bar{z}_1 \in \pi_{1,1}\big(\Omega(\mathbb{C}P^k)\big)\otimes\mathbb{Q}$, and $\eta \in \pi_{2k,2}\big(\Omega(\mathbb{C}P^k)\big)\otimes\mathbb{Q}$ is represented by the homogeneous bracket length two cycle, $\eta = -\frac{1}{2}\sum_{i=1}^{k}[\bar{z}_i,\bar{z}_{k+1-i}]$. With suitable re-indexing, the brackets and the above differential θ on $\mathbb{L}(x_{k+2},\ldots,x_{2k+1})$, can be identified with the Quillen model $\mathbb{L}\big(s^{-1}H_*(\mathbb{C}P^k;\mathbb{Q}),d_\theta\big)$. Now, since $E_{k+1}^{p,1} \cong \oplus_{s\ge 3}\Big(\pi_{|\bar{x}_p|-1,s}\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big)\otimes\mathbb{Q}\Big)\cdot\partial x_p$, any element in $E_{k+1}^{p,1}$ will have coefficient represented by brackets of length at least three. In particular, any $\bar{\delta}_{k+1}$-cocycle in $E_{k+1}^{p,1}$ will have a coefficient that can be identified with a length ≥ 3 cycle of $\mathbb{L}(s^{-1}H_*(\mathbb{C}P^k;\mathbb{Q}),d_\theta)$, and hence will be $\bar{\delta}_{k+1}$-exact by a suitable element of $E_{k+1}^{p-(k+1),0}$. Thus $H(E_{k+1}^{*,1},\bar{\delta}_{k+1}) \cong E_{k+2}^{*,1} = 0$.

END OF PROOF OF CLAIM.

Hence $E_{k+2}^{*,1} = E_\infty^{*,1} = 0$, so $H^1(\mathcal{L}_{H(\infty/k)},\delta) = 0$ and the conclusion follows from Theorem 1.3. ■

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Recall that a *shallow* space is one whose cohomological dimension is a small multiple of its connectivity [S-S, 8.4]. Now $H^{\mathrm{odd}}(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q}) = 0$, so that if $n < 4k+8$, then $\mathbb{C}P^n/\mathbb{C}P^k$ is essentially of shallow type and the intrinsic formality of $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ for n in this range can be obtained by applying any of the standard techniques applicable to shallow spaces [H-S, 5.16], [S-S, 8.4]. Nevertheless, we give the proof of this result to demonstrate how easily such results fall out of the spectral sequence machinery.

For the purposes of our analysis, it is sufficient to consider $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ through degree $2n$, and $\pi_*\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big)\otimes\mathbb{Q}$ through degree $2n-1$. In this range, the algebras $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ and $H^*(\mathbb{C}P^\infty/\mathbb{C}P^k;\mathbb{Q})$ are isomorphic, as are their respective Quillen models, and homotopy Lie algebras. Thus, many of the required calculations have already been done in Theorem 2.6. The notation above will be retained, so $\{\bar{x}_i\}_{k+1\leq i\leq n}$ is a basis of $s^{-1}H_*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$, and $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ has Quillen model $\mathbb{L}(\bar{x}_{k+1},\ldots,\bar{x}_n;d)$, where d is as described in the discussion that follows Theorem 2.3.

THEOREM 2.7. *If $n\leq 4k+8$ then $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is intrinsically formal.*

PROOF. We use the spectral sequence to show $H^1(\mathcal{L}_{H(n/k)},\delta) = 0$. As in the proof of Theorem 2.6, the differentials $\bar{\delta}_i$ for $1\leq i\leq k$ are all zero and $E_{k+1}^{p,q}\cong E_1^{p,q}$. We now show that $E_{k+2}^{*,1} = 0$.

To simplify notation we use $\pi_{r,s}$ to denote $\pi_{r,s}\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big)\otimes\mathbb{Q}$. Recall that the 1-line of the spectral sequence is given by $E_{k+1}^{p,1}\cong\bigoplus_{s\geq 3}\pi_{2(p-1),s}\cdot\partial x_p$. However, since all generators $\bar{x}_i$ have odd degree, $E_{k+1}^{p,1}\cong\bigoplus_{t\geq 2}\pi_{2(p-1),2t}\cdot\partial x_p$. As remarked above, $\pi_*\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big)\otimes\mathbb{Q}$ is isomorphic with $\pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big)\otimes\mathbb{Q}$ in the degrees of interest here, which in turn is isomorphic with $\langle\bar{x}_{k+1}\rangle\oplus\mathbb{L}(\bar{x}_{k+2},\ldots,\bar{x}_{2k+1})$ as a vector space. So the least p for which $E_{k+1}^{p,1}$ could be non-zero satisfies $2(p-1) = 8k+14$, or $p = 4k+8$. If $n < 4k+8$, therefore, then $E_{k+1}^{p,1} = 0$ for each p, since $p\leq n < 4k+8$, and so each $E_{k+2}^{p,1} = 0$ also. Likewise, if $n = 4k+8$ and $p < n$, then $E_{k+2}^{p,1} = 0$.

The only case remaining is $p = n = 4k+8$. In this case we have $E_{k+1}^{*,1}\cong E_{k+1}^{n,1}\cong\pi_{8k+14,4}\cdot\partial x_p\cong\mathbb{Q}$, with basis element $\eta = \Big[x_{k+2},\big[x_{k+2},[x_{k+2},x_{k+3}]\big]\Big]\cdot\partial x_n$. Note that $\bar{\delta}_{k+1}(\eta) = 0$ for lacunary reasons.

CLAIM. *In the case under consideration, $\bar{\delta}_{k+1}\colon E_{k+1}^{3k+7,0}\longrightarrow E_{k+1}^{n,1}$ is onto.*

PROOF OF CLAIM. $E_{k+1}^{3k+7,0}\cong\bigoplus_{s\geq 2}\pi_{6k+13,s}\cdot\partial x_{3k+7}$. In the underlying multi-complex the element $\zeta = \big[x_{k+3},[x_{k+2},x_{k+3}]\big]\cdot\partial x_{3k+7}\in\mathcal{L}_{H(n/k)}^{3k+7,0}$ satisfies

$$\begin{aligned}\delta_{k+1}(\zeta) &= -\Big[x_{k+1},\big[x_{k+3},[x_{k+2},x_{k+3}]\big]\Big]\cdot\partial x_n\\ &= \Big(\Big[x_{k+3},\big[x_{k+1},[x_{k+2},x_{k+3}]\big]\Big] - \big[[x_{k+1},x_{k+3}],[x_{k+2},x_{k+3}]\big]\Big)\cdot\partial x_n,\end{aligned}$$

by the Jacobi identity. The bracket structure in $E_1^{*,*}$ gives $[x_{k+1},x_{k+3}] = -\frac{1}{2}[x_{k+2},x_{k+2}]$ and $[x_{k+1},x_{k+2}] = 0$; so using the Jacobi identity, $\big[x_{k+1},[x_{k+2},x_{k+3}]\big]\cdot\partial x_n = 0$ in $E_{k+1}^{n,1}$. Consequently $\bar{\delta}_{k+1}(\zeta) = \frac{1}{2}\big[[x_{k+2},x_{k+2}],[x_{k+2},x_{k+3}]\big]\cdot\partial x_n = \eta$.

END OF PROOF OF CLAIM.

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Thus if $n = 4k+8$, then $E^{n,1}_{k+2} = 0$. We have shown that if $n \leq 4k+8$, then $E^{*,1}_{k+2} = 0$ and hence $H^1(\mathcal{L}_{H(n/k)}, \delta) = 0$. The result now follows from Theorem 1.3. ∎

We finish this section with a proof of case (iv) of Theorem 2.1. For any k, denote the spectral sequences for the algebras $H^*(\mathbb{C}P^\infty/\mathbb{C}P^k; \mathbb{Q})$ and $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$ by $\{(E_r)^\infty, \bar{\delta}^\infty_r\}$ and $\{(E_r)^n, \bar{\delta}^n_r\}$ respectively. As in the proof of Theorem 2.6, $\bar{\delta}^n_r = 0$ for $1 \leq r \leq k$, so that $(E^{p,q}_{k+1})^n \cong (E^{p,q}_1)^n$. We next give a technical result that is the basis for all further computation. In particular, it shows that in the spectral sequence calculation of $H^1(\mathcal{L}_{H(n/k)}, \delta)$, the only non-trivial differential is $\bar{\delta}^n_{k+1}$.

LEMMA 2.8. *Using the above notation, $(E^{p,1}_\infty)^n \cong (E^{p,1}_{k+2})^n$ for each p and if $p < n-k$, then $(E^{p,1}_\infty)^n \cong (E^{p,1}_{k+2})^n = 0$. Furthermore, suppose that $\alpha \cdot \partial x_p \in (E^{p,1}_{k+1})^n$ with $n-k \leq p \leq n$. Then $\alpha \cdot \partial x_p$ survives to $(E^{p,1}_{k+2})^n$, and thus to $(E^{p,1}_\infty)^n$, if and only if $\bar{\delta}^\infty_{k+1}(\alpha \cdot \partial x_p) \neq 0$.*

PROOF. We use the notation and calculations of Theorems 2.6 and 2.7. Since each generator $\bar{x}_p$ has odd degree, we have $E^{p,0}_{k+1} \cong \bigoplus_{t\geq 1} \pi_{2p-1,2t+1} \cdot \partial x_p$ and $E^{p,1}_{k+1} \cong \bigoplus_{t\geq 2} \pi_{2(p-1),2t} \cdot \partial x_p$. Now consider $p \leq n-k-1$. In this range of p, $\bar{\delta}^n_{k+1}$ and $\bar{\delta}^\infty_{k+1}$ can be identified. As in the proof of the claim in Theorem 2.6, we identify $(E^{*,1}_{k+1})^\infty$ and $\bar{\delta}^\infty_{k+1}$ with terms from the Quillen model of $\mathbb{C}P^k$, and conclude that $(E^{p,0}_{k+2})^\infty = 0$ and $(E^{p,1}_{k+2})^\infty = 0$, since the only non-exact cycles in the Quillen model of $\mathbb{C}P^k$ have bracket length 1 or 2. Thus if $p \leq n-k-1$, then $(E^{p,1}_{k+2})^n = 0$ and if $n-k \leq p \leq n$, then for each $i \geq k+2$, $\bar{\delta}^n_i \colon (E^{p-i,0}_i)^n \longrightarrow (E^{p,1}_i)^n$ is trivial since $(E^{p-i,0}_i)^n = 0$. On the other hand, if $n-k \leq p \leq n$, then $\bar{\delta}^n_i(E^{p,1}_i)^n = 0$ for $i \geq k+1$, for lacunary reasons. This proves the first assertion.

Now assume that $\alpha \cdot \partial x_p \in (E^{p,1}_{k+1})^n$ with $n-k \leq p \leq n$. By the proof of Theorem 2.6, $\bar{\delta}^\infty_i = 0$ for $1 \leq i \leq k$, and $(E^{*,1}_{k+2})^\infty = 0$. Hence $\bar{\delta}^\infty_{k+1}(\alpha \cdot \partial x_p) = 0$ if and only if there is some $\eta \cdot \partial x_{p-k-1} \in (E^{p-k-1,0}_{k+1})^\infty$ such that $\bar{\delta}^\infty_{k+1}(\eta \cdot \partial x_{p-k-1}) = \alpha \cdot \partial x_p$. But $(E^{p-k-1,0}_{k+1})^\infty = (E^{p-k-1,0}_{k+1})^n$ and $\bar{\delta}^\infty_{k+1}(\eta \cdot \partial x_{p-k-1}) = \bar{\delta}^n_{k+1}(\eta \cdot \partial x_{p-k-1})$, so the result follows. ∎

THEOREM 2.9. *If $n = 21, 22, 23$ or 30, then $H^*(\mathbb{C}P^n/\mathbb{C}P^2; \mathbb{Q})$ is intrinsically formal.*

PROOF. Once more we use the spectral sequence to show that $H^1(\mathcal{L}_{H(n/k)}, \delta) = 0$. As in the proof of Theorem 2.7, and according to the remarks above Lemma 2.8, it is sufficient to consider $E^{p,1}_3 \cong \bigoplus_{t\geq 2} \pi_{2p-2,2t} \cdot \partial x_p$ for $n-2 \leq p \leq n$. Length 4 brackets in $\pi_*\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^2)\big) \otimes \mathbb{Q}$ have degree ≤ 34, with $\big[\bar{x}_5, [\bar{x}_5, [\bar{x}_4, \bar{x}_5]]\big]$ having maximal degree. Length 6 brackets have degree ≥ 44, with $\mathrm{ad}^5(\bar{x}_4)(\bar{x}_5)$ having minimal degree. If $n = 21$ or 22, the inequalities $34 < 2p-2 < 44$ hold, and so $E^{p,1}_3 = 0$ for p in the range $n-2 \leq p \leq n$, and hence $(E^{*,1}_\infty)^n = 0$. If $n = 23$, then the critical range for p is $21 \leq p \leq 23$; here $E^{p,1}_3 = 0$ unless $p = 23$, and $E^{23,1}_3$ has basis consisting of the single element $\mathrm{ad}^5(x_4)(x_5) \cdot \partial x_{23}$. By direct calculation, this element vanishes under $\bar{\delta}^\infty_3$, so that $(E^{*,1}_\infty)^n = 0$ by Lemma 2.8. Finally, if $n = 30$ the argument is similar to the case $n = 23$ with the single basis element in the critical range being $\mathrm{ad}^7(x_4)(x_5) \cdot \partial x_{30} \in E^{30,1}_3$. Thus in all cases $(E^{*,1}_\infty)^n = 0$ and the result follows from Theorem 1.3. ∎

3. $H^1(\mathcal{L}_{H(n/k)}, \delta) \neq 0$ **and the algebras** $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$. Having determined certain values of n and k for which $H^1(\mathcal{L}_{H(n/k)}, \delta) = 0$, we now consider the remaining values for n and k and show that $H^1(\mathcal{L}_{H(n/k)}, \delta) \neq 0$ in all other cases—see Propositions 3.5, 3.6 and 3.9 below. This information is critical in Section 4 where we prove that an algebra $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$ is intrinsically formal if and only if $H^1(\mathcal{L}_{H(n/k)}, \delta) = 0$—a fact that is not true for all algebras H.

We consider three cases separately: $k \geq 4$, $k = 3$ and $k = 2$. In each case we identify particular elements in the spectral sequence that survive to E_∞, so that $H^1(\mathcal{L}, \delta) \neq 0$. If $k \geq 3$, then the following lemma, when combined with Lemma 2.8, provides a useful criterion in this regard. We use the same notation as in Section 2, with $\bar{x}$ denoting an element of $\mathbb{L}(s^{-1}H_*)$ and x denoting the negatively re-graded element of L_H.

LEMMA 3.1. *Let* $\bar{\beta} \in \mathbb{L}(\bar{x}_{k+2}, \ldots, \bar{x}_{2k+1}) \subset \pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$ *be an element of the following type:*

$$\bar{\beta} = [\bar{x}_{i_1}, [\bar{x}_{i_2}, [\ldots, [\bar{x}_{i_r}, \bar{x}_{i_{r+1}}]] \cdots]$$

with (i) $\max\{i_1, \ldots, i_{r-1}\} < i_r$ *and (ii)* $i_r \leq i_{r+1}$. *Then* $\bar{\delta}^\infty_{k+1}(\beta \cdot \partial x_p) \neq 0$.

The proof of Lemma 3.1 is a consequence of the following lemma, which gives a useful test for linear independence among certain types of elements in a free Lie algebra:

LEMMA 3.2 ([LU, 4.7]). *In the free Lie algebra* $\mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$, *let* χ *be an element* $\chi = \sum_{i=k+2}^{2k+1}[x_i, B_i]$; *where each* B_i *has the form*

$$B_i = \sum_J \lambda^i_J [x_{j_1}, [x_{j_2}, [\ldots, [x_{j_{r-1}}, x_{j_r}]] \cdots]$$

where $J = (j_1, \ldots, j_r)$, *satisfying* $\lambda^i_J = 0$ *unless* $\max\{i, j_1, \ldots, j_{r-2}\} < j_r$ *and* $j_{r-1} \leq j_r$. *Then* $\chi = 0$ *implies each* $B_i = 0$.

PROOF. This is proved by a straightforward argument using the universal enveloping algebra, which in this case is the tensor algebra $T(x_{k+2}, \ldots, x_{2k+1})$. ∎

PROOF OF LEMMA 3.1. Suppose that $\bar{\delta}^\infty_{k+1}(\beta \cdot \partial x_p) = 0$. Recall from the proof of Theorem 2.6 that $\bar{\delta}^\infty_{k+1}(\beta \cdot \partial x_p) = \theta(\beta) \cdot \partial x_{p+k+1}$, where $\theta(x_{k+2}) = 0$, $\theta(x_{k+3}) = -\frac{1}{2}[x_{k+2}, x_{k+2}], \ldots$, $\theta(x_{2k+1}) = -\frac{1}{2}\sum_{i=k+2}^{2k}[x_i, x_{3k+2-i}]$. Thus, if $\bar{\delta}^\infty_{k+1}(\beta \cdot \partial x_p) = 0$ with β as in the hypotheses, then $\theta(\beta) = 0$, which implies that

$$\begin{aligned} 0 = \sum_{j=1}^{r-1} &(-1)^{j+1}[x_{i_1}, [\ldots, [\theta(x_{i_j}), [\ldots, [x_{i_r}, x_{i_{r+1}}]] \cdots] \\ &+ (-1)^{r+1}[x_{i_1}, [\ldots, [x_{i_{r-1}}, [\theta(x_{i_r}), x_{i_{r+1}}]] \cdots] \\ &+ (-1)^{r+2}[x_{i_1}, [\ldots, [x_{i_r}, \theta(x_{i_{r+1}})]] \cdots]. \end{aligned}$$

If $i_r < i_{r+1}$, then the last term is linearly independent of all the others, since they all contain an entry $x_{i_{r+1}}$ of maximal index. Now $\theta(\beta) = 0$, so in this case $\theta(x_{i_1}) = \cdots = \theta(x_{i_{r-1}}) = \theta(x_{i_r}) = 0$, by Lemma 3.2. Thus $i_r = k + 2$, contradicting assumption (i) of the hypotheses. On the other hand, if $i_r = i_{r+1}$, then the term written as a sum is linearly independent of the last two elements—terms in the sum are brackets with two entries of maximal index, while the last two terms are brackets with one entry of maximal index. In this case, $\theta(\beta) = 0$ implies $(-1)^{r+1}2[x_{i_1}, [\ldots, [x_{i_{r-1}}, [\theta(x_{i_r}), x_{i_r}]] \cdots] = 0$, so that $\theta(x_{i_r}) = 0$ by Lemma 3.2. But since $i_r = i_{r+1}$, then $\theta(\beta) = 0$ implies $i_r = k+2$, again contradicting assumption (i). Now $i_r \leq i_{r+1}$, by assumption (ii), so $\theta(\beta) \neq 0$ and hence $\bar{\delta}^{\infty}_{k+1}(\beta \cdot \partial x_p) \neq 0$ as claimed. ∎

A bracket having the form as in the hypotheses of Lemma 3.1 will henceforth be referred to as a *β-type bracket.* An element $\beta \cdot \partial x_p$ in the spectral sequence will be *of β-type* if the coefficient β corresponds—under the re-grading—to a β-type bracket in $\pi_*\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$. Lemmas 2.8 and 3.1 imply that elements in the spectral sequence of β-type survive to $(E^{p,1}_{\infty})^n$ whenever $n-k \leq p \leq n$. In fact if $k \geq 3$, then such elements can be found in this critical range, for all but one value of n. Our proof of this fact follows from the next two lemmas.

LEMMA 3.3. *If $2kr+3r+2 \leq s \leq 4kr-r+2$, then $\pi_{2s}\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$ contains a β-type bracket of length $2r$.*

PROOF. If $s = 2kr+3r+2$, then $\mathrm{ad}^{2r-2}(\bar{x}_{k+2})([\bar{x}_{k+3}, \bar{x}_{k+3}]) \in \mathbb{L}_{2s}(\bar{x}_{k+2}, \ldots, \bar{x}_{2k+1}) \subset \pi_{2s}\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$ is a β-type bracket of length $2r$. For the next $k-2$ values of s, successively replace the right-most entry $\bar{x}_{k+3+i}$, by $\bar{x}_{k+3+i+1}$ for $0 \leq i \leq k-2$; obtaining a sequence of brackets ending with $\mathrm{ad}^{2r-2}(\bar{x}_{k+2})([\bar{x}_{k+3}, \bar{x}_{2k+1}])$. Each replacement increases degree by two and gives a β-type bracket of length $2r$ in $\pi_*\big(\Omega(\mathbb{C}P^n/\mathbb{C}P^k)\big) \otimes \mathbb{Q}$. For the next $k-2$ values of s, successively replace the second from right-most entry $\bar{x}_{k+3+i}$, by $\bar{x}_{k+3+i+1}$ for $0 \leq i \leq k-2$, ending with $\mathrm{ad}^{2r-2}(\bar{x}_{k+2})([\bar{x}_{2k+1}, \bar{x}_{2k+1}])$. Again, each replacement gives a bracket with the required properties. Now successively replace the right-most entry in $\bar{x}_{k+2}$ by $\bar{x}_{k+3}, \ldots, \bar{x}_{2k}$ ending with $\mathrm{ad}^{2r-3}(\bar{x}_{k+2})\Big(\big[\bar{x}_{2k}, [\bar{x}_{2k+1}, \bar{x}_{2k+1}]\big]\Big)$, and continue in this manner, working from right to left until the bracket $\mathrm{ad}^{2r-2}(\bar{x}_{2k})([\bar{x}_{2k+1}, \bar{x}_{2k+1}])$ in degree $8kr-2r+4$ is obtained. ∎

LEMMA 3.4. *If $(2r-1)k \geq 4r+2$ for all $r \geq 2$; then for all $n \geq 4k+9$, there is some p in the range $n-k \leq p \leq n$ with $(E^{p,1}_{k+1})^n$ containing an element of β-type.*

PROOF. The β-type element in $(E^{*,1}_{k+1})^{\infty}$ having coefficient a length four bracket of least negative degree $-(8k+16)$, is $\Big[x_{k+2}, \big[x_{k+2}, [x_{k+3}, x_{k+3}]\big]\Big] \cdot \partial x_{4k+9}$. The β-type element in $(E^{*,1}_{k+1})^{\infty}$ having coefficient a length four bracket of most negative degree $-16k$, is $\Big[x_{2k}, \big[x_{2k}, [x_{2k+1}, x_{2k+1}]\big]\Big] \cdot \partial x_{8k+1}$. Thus, by Lemma 3.3 $(E^{p,1}_{k+1})^{\infty}$ contains a β-type element for each p in the range $4k+9 \leq p \leq 8k+1$. That is, for each n in the range $4k+9 \leq n \leq 9k+1$, there exists some p with $n-k \leq p \leq n$ and a β-type element in $(E^{p,1}_{k+1})^n$. Similarly, β-type elements with length 6 coefficients begin with $\mathrm{ad}^4(x_{k+2})([x_{k+3}, x_{k+3}]) \cdot \partial x_{6k+12}$ in minimal cellular degree, and end with $\mathrm{ad}^4(x_{2k})([x_{2k+1}, x_{2k+1}]) \cdot \partial x_{12k}$ in maximal cellular degree. Therefore, for each n in the range $6k+12 \leq n \leq 13k$, there is some p with $n-k \leq p \leq n$ and a β-type element in $(E^{p,1}_{k+1})^n$. By assumption, $r=2$ implies $3k \geq 10$ so that $6k+12 \leq 9k+2 \leq 13k$; it follows that a β-type element with coefficient length 6 can be found in the appropriate range, when it is no longer possible to find β-type elements with coefficient length 4.

Inductively, given β-type elements with coefficient length $2r$, some $r \geq 2$; invoke the inequality of the hypothesis to show that β-type elements with coefficient length $2r+2$ can be found to extend the range of n, for which the conclusion holds, when it is no longer possible to find β-type elements of coefficient length $2r$. ∎

PROPOSITION 3.5. *If $4k+9 \leq n < \infty$ and $k \geq 4$, then $H^1(\mathcal{L}_{H(n/k)}, \delta) \neq 0$.*

PROOF. Since $k \geq 4$, k satisfies the hypothesis of Lemma 3.4 for all $r \geq 2$. The conclusion now follows from Lemmas 2.8 and 3.1. ∎

PROPOSITION 3.6. *If $21 \leq n < \infty$, then $H^1(\mathcal{L}_{H(n/3)}, \delta) \neq 0$.*

PROOF. When $k = 3$, Lemma 3.4 fails. However, if r is required to be at least three, the conclusion of Lemma 3.4 holds for all $n \geq 30$. Thus it is sufficient to consider the range $21 \leq n \leq 29$. As in the proof of Lemma 3.4, β-type elements of length 4 begin with $\Big[x_5, \big[x_5, [x_6, x_6]\big]\Big] \cdot \partial x_{21}$ and end with $\Big[x_6, \big[x_6, [x_7, x_7]\big]\Big] \cdot \partial x_{25}$. Hence for each $21 \leq n \leq 28$, there is a p with $n-3 \leq p \leq n$ and a β-type element in $(E_4^{p,1})^n$. For the remaining case $n = 29$, let $\alpha \cdot \partial x_{26} = \big[[x_6, x_7,], [x_7, x_7]\big] \cdot \partial x_{26}$. Then $\bar{\delta}_4^\infty(\alpha \cdot \partial x_{26}) = \theta(\alpha) \cdot \partial x_{30}$, where $\theta(x_6) = -\frac{1}{2}[x_5, x_5]$ and $\theta(x_7) = -[x_5, x_6]$ as in the proof of Theorem 2.6. Direct calculation shows that

$$\begin{aligned}\bar{\delta}_4^\infty(\alpha \cdot \partial x_{26}) &= \Big(\big[[\theta(x_6), x_7], [x_7, x_7]\big] + \text{ brackets with 2 entries in } x_7\Big) \cdot \partial x_{30} \\ &\neq 0.\end{aligned}$$

The conclusion follows from Lemma 2.8. ∎

We conclude this section with a discussion of the case $k = 2$. Observe that, as a vector space, $\pi_*\big(\Omega(\mathbb{C}P^\infty/\mathbb{C}P^2)\big) \otimes \mathbb{Q} \cong \langle \bar{x}_3 \rangle \oplus \mathbb{L}(\bar{x}_4, \bar{x}_5)$ and so contains exactly one length $2r$, β-type bracket for each $r \geq 2$. These brackets appear in degrees too sparsely distributed for our purposes. Instead, there is the following: For each $r \geq 2$, let $\bar{\omega}_{2r} = \mathrm{ad}(\bar{x}_5)\,\mathrm{ad}^{2r-2}(\bar{x}_4)(\bar{x}_5)$; the brackets $\{\mathrm{ad}^j([\bar{x}_4, \bar{x}_5])(\bar{\omega}_{2r})\}_{j \geq 0}$ have the suitable properties. Henceforth, a bracket in this family will be referred to as a γ-type bracket, and a corresponding element $\gamma \cdot \partial x_p$ in the spectral sequence will be referred to as a γ-type element.

LEMMA 3.7. *If $\gamma \cdot \partial x_p \in (E_3^{p,1})^\infty$ is a γ-type element, then $\bar{\delta}_3^\infty(\gamma \cdot \partial x_p) \neq 0$.*

PROOF. Recall from the proof of Theorem 2.6 that $\bar{\delta}_3^\infty(\gamma \cdot \partial x_p) = \theta(\alpha) \cdot \partial x_{p+3}$, where θ is induced from $\mathrm{ad}(x_3)$. Thus, it is sufficient to show that $\theta(\gamma) \neq 0$ for any γ-type bracket. Let $\mathcal{H}$ be a graded Hall basis for $\mathbb{L}(x_4, x_5)$ [N-M, 4.5]. Note that $\theta(x_4) = 0$ and $\theta(x_5) = -\frac{1}{2}[x_4, x_4]$, so that $\theta([x_4, x_5]) = 0$. Now $\theta(\omega_{2r}) = \mathrm{ad}(-\frac{1}{2}[x_4, x_4])\,\mathrm{ad}^{2r-2}(x_4)(x_5) = -\mathrm{ad}^{2r}(x_4)(x_5)$, by the Jacobi identity. But $\mathrm{ad}^{2r}(x_4)(x_5) \in \mathcal{H}$, so $\theta(\omega_{2r}) \neq 0$. Now if $\gamma = \mathrm{ad}^j([x_4, x_5])(\omega_{2r})$, then $\theta(\gamma) = \mathrm{ad}^j([x_4, x_5])\big(\theta(\omega_{2r})\big) = -\mathrm{ad}^j([x_4, x_5])\,\mathrm{ad}^{2r}(x_4)(x_5)$. But $\mathrm{ad}^j([x_4, x_5])\,\mathrm{ad}^{2r}(x_4)(x_5) \in \mathcal{H}$ and it follows that $\theta(\gamma) \neq 0$. ∎

LEMMA 3.8. *For $n \geq 38$, there exists some p in the range $n-2 \leq p \leq n$, for which $(E_3^{p,1})^n$ contains a γ-type element.*

PROOF. In most cases we can take p equal to n: Since $n \geq 38$, write $n = 10+7r+m$ for $r \geq 4$ and $0 \leq m < 7$. If $2r+2-2m > 0$, then consider the γ-type bracket $\gamma = \mathrm{ad}^m([\bar{x}_4,\bar{x}_5])(\bar{\omega}_{2r+2-2m})$ in degree $18+14r+2m$. This gives a corresponding γ-type element $\gamma \cdot \partial x_{10+7r+m} \in (E_3^{n,1})^n$. This leaves the three cases $n = 43, 44$ and 51, corresponding to $(r,m) = (4,5), (4,6)$ and $(5,6)$ respectively. In these cases, the gamma-type brackets $\mathrm{ad}^3([\bar{x}_4,\bar{x}_5])(\bar{\omega}_4)$, $\mathrm{ad}^4([\bar{x}_4,\bar{x}_5])(\bar{\omega}_4)$ and $\mathrm{ad}^4([\bar{x}_4,\bar{x}_5])(\bar{\omega}_5)$, respectively, give gamma-type elements in $(E_3^{n-2,1})^n$. ■

PROPOSITION 3.9. *If $17 \leq n \leq 20$, $24 \leq n \leq 29$ or $31 \leq n < \infty$, then $H^1(\mathcal{L}_{H(n/2)}, \delta) \neq 0$.*

PROOF. Lemmas 2.8, 3.7 and 3.8 give the result for $n \geq 38$. The γ-type bracket $\bar{\omega}_4$ has degree 32, giving a corresponding γ-type element $\omega_4 \cdot \partial x_{17}$. Thus, for $n = 17, 18$ or 19, there exists a γ-type element in $(E_3^{p,1})^n$, for some p in the range $n-2 \leq p \leq n$. Also, $\bar{\omega}_6$ has degree 46 and gives a corresponding element $\omega_6 \cdot \partial x_{24}$. Thus for $n = 24, 25$ or 26, there exists a γ-type element in $(E_3^{p,1})^n$ for some p in the range $n-2 \leq p \leq n$. Consider the γ-type brackets $\mathrm{ad}^j([\bar{x}_4,\bar{x}_5])(\bar{\omega}_{8-2j})$ for $j = 0,1,2,3$, which range in even degree from 60 through 66. These give corresponding γ-type elements $\gamma \cdot \partial x_p$ for $31 \leq p \leq 34$. Thus for n in the range $31 \leq n \leq 36$, there exists a γ-type element in $(E_3^{p,1})^n$ for some p in the range $n-2 \leq p \leq n$. The discussion above, together with Lemmas 2.8, 3.7 and 3.8, implies the result for all n in the hypotheses except for $n = 20, 27, 28$ and 37. For these, set $u = [\bar{x}_4,\bar{x}_5]$ and $v = [\bar{x}_5,\bar{x}_5]$ and define $\bar{\alpha}_1 = [u,v]$, $\bar{\alpha}_2 = \big[v,[u,v]\big]$ and $\bar{\alpha}_3 = \Big[v,\big[u,[u,v]\big]\Big]$, in degrees 34, 52 and 68 respectively. These give corresponding elements $\alpha_1 \cdot \partial x_{18}$, $\alpha_2 \cdot \partial x_{27}$ and $\alpha_3 \cdot \partial x_{35}$. Direct calculation shows that $\theta(\alpha_i) \neq 0$ for each i; so by Lemma 2.8, $\alpha_1 \cdot \partial x_{18}$ survives to $(E_\infty^{18,1})^n$ for $18 \leq n \leq 20$, $\alpha_2 \cdot \partial x_{27}$ survives to $(E_\infty^{27,1})^n$ for $27 \leq n \leq 29$ and $\alpha_3 \cdot \partial x_{35}$ survives to $(E_\infty^{35,1})^n$ for $35 \leq n \leq 37$, and this completes the proof. ■

4. **Perturbations and the algebras $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$.**

This section identifies perturbations of d in the Quillen model of H with representatives of classes in $H(\mathcal{L}_{H(n/k)}, \delta)$. We apply Theorem 1.2 to obtain our concluding results:

THEOREM 4.1. *$H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is realized by two distinct rational homotopy types if:*

(i) $n = 4k+9$;
(ii) $k = 3$ *and* $n = 29$; *or*
(iii) $k = 2$ *and* $n = 20$.

THEOREM 4.2. *If n and k fail to satisfy the hypotheses of Theorem 2.1 or Theorem 4.1, then $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is realized by a countably infinite number of rational homotopy types.*

The proofs of Theorems 4.1 and 4.2 appear in 4.17 and 4.18 below. Together with Theorem 2.1, this completes Theorem 1.1.

The following four lemmas and Corollary 4.7 are technical results that give information about $H^1(\mathcal{L}_{H(n/k)}, \delta)$ and the structure of $V/\exp(\mathrm{ad}\,\mathcal{L}^0_{H(n/k)})$. If $\theta \in \mathcal{L}_{H(n/k)}$ is a derivation, then we will denote its homology class in $H^1(\mathcal{L}_{H(n/k)}, \delta)$ by $[\theta]$.

LEMMA 4.3. *$H^1(\mathcal{L}_{H(n/k)}, \delta)$ is spanned by classes of the form $[\alpha \cdot \partial x_p]$, where $n-k \leq p \leq n$ and $\alpha \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$. Furthermore, if $Q = \sum_{p=n-k}^{n} \alpha_p \cdot \partial x_p$ with $\alpha_p \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$, then Q is both a δ-cocycle and a perturbation.*

PROOF. As in the proof of Theorem 2.7, $(E_\infty^{p,1})^n = 0$ for $p < n-k$. For $n-k \leq p \leq n$, consider a basic derivation $\alpha \cdot \partial x_p$ in the multi-complex $(\mathcal{L}_{H(n/k)}, \delta) \subset (\mathrm{Der}\, L_{H(n/k)}, \delta)$. Recall from Section 2 that $\delta = \sum_{i \geq 0} \delta_i$. In this range, $\delta(\alpha \cdot \partial x_p) = \delta_0(\alpha \cdot \partial x_p) = (d\alpha) \cdot \partial x_p$. Hence representatives $\alpha \cdot \partial x_p$ of non-vanishing classes in $(E_\infty^{p,1})^n$ are δ-cycles that represent non-vanishing classes in $H^1(\mathcal{L}_{H(n/k)}, \delta)$. Furthermore, any non-zero class in $(E_1^{p,1})^n$ can be represented by a δ_0-cocycle $\alpha \cdot \partial x_p$, with $\alpha \in H(L_{H(n/k)}, d)$, and α of bracket length at least three. But $H(L_{H(n/k)}, d) \cong \langle x_{k+1} \rangle \oplus \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$ as vector spaces in this range, so α can be chosen from $\mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$.

Now suppose $\alpha \cdot \partial x_p \in \mathcal{L}^1_{H(n/k)}$, with $n-k \leq p \leq n$, and $\alpha \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$. By considering the connectivity of $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$, $d(x_i)$ contains no entries in x_p, with $n-k \leq p \leq n$, for any $k+1 \leq i \leq n$. Hence $(\alpha \cdot \partial x_p)(d) = 0$. But $d(x_i) = 0$, for $k+1 \leq i \leq 2k+1$, so $d(\alpha \cdot \partial x_p) = 0$ and thus $\alpha \cdot \partial x_p$ is a δ-cocycle. Now consider $\alpha' \cdot \partial x_q \in \mathcal{L}^1_{H(n/k)}$, with $n-k \leq q \leq n$, and $\alpha' \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$. Again by the connectivity of $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$, α' contains no entries in x_p, so $(\alpha \cdot \partial x_p)(\alpha' \cdot \partial x_q) = \big(\alpha \cdot \partial x_p(\alpha')\big) \cdot \partial x_q = 0$. For $Q = \sum_{p=n-k}^{n} \alpha_p \cdot \partial x_p$, this implies $Q^2 = 0$. But for Q of this form, Q is a sum of δ-cocycles, by the above, and hence is a δ-cocycle itself. Thus $(d+Q)^2 = d^2 + \delta Q + Q^2 = 0$, so Q is a perturbation. ∎

LEMMA 4.4. *Let $\hat{P} \in \mathcal{L}^1_{H(n/k)}$, with $\hat{P} = \sum_{p=n-k}^{n} \alpha_p \cdot \partial x_p$ and $\alpha_p \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$. If $\eta \in \mathcal{L}^0_{H(n/k)}$, then $[\eta, \hat{P}] = 0$.*

PROOF. $\eta \in \mathcal{L}^0_{H(n/k)}$ is a degree zero derivation of $L_{H(n/k)}$ that increases bracket length by at least one. In this case, however, each generator of $L_{H(n/k)}$ has odd degree so that η increases bracket length by at least two. As in the previous lemma, the connectivity of $H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$ implies that, for $k+1 \leq i \leq n$ and $n-k \leq p \leq n$, $\eta(x_i)$ contains no term in x_p. Thus $\hat{P}\eta = 0$. On the other hand, $\hat{P}(x_i)$ is a sum of brackets from $\mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$. Again, connectivity implies that, for $k+2 \leq i \leq 2k+1$, $\eta(x_i) = 0$ so that $\eta\hat{P} = 0$ as well and it follows that $[\eta, \hat{P}] = \eta\hat{P} - \hat{P}\eta = 0$. ∎

The next two lemmas relate to the 'exponential action of $\mathcal{L}^0$' on the set of perturbations $V = \{P \mid (d+P)^2 = 0\}$. For details of this see [S-S]. This action can be described briefly as follows: For $\theta \in \mathcal{L}^0$ and $P \in V$, send

$$P \longmapsto \exp(\operatorname{ad}\theta)(d+P) - d = \sum_{m\geq 0} \frac{1}{m!}\operatorname{ad}^m(\theta)(d+P) - d$$
$$= P + [\theta, d+P] + \frac{1}{2!}\big[\theta, [\theta, d+P]\big] + \cdots .$$

This defines an equivalence relation on V by $Q \sim P$ if and only if $Q = \exp(\operatorname{ad}\theta)(d{+}P){-}d$, for some $\theta \in \mathcal{L}^0$; we denote the orbit of P by $\{P\}$.

Let $\hat{V} = \big\{\hat{P} \mid \hat{P} = \sum_{p=n-k}^{n} \alpha_p \cdot \partial x_p$ with $\alpha_p \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})\big\}$; by Lemma 4.3 $\hat{V} \subset V$.

LEMMA 4.5. *If $P \in V$, then there exists some $\hat{P} \in \hat{V}$ such that $\{P\} = \{\hat{P}\}$.*

PROOF. Write $P = P_2 + P_3 + \cdots$, where P_i extends bracket length by i. We show, by induction on bracket length, that $\{P\} = \{\hat{P}\}$ for some $\hat{P} = \hat{P}_2 + \hat{P}_3 + \cdots$ with $\hat{P}_i \in \hat{V}$ for all i. Assume inductively that $\{P\} = \{\hat{P}_{(m-1)}\}$ and $\hat{P}_{(m-1)} = \hat{P}_2 + \cdots + \hat{P}_{m-1} + P_m + \cdots$, with $\hat{P}_i \in \hat{V}$ for $i \leq m-1$. Since $\hat{P}_{(m-1)}$ is a perturbation, $(d + \hat{P}_{(m-1)})^2 = 0$, so that $\delta P_m + \frac{1}{2}[\hat{P}_2, \hat{P}_{m-1}] + \cdots + \frac{1}{2}[\hat{P}_{m-1}, \hat{P}_2] = 0$ by equating homogeneous components. But $[\hat{P}_i, \hat{P}_j] = 0$ for each i,j, as in the proof of Lemma 4.3, thus $\delta P_m = 0$. So consider the class $[P_m] \in H^1(\mathcal{L}_{H(n/k)}, \delta)$. By Lemma 4.3 $P_m = \hat{P}_m + \delta\eta_{m-1}$, for some $\hat{P}_m \in \hat{V}$ and $\eta_{m-1} \in \mathcal{L}^0_{H(n/k)}$. Note that η_{m-1} increases bracket length by at least $m-1$. Now $[\eta_{m-1}, \hat{P}_i] = 0$ for $2 \leq i \leq m-1$, by Lemma 4.4, so

$$\begin{aligned}\exp(\operatorname{ad}\eta_{m-1})(d + \hat{P}_{(m-1)}) &= d + \hat{P}_{(m-1)} + [\eta_{m-1}, d] \\ &\qquad + \text{ terms that extend by } \geq m+1 \\ &= d + \hat{P}_2 + \cdots + \hat{P}_{m-1} + \hat{P}_m \\ &\qquad + \text{ terms that extend by } \geq m+1\end{aligned}$$

because $P_m - [d, \eta_{m-1}] = \hat{P}_m$. By the inductive hypothesis $\{P\} = \{\hat{P}_{(m-1)}\}$, so that $\exp(\operatorname{ad}\eta)(d+P) = d + \hat{P}_{(m-1)}$ for some $\eta \in \mathcal{L}^0_{H(n/k)}$. Now

$$\exp(\operatorname{ad}\eta_{m-1})\exp(\operatorname{ad}\eta)(d+P) = \exp(\operatorname{ad}\eta')(d+P),$$

where η' is related to η_{m-1} and η by the Baker-Campbell-Hausdorff formula, $\eta' = \eta_{m-1} + \eta + \frac{1}{2}[\eta_{m-1}, \eta] + \cdots$ [Ja, p. 174]. Thus putting $\hat{P}_{(m)} = \exp(\operatorname{ad}\eta_{m-1})(d + \hat{P}_{(m-1)})$, we have $\{P\} = \{\hat{P}_{(m)}\}$ and the inductive step is complete. Induction starts with $m = 1$, where $P = P_{(0)}$, and the result follows. ∎

LEMMA 4.6. *If $\hat{P} \in \hat{V}$, then $\{\hat{P}\} = \{0\}$ if and only if $[\hat{P}] = 0 \in H^1(\mathcal{L}_{H(n/k)}, \delta)$.*

PROOF. If $\hat{P} = \delta\eta$ for some $\eta \in \mathcal{L}^0_{H(n/k)}$, then $\big[\eta, [\eta, d]\big] = -[\eta, \hat{P}] = 0$, by Lemma 4.4, so that $\exp(\operatorname{ad}\eta)(d + \hat{P}) = d$.

Conversely, if $\exp(\operatorname{ad}\theta)(d{+}\hat{P}) = d$ for some $\theta \in \mathcal{L}^0_{H(n/k)}$, then use the fact that $[\theta, \hat{P}] = 0$, from Lemma 4.4, to re-write this as $\exp(\operatorname{ad}\theta)(d) = d - \hat{P}$. Write $\hat{P} = \hat{P}_2 + \hat{P}_3 + \cdots$ and

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$\theta = \theta_1 + \theta_2 + \cdots$, where $\hat{P}_i$ and θ_i both extend bracket length by i. Assume inductively that for some m and all $1 \leq i \leq m-1$, there exists $\eta_i \in \mathcal{L}^0_{H(n/k)}$ with $\delta\eta_i = \hat{P}_{i+1}$, and that

$$(\dagger) \qquad \exp(\operatorname{ad}\psi_{(m)})(d) = d - \hat{P}_{m+1} - \hat{P}_{m+2} - \cdots,$$

where $\psi_{(m)} \in \mathcal{L}^0_{H(n/k)}$ extends bracket length by at least m. Write $\psi_{(m)} = \psi_m + \psi_{m+1} + \cdots$, so that $\delta\psi_m = \hat{P}_{m+1}$ by equating homogeneous components of ($\dagger$). Now apply $\exp(\operatorname{ad} -\psi_m)$ to ($\dagger$) and obtain $\exp(\operatorname{ad}\psi_{(m+1)})(d) = d - \hat{P}_{m+2} - \hat{P}_{m+3} - \cdots$, where $\psi_{(m+1)}$ extends bracket length by at least $m+1$, from the Baker-Campbell-Hausdorff formula and Lemma 4.4. This completes the induction step. Induction starts with $m = 1$ and $\psi_{(1)} = \theta$. Thus there exists η_i with $\delta\eta_i = \hat{P}_i$ for all $i \geq 2$ and so $\delta\eta = \hat{P}$ with $\eta = \sum_{i\geq 2}\eta_i$, as desired. ∎

COROLLARY 4.7. *Let $\hat{P}, \hat{Q} \in \hat{V}$. Then $\{\hat{P}\} = \{\hat{Q}\}$ if and only if $[\hat{P}] = [\hat{Q}]$ in $H^1(\mathcal{L}_{H(n/k)}, \delta)$.*

PROOF. $\{\hat{P}\} = \{\hat{Q}\}$ if and only if $\exp(\operatorname{ad}\theta)(d + \hat{P}) - \hat{Q} = d$ for some $\theta \in \mathcal{L}^0_{H(n/k)}$. By Lemma 4.4, this holds if and only if $\exp(\operatorname{ad}\theta)(d + \hat{P} - \hat{Q}) = d$. This is true if and only if $\hat{P} - \hat{Q} = \delta\eta$, for some $\eta \in \mathcal{L}^0_{H(n/k)}$, by Lemma 4.6. ∎

The set $V/\exp(\operatorname{ad}\mathcal{L}^0_{H(n/k)})$ has a vector space structure which we now describe. If $\{P\} \in V/\exp(\operatorname{ad}\mathcal{L}^0_{H(n/k)})$, use Lemma 4.5 to choose a class representative $\hat{P} \in \hat{V}$ and write $\hat{P} = \sum_{p=n-k}^{n}\alpha_p \cdot \partial x_p$, with $\alpha_p \in \mathbb{L}(x_{k+2}, \ldots, x_{2k+1})$. Define scalar multiplication by $\lambda\{\hat{P}\} = \{\lambda\hat{P}\}$, where $\{\lambda\hat{P}\} = \sum_{p=n-k}^{n}(\lambda\alpha_p)\cdot\partial x_p$. This is well-defined by Corollary 4.7: If $\hat{Q} \in \hat{V}$ is another class representative, then $[\hat{P}] = [\hat{Q}]$ by Corollary 4.7, so $[\lambda\hat{P}] = [\lambda\hat{Q}]$ and hence $\{\lambda\hat{P}\} = \{\lambda\hat{Q}\}$ again by Corollary 4.7.

THEOREM 4.8. *There is an isomorphism of vector spaces*

$$\Phi\colon \frac{V}{\exp(\operatorname{ad}\mathcal{L}^0_{H(n/k)})} \longrightarrow H^1(\mathcal{L}_{H(n/k)}, \delta).$$

PROOF. Define $\Phi(\{P\}) = [\hat{P}]$, where $\hat{P}$ is a representative of $\{P\}$ chosen from $\hat{V}$. This is well-defined by Corollary 4.7 and is easily seen to be an isomorphism of vector spaces by Lemmas 4.5 and 4.6. ∎

The final stage of the Schlessinger-Stasheff classification program identifies perturbations that "differ by a change of basis in H". More precisely, $\operatorname{Aut}H$—the group of algebra automorphisms of the algebra H—induces an action on $V/\exp(\operatorname{ad}\mathcal{L}^0_{H(n/k)})$ whose orbits consist of perturbations representing the same rational homotopy type. For details see [S-S]. This action can be described as follows: If $\phi \in \operatorname{Aut}H$, consider the dual coalgebra automorphism ϕ_* of H_* and desuspend to a vector space automorphism $s^{-1}\phi_*$ of $s^{-1}H_*$. Extend to a free graded Lie algebra automorphism $s^{-1}\phi_*$ of $\mathbb{L}(s^{-1}H_*)$ and negatively regrade so that $s^{-1}\phi_*$ is an automorphism of L_H. We will abuse notation and denote $s^{-1}\phi_*$ by ϕ_*. Then for $\{P\} \in V/\exp(\operatorname{ad}\mathcal{L}^0)$, send $\{P\} \mapsto \{(\phi_*)P(\phi_*)^{-1}\}$.

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On the other hand, conjugation also induces an action on $H^1(\mathcal{L}_H, \delta)$, given by $[P] \mapsto [(\phi_*)P(\phi_*)^{-1}]$. When $H = H^*(\mathbb{C}P^n/\mathbb{C}P^k; \mathbb{Q})$, the vector space isomorphism of Theorem 4.8 is clearly compatible with these actions. Hence the bijection of Theorem 1.2 reduces to:

COROLLARY 4.9. *There is a bijection of sets*

$$\left\{ \begin{matrix} \textit{Rational Homotopy Types} \\ \textit{That Realize } H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q}) \end{matrix} \right\} \longleftrightarrow \left\{ \frac{H^1(\mathcal{L}_{H(n/k)},\delta)}{\mathrm{Aut}\, H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})} \right\}.$$

COROLLARY 4.10. *$H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is intrinsically formal if and only if $H^1(\mathcal{L}_{H(n/k)},\delta) = 0$.*

PROOF. Denote $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ by $H(n/k)$. The orbit of $[0]$ in $H^1(\mathcal{L}_{H(n/k)},\delta)$ under the action induced by $\mathrm{Aut}\, H(n/k)$ is just $[0]$. Hence $H^1(\mathcal{L}_{H(n/k)},\delta)/\mathrm{Aut}\, H(n/k) = 0$ implies that $H^1(\mathcal{L}_{H(n/k)},\delta) = 0$. The converse is Theorem 1.3. ∎

For each $n < \infty$, $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$. Thus $H^1(\mathcal{L}_{H(n/k)},\delta)$ is a countable set since it has finite rank. Hence by Corollary 4.9 there are a countable number of rational homotopy types that realize $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$.

REMARK 4.11. If H is an algebra with $H^i = 0$ for $1 \le i \le k-1$ and $i \ge 4k-2$, then the sequence of lemmas leading to Corollary 4.9 carry through with much simplification. This proves the following:

THEOREM 4.12 ([S-S, 8.5]). *Let H be an algebra with $H^i = 0$ for $1 \le i \le k-1$ and $i \ge 4k-2$. There exists a bijection of sets:*

$$\left\{ \begin{matrix} \textit{Rational Homotopy Types} \\ \textit{That Realize } H \end{matrix} \right\} \longleftrightarrow \left\{ \frac{H^1(\mathcal{L}_H,\delta)}{\mathrm{Aut}\, H} \right\}.$$ ∎

Corollary 4.9 and Theorem 4.12, whilst independent, are analogous results. Since [S-S, 8.5] is a statement about shallow spaces, it is reasonable to say that the algebras $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ display shallow-like behaviour.

We conclude with a discussion of the action induced by $\mathrm{Aut}\, H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ and an analysis of the cases in Theorem 1.1 that remain.

LEMMA 4.13. *Let $n \ge 4k+1$ and let $\{y_i\}_{k \le i \le n}$ be the additive basis for $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ introduced after Proposition 2.2. Then $\phi \in \mathrm{Aut}\, H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ if and only if $\phi(y_i) = \lambda^i y_i$ for some $\lambda \ne 0 \in \mathbb{Q}$.*

PROOF. Recall from Proposition 2.2 and as in Theorem 2.3, that $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q}) \cong \mathbb{Q}[y_{k+1},\ldots,y_{2k+1}]/\mathcal{S}$, where $\mathcal{S}$ is the ideal generated by $\{y_{k+1+i}y_{k+1+j} - y_{k+1}y_{k+1+i+j}\}$ for $0 \le i+j \le k$, $\{y_{k+1+i}y_{k+1+j} - y_{k+1}^2 y_{i+j}\}$ for $k+1 \le i+j \le 2k$, and all products y_iy_j with $i+j > n$. Let $\phi \in \mathrm{Aut}\, H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ and write $\phi(y_i) = \lambda_i y_i$ for each i. Put $j = 1$ in the first set of relations. If $n \geq 3k+2$, then $\phi(y_{k+1+i}y_{k+2} - y_{k+1}y_{k+2+i}) = 0$ implies that $\lambda_{k+i+2} = \lambda_{k+i+1}(\lambda_{k+2}/\lambda_{k+1})$ for $-1 \leq i \leq k-1$. Recursive application of this gives

$$(\dagger) \qquad \lambda_{k+i+2} = \lambda_{k+2}\left(\frac{\lambda_{k+2}}{\lambda_{k+1}}\right)^{i}$$

for $-1 \leq i \leq k-1$. On the other hand, by setting $i = k$ and $j = k-1$ in the second set of relations; if $n \geq 4k+1$, then $\phi(y_{2k+1}y_{2k} - y_{k+1}^2 y_{2k-1}) = 0$ implies that

$$(\ddagger) \qquad \lambda_{2k+1} \cdot \lambda_{2k} = \lambda_{k+1}^2 \cdot \lambda_{2k-1}.$$

Using $(\dagger)$, re-write each factor in $(\ddagger)$ in terms of λ_{k+1} and λ_{k+2}, both of which are non-zero rationals, to obtain $\lambda_{k+2} = (\lambda_{k+1})^{\frac{k+2}{k+1}}$. Hence, $\lambda_{k+1} = \lambda^{k+1}$ for some $\lambda \in \mathbb{Q}^*$, and so for each i, $\phi(y_i) = \lambda^i$ as required. ■

Recall that each $\phi \in \operatorname{Aut} H$ induces an action $P \longmapsto (\phi_*)P(\phi_*)^{-1}$ on perturbations. Since $H^{\mathrm{odd}}(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q}) = 0$, homogeneous length perturbations always extend bracket length by an odd number.

PROPOSITION 4.14. *Let P_{2r-1} be a perturbation of the Quillen model of $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ that extends bracket length by $2r-1$. If $\phi_\lambda \in \operatorname{Aut} H$ is given by $\phi_\lambda(y_i) = \lambda^i y_i$, then $(\phi_{\lambda *})(P_{2r-1})(\phi_{\lambda *})^{-1} = \lambda^{r-1}P_{2r-1}$.*

PROOF. Suppose $P_{2r-1}(\bar{x}_i) = \sum_I a_I\Big[\bar{x}_{i_1}, \Big[\ldots, [\bar{x}_{i_{2r-1}}, \bar{x}_{i_{2r}}]\Big]\cdots\Big]$. Since P_{2r-1} is of degree -1, $\sum_{j=1}^{2r} i_j = i + r - 1$ for each $I = (i_1, \ldots, i_{2r})$. Now $(\phi_{\lambda *})^{-1}(\bar{x}_i) = \lambda^{-i}\bar{x}_i$, so $(\phi_{\lambda *})(P_{2r-1})(\phi_{\lambda *})^{-1}(\bar{x}_i) = \lambda^{-i}(\phi_{\lambda *})(P_{2r-1})(\bar{x}_i)$. But $(\phi_{\lambda *})(\bar{x}_{i_j}) = \lambda^{i_j}\bar{x}_{i_j}$, and so $(\phi_{\lambda *})(P_{2r-1})(\phi_{\lambda *})^{-1}(\bar{x}_i) = \lambda^{r-1}P_{2r-1}(\bar{x}_i)$. ■

By Proposition 4.14, $\operatorname{Aut} H$ acts linearly on perturbations that extend bracket length by three, and non-linearly on perturbations that extend bracket length by more than three. This fact, in light of Corollary 4.9 and subsequent remarks, suggests that for a fixed k, most of the algebras $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$ are realized by a countably infinite number of rational homotopy types. The following technical corollaries make this precise:

COROLLARY 4.15. *Let $[P_{2r-1}]$ be non-zero in $H^1(\mathcal{L}_{H(n/k)}, \delta)$ and suppose P_{2r-1} extends bracket length by $2r-1$. If $r \geq 3$, then there are a countably infinite number of rational homotopy types that realize $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$.*

PROOF. Consider $\phi_\lambda \in \operatorname{Aut} H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$. By Proposition 4.14 $(\phi_{\lambda *})[P_{2r-1}](\phi_{\lambda *})^{-1} = \lambda^{r-1}[P_{2r-1}]$ so that for $s, t \in \mathbb{Q}$, $s[P_{2r-1}]$ and $t[P_{2r-1}]$ lie in the same orbit if and only if $s = \lambda^{r-1}t$. Thus $r \geq 3$ implies $H^1(\mathcal{L}_{H(n/k)}, \delta)$ has infinitely many distinct orbits, and the conclusion follows from Corollary 4.9. ■

COROLLARY 4.16. *If $\operatorname{rank}\big(H^1(\mathcal{L}_{H(n/k)}, \delta)\big) \geq 2$ as a vector space over $\mathbb{Q}$, then there exist a countably infinite number of rational homotopy types that realize $H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$.*

PROOF. If $[P]$ and $[Q]$ are linearly independent in $H^1(\mathcal{L}_{H(n/k)}, \delta)$, use Lemma 4.3 to choose respective representatives $\alpha \cdot \partial x_p, \beta \cdot \partial x_q \in \hat{V}$, where α and β have homogeneous bracket length. If either α or β have length 6 or more, the result follows from Corollary 4.15. So suppose that both α and β have length 4, and consider a linear combination $s[P]+t[Q]$, where $s,t\in\mathbb{Q}$. If $\phi_\lambda\in\operatorname{Aut}H^*(\mathbb{C}P^n/\mathbb{C}P^k;\mathbb{Q})$, then $(\phi_{\lambda*})(s[P]+t[Q])(\phi_{\lambda*})^{-1}=\lambda(s[P]+t[Q])$ by Proposition 4.14, and $s[P]+t[Q]$ lies in the same orbit as $s'[P]+t'[Q]$ if and only if $s'=\lambda s$ and $t'=\lambda t$. Thus $H^1(\mathcal{L}_{H(n/k)},\delta)$ has infinitely many orbits and the result follows from Corollary 4.9. ∎

4.17 PROOF OF THEOREM 4.1. (i) If $n=4k+9$, the proofs of Propositions 3.5, 3.6 and 3.9 imply that $(E_\infty^{p,1})^n=0$ for $p<n$, and that $(E_\infty^{n,1})^n$ contains a non-zero β-type element $\Big[x_{k+2},\big[x_{k+2},[x_{k+3},x_{k+3}]\big]\Big]\cdot\partial x_n$. Denote this element by $\beta\cdot\partial x_n$. For $k=2$, the Jacobi identity implies $(E_\infty^{n,1})^n$, and hence $H^1(\mathcal{L}_{H(n/k)},\delta)$, has rank 1. For $k\geq 3$, $(E_{k+1}^{n,1})^n$ is spanned by $\beta\cdot\partial x_n$ and $\zeta\cdot\partial x_n$, where ζ is the bracket $\Big[x_{k+2},\big[x_{k+2},[x_{k+2},x_{k+4}]\big]\Big]$—again using the Jacobi identity and degree constraints. Recall the action of $\bar\delta_{k+1}^\infty$ as described in Theorem 2.6. This gives $\bar\delta_{k+1}^\infty(\beta\cdot\partial x_n+\zeta\cdot\partial x_n)=0$; so Lemma 2.8 implies $(E_\infty^{n,1})^n$, and hence $H^1(\mathcal{L}_{H(n/k)},\delta)$, has rank 1. In either case, $H^1(\mathcal{L}_{H(n/k)},\delta)$ is generated by a single perturbation that extends bracket length by 3; so Proposition 4.14 implies that $\operatorname{Aut}H$ acts linearly, with the desired conclusion.

(ii) In the proof of Proposition 3.6 it is shown that $\alpha\cdot\partial x_{26}\in(E_4^{26,1})^{29}$ survives to $(E_\infty^{26,1})^{29}$, where α is the bracket $\big[[x_6,x_7],[x_7,x_7]\big]$. Furthermore, if $\beta\cdot\partial x_p\in(E_4^{p,1})^n$, for any n, and β is a bracket of length 6 or more, then $30\leq p\leq n$. So the Jacobi identity implies $(E_\infty^{26,1})^{29}$ has rank 1, and degree constraints imply $(E_\infty^{p,1})^{29}=0$ for $p=27,28$ and 29. Thus $(E_\infty^{*,1})^{29}$, and hence $H^1(\mathcal{L}_{H(n/k)},\delta)$ has rank 1 and is generated by a single perturbation that extends bracket length by 3. As before, Proposition 4.14 gives the desired conclusion.

(iii) In the proof of Proposition 3.9 it is shown that $\alpha_1\cdot\partial x_{18}\in(E_3^{18,1})^{20}$ survives to $(E_\infty^{18,1})^{20}$, where α_1 is the bracket $\big[[x_4,x_5],[x_5,x_5]\big]$. An argument identical to that for part (ii) above now completes the proof. ∎

4.18 PROOF OF THEOREM 4.2. (i) First, suppose $k\geq 3$ and $n\geq 6k+12$. By Lemmas 2.8 and 3.1 and Corollary 4.15, it is sufficient to identify a β-type bracket β, of length ≥ 6 such that $2n-2k-2\leq|\beta|\leq 2n-2$. For a given k, the length 6 β-type bracket having minimal degree $12k+22$ is $\operatorname{ad}^4(\bar x_{k+2})([\bar x_{k+3},\bar x_{k+3}])$. By Lemma 3.3, each even degree $\geq 12k+22$ contains a β-type bracket of length ≥ 6. Since $n\geq 6k+12$, or equivalently $2n-2\geq 12k+22$, the result follows.

Now suppose $k\geq 4$ and $4k+10\leq n\leq 6k+11$. By Corollary 4.16 it is sufficient to show $H^1(\mathcal{L}_{H(n/k)},\delta)$ has rank at least two. Recall from Section 2 that $\operatorname{rank}\big(H^1(\mathcal{L}_{H(n/k)},\delta)\big)=\operatorname{rank}\big((E_\infty^{*,1})^n\big)=\operatorname{rank}\big(\oplus_{p=n-k}^n(E_\infty^{p,1})^n\big)$; it is sufficient to show $(E_\infty^{p,1})^n\neq 0$ for at least two values of p in the range $n-k\leq p\leq n$. Now $(E_{k+1}^{p,1})^n\cong\{\oplus_{s\geq 3}\pi_{2(p-1),s}\}\cdot\partial x_p$, and in the range $n-k\leq p\leq n$, β-type elements $\beta\cdot\partial x_p$ survive to $(E_\infty^{p,1})^n$. By Lemma 3.3 there exist β-type brackets of length 4 in every even degree from $8k+16$ through $16k$. These give corresponding β-type elements $\beta\cdot\partial x_p\in(E_{k+1}^{p,1})^n$ for $4k+9\leq p\leq 8k+1$; so if $4k+10\leq n\leq 9k$, then $(E_\infty^{p,1})^n\neq 0$ for two or more values of p. By assumption,

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$k \geq 4$ so that $6k + 11 \leq 9k$; thus $(E_\infty^{p,1})^n \neq 0$ for two or more values of p whenever $4k + 10 \leq n \leq 6k + 11$.

(ii) By the first part of (i) above, it is sufficient to consider $22 \leq n \leq 28$. If $22 \leq n \leq 27$, then an argument identical to that of the first paragraph of (i) above gives the result. For $n = 28$, recall that the proof of Proposition 3.6 identified the element $\alpha \cdot \partial x_{26} \in (E_4^{26,1})^{28}$ surviving to $(E_\infty^{26,1})^{28}$, where α is the bracket $\left[[x_6, x_7], [x_7, x_7]\right]$. Furthermore, the β-type element $\left[x_6, \left[x_6, [x_7, x_7]\right]\right] \cdot \partial x_{25}$ survives to $(E_\infty^{25,1})^{28}$ by Lemmas 3.1 and 2.8. Hence, as above, $(E_\infty^{*,1})^{28}$ has rank at least two and the result follows from Corollary 4.16.

(iii) As in (i) above, it is sufficient by Corollary 4.15 to identify an element $\alpha \cdot \partial x_p$ surviving to $(E_\infty^{p,1})^n$, such that $n - 2 \leq p \leq n$ and α has length at least 6. Proposition 3.9 identifies such an element $\alpha \cdot \partial x_p$ for each $n \geq 24$, except $n = 30$. The bracket α in these cases must have length at least 6 for degree reasons. For the cases $n = 18$ and 19, recall the proof of Proposition 3.9 in which elements $\omega_4 \cdot \partial x_{17} \in (E_3^{17,1})^n$ and $\alpha_1 \cdot \partial x_{18} \in (E_3^{18,1})^n$ were identified, both surviving to $(E_3^{*,1})^n$ for $n = 18$ or 19. Thus $H^1(\mathcal{L}_{H(n/2)}, \delta)$ has rank at least two in these cases, and the conclusion follows from Corollary 4.16. ∎

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