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Piotr Achinger

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Abstract

A technical ingredient in Faltings' original approach to p-adic comparison theorems involves the construction of $K(\pi, 1)$-neighborhoods for a smooth scheme X over a mixed characteristic discrete valuation ring with a perfect residue field: every point $x \in X$ has an open neighborhood U whose generic fiber is a $K(\pi, 1)$ scheme (a notion analogous to having a contractible universal cover). We show how to extend this result to the logarithmically smooth case, which might help to simplify some proofs in p-adic Hodge theory. The main ingredient of the proof is a variant of a trick of Nagata used in his proof of the Noether normalization lemma.

1. Introduction

This paper contains several results about the étale topology of schemes over discrete valuation rings, mostly with applications to p-adic Hodge theory in mind. We prove the existence of $K(\pi, 1)$-neighborhoods for a log smooth scheme over a mixed characteristic discrete valuation ring (Theorem 6.1), and a comparison theorem between étale cohomology and the cohomology of Faltings' topos (Corollary 9.6). These could be used to simplify some arguments Faltings' second paper on p-adic Hodge theory [Fal02] using the approach of the first one [Fal88]. A reader familiar with the notion of a $K(\pi, 1)$ in the étale topology and with Faltings' topos might want to skip ahead to § 1.2.

1.1 Motivation and background

When studying differentiable manifolds, one benefits from the fact that the underlying topological space is locally contractible. This is not the case in algebraic geometry: a smooth complex algebraic variety often does not admit a Zariski open cover by subvarieties which are contractible in the classical topology (a curve of nonzero genus, for example). On the other hand, as noticed by Artin in the course of the proof of the comparison theorem [SGA4, Example XI, 4.4], one can find a Zariski open cover by $K(\pi, 1)$ spaces (see Theorem 3.3 for the precise statement). Recall that a path connected topological space is called a $K(\pi, 1)$ space if its universal cover is weakly contractible. This is equivalent to the vanishing of all of its higher homotopy groups.

The notion of a $K(\pi, 1)$ space has a natural counterpart in algebraic geometry, defined in terms of étale local systems. Let Y be a connected scheme with a geometric point $\overline{y}$. If $\mathscr{F}$ is a locally constant constructible abelian sheaf on $Y_{\text{ét}}$, the stalk $\mathscr{F}_{\overline{y}}$ is a representation of the fundamental group $\pi_1(Y, \overline{y})$, and we have natural maps

$$\rho^i : H^i(\pi_1(Y, \overline{y}), \mathscr{F}_{\overline{y}}) \longrightarrow H^i(Y_{\text{ét}}, \mathscr{F}).$$

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We call Y a $K(\pi,1)$ if for every n invertible on Y, and every $\mathscr{F}$ as above with $n\cdot\mathscr{F}=0$, the maps ρ^i are isomorphisms for all $i\geqslant 0$. See § 3 for a slightly more general definition and a discussion of this notion.

In a similar way as in Artin's comparison theorem, coverings by $K(\pi,1)$ play a role in Faltings' approach to p-adic comparison theorems [Fal88, Fal02, Ols09]. Let K be a finite extension of $\mathbb{Q}_p$ and let X_K be a smooth scheme over K. Loosely speaking, p-adic Hodge theory seeks to relate $H^i(X_{\overline{K}},\mathbb{Q}_p)$ to $H^i_{dR}(X_K/K)$ and other cohomology groups (see [Fon82]). In [Fal88, Fal02], as a step towards this comparison, under the assumption that there is a smooth model $X/\mathscr{O}_K$, Faltings defines an intermediate cohomology theory $\mathscr{H}^\bullet(X)$ as the cohomology of a certain topos $\widetilde{E}$ (following Abbes and Gros [AG15], we call it the *Faltings' topos*). This is the topos associated to a site E whose objects are morphisms $V\longrightarrow U$ over $X_{\overline{K}}\longrightarrow X$ with $U\longrightarrow X$ étale and $V\longrightarrow U_{\overline{K}}$ finite étale (see Definition 9.1 for the definition). The association $(V\longrightarrow U)\mapsto V$ induces a morphism of topoi

$$\Psi: X_{\overline{K},\text{ét}}\longrightarrow \widetilde{E}.$$

To compare $H^i(X_{\overline{K}},\mathbb{Q}_p)$ and $\mathscr{H}^\bullet(X)$, the first step is to investigate the higher direct images $R^i\Psi_*$. In this direction, Faltings shows the following generalization of Artin's result ([Fal88, Lemma II 2.1], see Theorem 3.4): *every point $x\in X$ has an open neighborhood U for which $U_{\overline{K}}$ is a $K(\pi,1)$*. It follows that *$R^i\Psi_*\mathscr{F}=0$ for $i>0$ and every locally constant constructible abelian sheaf $\mathscr{F}$ on $X_{\overline{K}}$*. It is these two results that we are going to generalize.

1.2 Contents of the paper

Let V be a discrete valuation ring with perfect residue field k and fraction field K of characteristic zero. Choose an algebraic closure $\overline{K}$ of K, and let

$$S=\operatorname{Spec} V,\quad s=\operatorname{Spec} k,\quad \eta=\operatorname{Spec} K,\quad \overline{\eta}=\operatorname{Spec}\overline{K}.$$

For a scheme X over S and an open subscheme $X^\circ\subseteq X$, we denote by $\widetilde{E}$ the Faltings' topos of $X^\circ_{\overline{\eta}}\longrightarrow X$ (see Definition 9.1), and by $\Psi: X^\circ_{\overline{\eta},\text{ét}}\longrightarrow\widetilde{E}$ the morphism of topoi Definition 9.1(c). For a geometric point $\overline{x}$ of X, we denote by $X_{(\overline{x})}$ the strict localization of X at $\overline{x}$. Consider the following four statements:

(A) *X has a basis of the étale topology consisting of U for which $U\times_X X^\circ_{\overline{\eta}}$ is a $K(\pi,1)$;*

(B) *for every geometric point $\overline{x}\in X$, $(X_{(\overline{x})}\times_{S_{(f(\overline{x}))}}\overline{\eta})\times_X X^\circ$ is a $K(\pi,1)$;*

(C) *$R^i\Psi_*\mathscr{F}=0$ $(i>0)$ for every locally constant constructible abelian sheaf $\mathscr{F}$ on $X^\circ_{\overline{\eta}}$;*

(D) *for every locally constant constructible abelian sheaf $\mathscr{F}$ on $X^\circ_{\overline{\eta}}$, the natural maps*

$$H^i(\widetilde{E},\Psi_*(\mathscr{F}))\longrightarrow H^i(X^\circ_{\overline{\eta},\text{ét}},\mathscr{F}).$$

are isomorphisms for all $i\geqslant 0$.

Then (A) ⇒ (B) ⇒ (C) ⇒ (D), and the aforementioned theorem of Faltings (Theorem 3.4) states that (A) holds if X is smooth over S (and $X^\circ=X$). Faltings has also shown [Fal88, Lemma II 2.3] that (B) is true if X is smooth over S and X° is the complement of a normal crossings divisor relative to S.

It is natural to ask whether these two results remain true if we do not require that X be smooth over S (we still want X_η, or at least X°_η, to be smooth over η). In general, the answer is no, even for X regular (see § 7 for a counterexample). Note that the scheme $X_{(\overline{x})}\times_{S_{(f(\overline{x}))}}\overline{\eta}$ in (B) is the algebraic analogue of the Milnor fiber.

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The most natural and useful generalization, hinted at in [Fal02, Remark on p. 242], and brought to our attention by Ahmed Abbes, seems to be the case of X log smooth over S, where we endow S with the 'standard' log structure $\mathscr{M}_S \longrightarrow \mathscr{O}_S$, i.e. the compactifying log structure induced by the open immersion $\eta \hookrightarrow S$. Our first main result confirms this expectation.

THEOREM (Theorem 6.1). *Assume that* $\operatorname{char} k = p > 0$. *Let* $(X, \mathscr{M}_X)$ *be a log smooth log scheme over* $(S, \mathscr{M}_S)$ *such that* X_η *is smooth over* η. *Then* (A) *holds for* $X^\circ = X$.

Note that in the applications, in the above situation one usually cares about the case $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$ (the biggest open on which the log structure is trivial). While the theorem deals with $X^\circ = X$, we are able to deduce corollaries about the other case as well (see the next section).

The strategy is to reduce to the smooth case (idea due to R. Lodh) by finding an étale neighborhood U' of $\overline{x}$ in X and a map

$$f : U' \longrightarrow W'$$

to a smooth S-scheme W' such that $f_\eta : U'_\eta \longrightarrow W'_\eta$ is finite étale. In such a situation, by Faltings's result (Theorem 3.4), there is an open neighborhood W of $f(x)$ (x being the underlying point of $\overline{x}$) in W' such that W_η is a $K(\pi, 1)$. Then $U = f^{-1}(W)$ is an étale neighborhood of $\overline{x}$, and since $U_\eta \longrightarrow W_\eta$ is finite étale, U_η is a $K(\pi, 1)$ as well.

The proof of the existence of f makes use of the technique of Nagata's proof of the Noether normalization lemma, combined with the observation that the exponents used in that proof can be taken to be divisible by high powers of p (see § 5.1). Therefore our proof applies only in mixed characteristic. While we expect the result to be true regardless of the characteristic, we point out an additional difficulty in equal characteristic zero in § 6.3.

We also treat the equicharacteristic zero case and the case with boundary. More precisely, we use Theorem 6.1 and log absolute cohomological purity to prove the following.

THEOREM (Theorem 9.5 and Corollary 9.6). *Let* $(X, \mathscr{M}_X)$ *be a log smooth log scheme over* $(S, \mathscr{M}_S)$ *such that* X_η *is smooth over* η, *and let* $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$ *be the biggest open subset on which* $\mathscr{M}_X$ *is trivial. If* $\operatorname{char} k = 0$, *assume moreover that* $(X, \mathscr{M}_X)$ *is saturated. Then* (B)–(D) *above hold for* X *and* X°.

1.3 Outline

Sections 2–4 are preliminary. Section 2 contains some abstract nonsense on cohomology groups of topoi, which is then used in § 3 where we review the notion of a $K(\pi, 1)$ scheme. Section 4 provides a review of the relevant logarithmic geometry.

Sections 5–7 constitute the heart of the paper. Section 5 deals with a variant of Noether normalization and proves the key Proposition 5.10. The proof of our main theorem, Theorem 6.1, is the subsequent § 6. Section 7 gives an example of a regular scheme for which the assertion of Theorem 6.1 does not hold.

Section 8 deals with the equicharacteristic zero case. The final section, § 9, reviews the definition of Faltings' topos and proves our second main result, the comparison Theorem 9.5.

2. Functoriality properties of the cohomology pullback maps

This section checks a certain functoriality property of cohomology of topoi, needed in § 3. The reader should feel no discomfort in skipping this part. The result we need is that given a

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commutative diagram of topoi

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array} \tag{2.0.1}$$

and a sheaf $\mathscr{F}$ on X, there exist certain natural commutative diagrams (2.2.2)

$$\begin{array}{ccc} H^i(Y, f_*\mathscr{F}) & \longrightarrow & H^i(X, \mathscr{F}) \\ \downarrow & & \downarrow \\ H^i(Y', g^*f_*\mathscr{F}) & \longrightarrow & H^i(X', g'^*\mathscr{F}) \end{array}$$

for all $i \geqslant 0$.

2.1 Base change maps

Suppose we are given a commutative diagram of morphisms of topoi as in (2.0.1), that is, a diagram of morphisms together with a chosen isomorphism

$$\iota : f_* g'_* \simeq g_* f'_*. \tag{2.1.1}$$

By adjunction, this also induces an isomorphism (also denoted ι)

$$\iota : f'^* g^* \simeq g'^* f^*. \tag{2.1.2}$$

Applying f_* to the unit $\eta : \mathrm{id} \longrightarrow g'_* g'^*$ and composing with (2.1.1) yields a map

$$f_* \longrightarrow f_* g'_* g'^* \simeq g_* f'_* g'_*,$$

which (using the adjunction between g^* and g_*) gives us a map

$$\varphi : g^* f_* \longrightarrow f'_* g'^* \tag{2.1.3}$$

called the base change map.

Similarly, applying g'^* to the counit $\varepsilon : f^* f_* \longrightarrow \mathrm{id}$, and composing with (2.1.2) yields a map

$$f'^* g^* f_* \simeq g'^* f^* f_* \longrightarrow g'^*,$$

which (using the adjunction between f'^* and f'_*) gives us another map $g^* f_* \longrightarrow f'_* g'^*$ that is equal to (2.1.3) by [SGA4, Example XVII, Proposition 2.1.3].

2.2 Cohomology pullback maps

Recall that if $f : X \longrightarrow Y$ is a morphism of topoi, there is a natural map of δ-functors from the category of abelian sheaves on Y to the category of abelian groups:

$$f^* : H^i(Y, -) \longrightarrow H^i(X, f^*(-)).$$

Indeed, the right-hand side is a δ-functor because f^* is exact, the transformation is defined for $i = 0$, and $H^i(Y, -)$ is a universal δ-functor.

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The formation of this map is compatible with composition; that is, if $g : Z \longrightarrow X$ is another map, the diagram (of δ-functors of the above type)

$$\begin{array}{ccc} H^i(Y,-) & \xrightarrow{f^*} & H^i(X, f^*(-)) \\ {\scriptstyle (fg)^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ H^i(Z,(fg)^*(-)) & = & H^i(Z, g^*f^*(-)) \end{array}$$

commutes.

Applying this to the situation of § 2.1 and composing with the map induced by the counit $\varepsilon : f^*f_* \longrightarrow \mathrm{id}$, we get a system of natural transformations

$$\mu : H^i(Y, f_*(-)) \xrightarrow{f^*} H^i(X, f^*f_*(-)) \xrightarrow{\varepsilon} H^i(X,-). \tag{2.2.1}$$

These coincide with the edge homomorphisms in the Leray spectral sequence for f by [EGAIII, 0_{III} 12.1.7].

Let μ' be the following composition:

$$\mu' : H^i(Y', g^*f_*(-)) \xrightarrow{\varphi} H^i(Y', f'_*g'^*(-)) \xrightarrow{f'^*} H^i(X', f'^*f'_*g'^*(-)) \xrightarrow{\varepsilon'} H^i(X', g'^*(-)),$$

where $\varepsilon' : f'^*f'_* \longrightarrow \mathrm{id}$ is the counit.

The goal of this section is to show that the diagram

$$\begin{array}{ccc} H^i(Y, f_*(-)) & \xrightarrow{\mu} & H^i(X,(-)) \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g'^*} \\ H^i(Y', g^*f_*(-)) & \xrightarrow[\mu']{} & H^i(X', g'^*(-)) \end{array} \tag{2.2.2}$$

commutes.

2.3 Commutativity of (2.2.2)

The assertion of § 2.1 will follow from the commutativity of the following diagram.

$$\begin{array}{ccccc} H^i(Y, f_*(-)) & \xrightarrow{f^*} & H^i(X, f^*f_*(-)) & \xrightarrow{\varepsilon} & H^i(X,-) \\ & & {\scriptstyle g'^*}\downarrow & \texttt{(III)} & \downarrow{\scriptstyle g'^*} \\ {\scriptstyle g^*}\downarrow & \texttt{(I)} & H^i(X', g'^*f^*f_*(-)) & \xrightarrow[g'^*(\varepsilon)]{} & H^i(X', g'^*(-)) \\ & & \| {\scriptstyle \iota} & & \\ H^i(Y', g^*f_*(-)) & \xrightarrow{f'^*} & H^i(X', f'^*g^*f_*(-)) & \texttt{(IV)} & \\ {\scriptstyle \varphi}\downarrow & \texttt{(II)} & \downarrow{\scriptstyle f'^*(\varphi)} & & \\ H^i(Y', f'^*g'^*(-)) & \xrightarrow[f'^*]{} & H^i(X', f'^*f'_*g'^*(-)) & \xrightarrow{\varepsilon'} & H^i(X', g'^*(-)) \end{array}$$

Square (I) commutes by the functoriality of f^* (see § 2.2). Squares (II) and (III) commute simply because f^* is a natural transformation.

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It remains to prove that (IV) commutes. This in turn will follow from the commutativity of the following diagram.

$$\begin{array}{ccc} f'^*g^*f_* & \overset{\iota}{=\!=\!=} & g'^*f^*f_* \\ {\scriptstyle f'^*(\varphi)}\downarrow & & \downarrow{\scriptstyle g'^*(\varepsilon)} \\ f'^*f'_*g'^* & \underset{\varepsilon'}{\longrightarrow} & g'^* \end{array}$$

By the discussion of § 2.1, the composition $g'^*(\varepsilon) \circ \iota$ above is adjoint (under the adjunction between f'^* and f'_*) to the base change map $\varphi : g^*f_* \longrightarrow f'_*g'^*$. It suffices to show that $\varepsilon \circ f'^*(\varphi)$ is also adjoint to the base change map. This follows precisely from the triangle identities for the adjunction between f'^* and f'_*.

3. $K(\pi, 1)$ schemes

This section recalls the definition of a $K(\pi, 1)$ space in algebraic geometry, establishes some basic properties that apparently do not appear in the literature, and states the theorems of Artin and Faltings which assert the existence of coverings of smooth schemes by $K(\pi, 1)$ schemes.

3.0.1. *Basic assumption.* We will often consider schemes which are coherent (i.e. quasi-compact and quasi-separated) and have a finite number of connected components (see [AG15, 9.6] for some criteria).

Let Y be a scheme satisfying assumption 3.0.1. We denote by $\mathrm{Fét}(Y)$ the full subcategory of the étale site $\mathrm{Ét}(Y)$ consisting of *finite* étale maps $Y' \longrightarrow Y$, endowed with the induced topology, and by $Y_{\mathrm{fét}}$ the corresponding topos (cf. [AG15, 9.2]). Note that the maps in $\mathrm{Fét}(Y)$ are also finite étale. The inclusion functor induces a morphism of topoi (cf. [AG15, 9.2.1])

$$\rho : Y_{\mathrm{ét}} \longrightarrow Y_{\mathrm{fét}}.$$

The pullback ρ^* identifies $Y_{\mathrm{fét}}$ with the category of sheaves on $Y_{\mathrm{ét}}$ equal to the union of their locally constant subsheaves (cf. [Ols09, 5.1] and [AG15, 9.17]). If Y is connected and $\overline{y} \longrightarrow Y$ is a geometric point, we have an equivalence of topoi $Y_{\mathrm{fét}} \simeq B\pi_1(Y, \overline{y})$ where $B\pi_1(Y, \overline{y})$ is the classifying topos of $\pi_1(Y, \overline{y})$.

Definition 3.1 (cf. [Ols09, Definition 5.3]). Let $\wp$ be a set of prime numbers. A scheme Y satisfying § 3.0.1 is called a $K(\pi, 1)$ *for $\wp$-adic coefficients* if for every integer n with $\mathrm{ass}\, n \subseteq \wp$, and every sheaf of $\mathbb{Z}/(n)$-modules F on $Y_{\mathrm{fét}}$, the natural map

$$F \longrightarrow R\rho_*\rho^*F$$

is an isomorphism. If $\wp$ is the set of primes invertible on Y, we simply call Y a $K(\pi, 1)$.

The above condition is equivalent to saying that if $\mathscr{F}$ is a locally constant constructible sheaf of $\mathbb{Z}/(n)$-modules on $Y_{\mathrm{ét}}$, then $R^i\rho_*\mathscr{F} = 0$ for $i > 0$ (cf. [AG15, 9.17]).

Proposition 3.2. *Let $\wp$ be a set of prime numbers, and let Y be a scheme satisfying § 3.0.1.*

(a) *The scheme Y is a $K(\pi, 1)$ for $\wp$-adic coefficients if and only if for every integer n with $\mathrm{ass}\, n \subseteq \wp$, every locally constant constructible sheaf of $\mathbb{Z}/(n)$-modules $\mathscr{F}$ on $Y_{\mathrm{ét}}$, and every class $\zeta \in H^i(Y, \mathscr{F})$ with $i > 0$, there exists a finite étale surjective map $f : Y' \longrightarrow Y$ such that $f^*(\zeta) = 0 \in H^i(Y', f^*\mathscr{F})$.*

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(b) *Let $f : X \longrightarrow Y$ be a finite étale surjective map. Then X satisfies § 3.0.1, and Y is a $K(\pi,1)$ for $\wp$-adic coefficients if and only if X is.*

(c) *Suppose that Y is of finite type over a field F and that F' is a field extension of F. Denote $X = Y_{F'}$. Then X satisfies § 3.0.1, and Y is a $K(\pi,1)$ if and only if X is.*

Proof. (a) Let $\mathscr{F}$ be a locally constant constructible sheaf of $\mathbb{Z}/(n)$-modules on $Y_{\text{ét}}$. Then $R^i\rho_*\mathscr{F}$ is the sheaf of $\text{Fét}(Y)$ associated to the presheaf

$$(f : X \longrightarrow Y) \quad \mapsto \quad H^i(X, f^*\mathscr{F}).$$

It follows that $R^i\rho_*\mathscr{F} = 0$ if and only if the following condition (a′) holds: *for every finite étale $(f : X \longrightarrow Y) \in \text{Fét}(Y)$ and every $\zeta \in H^i(X, f^*\mathscr{F})$, there exists a cover $\{g_i : (f_i : X_i \longrightarrow Y) \longrightarrow (f : X \longrightarrow Y)\}_{i\in I}$ such that $f_i^*\zeta = 0 \in H^i(X_i, g_i^* f^*\mathscr{F}) = H^i(X_i, f_i^*\mathscr{F})$.* In case Y is connected, each g_i with X_i nonempty is finite étale surjective, and hence in such case (a′) implies (a) by considering $Y' = Y$. The general case follows by considering the connected components of Y separately.

We prove that the condition in (a) implies (a′). In the situation of (a′), let $\mathscr{F}_0 = f^*\mathscr{F}$ for brevity, and consider the sheaf $f_*\mathscr{F}_0$. As f is finite étale, $f_*\mathscr{F}_0$ is locally constant constructible and $R^jf_*\mathscr{F}_0 = 0$ for $j > 0$, therefore the natural map, (2.2.1)

$$\mu : H^i(Y, f_*\mathscr{F}_0) \longrightarrow H^i(X, \mathscr{F}_0), \tag{3.2.1}$$

is an isomorphism. Let $\zeta' \in H^i(Y, f_*\mathscr{F})$ map to ζ under (3.2.1). By (a), there exists a finite étale surjective map $g : Y' \longrightarrow Y$ with $g^*\zeta' = 0 \in H^i(Y', g^*f_*\mathscr{F}_0)$. Form a cartesian diagram as follows.

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

Then g' is finite étale and surjective. Moreover, by § 2.1, the diagram

$$\begin{array}{ccccc} & & \zeta' & \longmapsto & \zeta \\ & & \cap & & \cap \\ \zeta' & \in & H^i(Y, f_*\mathscr{F}_0) & \xrightarrow{\mu} & H^i(X, \mathscr{F}_0) \\ \Big\downarrow & & {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle g'^*} \\ 0 & \in & H^i(Y', g^*f_*\mathscr{F}_0) & \xrightarrow[\mu']{} & H^i(X', g'^*\mathscr{F}_0) \end{array}$$

commutes, and hence $g'^*\zeta = 0$.

(b) The argument is similar to (a). If X is a $K(\pi,1)$, then Y is a $K(\pi,1)$ as well by the characterization of (a). Suppose that Y is a $K(\pi,1)$ and let $\mathscr{F}_0$ be a locally constant constructible sheaf of $\mathbb{Z}/(n)$-modules on $X_{\text{ét}}$, $\zeta \in H^i(X, \mathscr{F}_0)$ $(i > 0)$. Apply the same reasoning as in (a).

(c) If F'/F is a finite separable extension, this follows from (b) as then $X \longrightarrow Y$ is finite étale and surjective. If F' is a separable closure of F, the assertion follows from the characterization in (a) and usual limit arguments. If F'/F is finite and purely inseparable, $X \longrightarrow Y$ induces equivalences $X_{\text{ét}} \simeq Y_{\text{ét}}$ and $X_{\text{fét}} \simeq Y_{\text{fét}}$, so there is nothing to prove. If F' and F are both

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algebraically closed, the assertion follows from [SGA4, Example XVI, 1.6]. If F'/F is arbitrary, pick an algebraic closure $\overline{F}'$ and let $\overline{F}$ be the algebraic closure of F in $\overline{F}'$. We now have a 'path' from F to F' of the form

$$F \subseteq F^{\mathrm{sep}} \subseteq \overline{F} \subseteq \overline{F}' \supseteq (F')^{\mathrm{sep}} \supseteq F'$$

and the assertion follows from the preceding discussion. □

Finally, we state the two key results about the existence of $K(\pi, 1)$ neighborhoods mentioned in the introduction.

Theorem 3.3 (Artin, follows from [SGA4, Example XI, 3.3], cf. [Ols09, Lemma 5.5]). *Let Y be a smooth scheme over a field of characteristic zero, y a point of Y. There exists an open neighborhood U of y which is a $K(\pi, 1)$.*

Note that the use of Abhyankar's lemma in the proof of [Ols09, Lemma 5.5] is why we need the characteristic zero assumption.

Theorem 3.4 (Faltings, [Fal88, Lemma II 2.1], cf. [Ols09, Theorem 5.4]). *Let S be as in § 1.2, let Y be a smooth S-scheme, and let y be a point of Y. There exists an open neighborhood U of y for which U_η is a $K(\pi, 1)$.*

4. Some logarithmic geometry

In this section, we review the relevant facts from log geometry and investigate the local structure of a log smooth S-scheme (with the standard log structures on X and S). We also state the logarithmic version of absolute cohomological purity, used in § 8.

4.1 Conventions about log geometry

If P is a monoid, $\overline{P}$ denotes the quotient of P by its group P^* of invertible elements, and $P \longrightarrow P^{gp}$ is the universal (initial) morphism form P into a group. P is called *fine* if it is finitely generated and integral (i.e. $P \longrightarrow P^{gp}$ is injective). A *face* of a monoid P is a submonoid $F \subseteq P$ satisfying $x+y \in F \Rightarrow x, y \in F$. For an integral monoid P and face F, the *localization* of P at F is the submonoid P_F of P^{gp} generated by P and $-F$. It satisfies the obvious universal property. If Q is a submonoid of an integral monoid P, the quotient P/Q is defined to be the image of P in P^{gp}/Q^{gp}.

For a monoid P, $\mathbb{A}_P = \operatorname{Spec}(P \longrightarrow \mathbb{Z}[P])$ is the log scheme associated to P; for a homomorphism $\theta : P \longrightarrow Q$, $\mathbb{A}_\theta : \mathbb{A}_Q \longrightarrow \mathbb{A}_P$ is the induced morphism of log schemes. A morphism $(X, \mathscr{M}_X) \longrightarrow (Y, \mathscr{M}_Y)$ of log schemes is *strict* if the induced map $f^\flat : f^*\mathscr{M}_Y \longrightarrow \mathscr{M}_X$ is an isomorphism. A strict map to some $\mathbb{A}_P$ is called a *chart*. A log scheme is *fine* if étale locally it admits a chart with target $\mathbb{A}_P$ for a fine monoid P. If $j : U \longrightarrow X$ is an open immersion, the *compactifying log structure* on X associated to U is the preimage of $j_*\mathscr{O}_U^*$ under the restriction map $\mathscr{O}_X \longrightarrow j_*\mathscr{O}_U$. For a log scheme $(X, \mathscr{M}_X)$, we denote by $(X, \mathscr{M}_X)_{\mathrm{tr}}$ the complement of the support of $\overline{\mathscr{M}}_X$. It is an open subset of X if $(X, \mathscr{M}_X)$ is fine, the biggest open subset on which $\mathscr{M}_X$ is trivial. See [Kat89, ACG+13].

We recall Kato's structure theorem for log smooth morphisms (which for our purposes might as well serve as a definition).

Theorem 4.2 [Kat89, Theorem 3.5]. *Let $f : (X, \mathscr{M}_X) \longrightarrow (S, \mathscr{M}_S)$ be a morphism of fine log schemes. Assume that we are given a chart $\pi : (S, \mathscr{M}_S) \longrightarrow \mathbb{A}_Q$ with Q a fine monoid. Then f is log smooth if and only if, étale locally on X, there exists a fine monoid P, a map $\rho : Q \longrightarrow P$*

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such that the kernel and the torsion part of the cokernel of $\rho^{gp} : Q^{gp} \longrightarrow P^{gp}$ are finite groups of order invertible on S, and a commutative diagram

$$\begin{array}{ccccc} (X, \mathscr{M}_X) & \longrightarrow & \mathbb{A}_{P,\rho,\pi} & \longrightarrow & \mathbb{A}_P \\ & \searrow^{f} & \downarrow & & \downarrow{\scriptstyle \mathbb{A}_\rho} \\ & & (S, \mathscr{M}_S) & \xrightarrow[\pi]{} & \mathbb{A}_Q, \end{array}$$

where the square is cartesian and $(X, \mathscr{M}_X) \longrightarrow \mathbb{A}_{P,\rho,\pi}$ is strict and étale (as a morphism of schemes).

4.3 Charts

Suppose that $f : (X, \mathscr{M}_X) \longrightarrow \mathbb{A}_P$ is a chart with P a fine monoid, and $\overline{x} \longrightarrow X$ is a geometric point. Let $F \subseteq P$ be the preimage of zero under the induced homomorphism $P \longrightarrow \overline{\mathscr{M}}_{X,\overline{x}}$. Then F is a face of P, and P injects into the localization P_F. Moreover, the induced map $P/F \longrightarrow \overline{\mathscr{M}}_{X,\overline{x}}$ is an isomorphism.

As P is finitely generated, F is finitely generated as a face, and hence the natural map $\mathbb{A}_{P_F} \longrightarrow \mathbb{A}_P$ is an open immersion: if F is generated as a face by an element $a \in P$, then $\mathbb{A}_{P_F} = D(a)$. Let us form a cartesian diagram.

$$\begin{array}{ccc} (U, \mathscr{M}_X|_U) & \longrightarrow & \mathbb{A}_{P_F} \\ \downarrow & & \downarrow \\ (X, \mathscr{M}_X) & \xrightarrow[f]{} & \mathbb{A}_P \end{array}$$

Then $U = D(f^*a)$ and $\overline{x}$ lies in U because f^*a is an invertible element of $\mathscr{O}_{X,x}$ by the construction of F.

It follows that any chart $f : (X, \mathscr{M}_X) \longrightarrow \mathbb{A}_P$ as above can be locally replaced by one for which the homomorphism $\overline{P} \longrightarrow \overline{\mathscr{M}}_{X,\overline{x}}$ is an isomorphism, without changing the local properties of f (e.g. without sacrificing étaleness if f is étale).

4.4 Charts in our setting

In the situation of § 1.2, let $f : (X, \mathscr{M}_X) \longrightarrow (S, \mathscr{M}_S)$ be a log smooth morphism. Applying Theorem 4.2 to a chart $\pi : (S, \mathscr{M}_S) \longrightarrow \mathbb{A}_{\mathbb{N}}$ given by a uniformizer π of V, we conclude that, étale locally on X, there exists a strict étale $g : (X, \mathscr{M}_X) \longrightarrow \mathbb{A}_{P,\rho,\pi}$ where

$$\mathbb{A}_{P,\rho,\pi} = \operatorname{Spec}\left(P \longrightarrow \frac{V[P]}{(\pi - \rho)}\right) \tag{4.4.1}$$

for a fine monoid P and a non-invertible element $\rho \in P$ with the property that $(P/\rho)^{gp}$ is p-torsion free.

Assume that X_η is smooth over η. Localizing P, we can assume that the scheme underlying $(\mathbb{A}_{P,\rho,\pi})_\eta$ is smooth over η as well. But $(\mathbb{A}_{P,\rho,\pi})_\eta$ is isomorphic to $\operatorname{Spec}(P/\rho \longrightarrow K[P/\rho])$. Note that for a fine monoid M, $\operatorname{Spec} K[M]$ is smooth over K if and only if $\overline{M}$ is a free monoid. It follows that $\overline{P}/\rho$ is free, and therefore the stalks of $\mathscr{M}_{X/S} := \mathscr{M}_X / f^\flat \mathscr{M}_S$ are free monoids. Moreover, every geometric point $\overline{x}$ of X has an étale neighborhood U such that $(U_\eta, \mathscr{M}_X|_{U_\eta})_{\mathrm{tr}}$ is the complement of a divisor with strict normal crossings on U_η.

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4.5 Absolute cohomological purity

We will need the following result (cf. [Nak98, Proposition 2.0.2]). Let $(X, \mathscr{M}_X)$ be a regular [Kat94, Definition 2.1] (in particular, fine and saturated) log scheme such that X is locally noetherian. Let $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$ be the biggest open subset on which $\mathscr{M}_X$ is trivial, and let $j : X^\circ \longrightarrow X$ be the inclusion. Let n be an integer invertible on X. Then for any $q \geqslant 0$, we have a natural isomorphism

$$R^q j_*(\mathbb{Z}/n\mathbb{Z}) \simeq \bigwedge^q \overline{\mathscr{M}}_X^{gp} \otimes \mathbb{Z}/n\mathbb{Z}(-1). \tag{4.5.1}$$

5. η-étale maps and Noether normalization

This section contains the key technical point used in the proof of Theorem 6.1. First, we prove a (slightly spiced-up) relative version of the Noether normalization lemma (Proposition 5.4). Then we study η-étale maps $f : X \longrightarrow Y$ over S, that is, maps which are étale in an open neighborhood of the closed fiber X_s of X. The main result is Proposition 5.10, which asserts that *in mixed characteristic* we can often replace an η-étale map $f' : X \longrightarrow \mathbb{A}^d_S$ by a quasi-finite η-étale map $f : X \longrightarrow \mathbb{A}^d_S$.

5.1 Relative Noether normalization

Lemma 5.2. *Let F be a field, and let $a \in F[x_1, \ldots, x_n]$ be a nonzero polynomial. For large enough m, the polynomial*

$$a(x_1 - x_n^m, x_2 - x_n^{m^2}, \ldots, x_{n-1} - x_n^{m^{n-1}}, x_n),$$

treated as a polynomial in x_n over $F[x_1, \ldots, x_{n-1}]$, has a constant leading coefficient.

Proof. The proof is standard (cf. e.g. [Mum99, § 1] or [Sta14, Tag 051N]). □

Definition 5.3. Let $f : X \longrightarrow Y$ be a map of schemes over some base scheme S. We call f *fiberwise finite relatively to S* if for every point $s \in S$, the induced map $X_s \longrightarrow Y_s$ is finite.

Let $V, S, \ldots$ be as in § 1.2 (the assumptions on K and k are unnecessary here). The following is a relative variant of Noether normalization. In the applications we will take N to be a high power of p.

Proposition 5.4. *Let $X = \operatorname{Spec} R$ be a flat affine S-scheme of finite type, let $d \geqslant 0$ be an integer such that $\dim(X/S) \leqslant d$, and let $x_1, \ldots, x_d \in R$. For any integer $N \geqslant 1$, there exist $y_1, \ldots, y_d \in R$ such that the following hold.*

(i) *The map $f = (f_1, \ldots, f_d) : X \longrightarrow \mathbb{A}^d_S$, $f_i = x_i + y_i$, is fiberwise finite relatively to S.*

(ii) *The y_i belong to the subring generated by Nth powers of elements of R.*

Proof. Write $R = V[x_1, \ldots, x_d, x_{d+1}, \ldots, x_n]/I$. The proof is by induction on $n - d$. If $n = d$, then the map $(x_1, \ldots, x_d) : X \longrightarrow \mathbb{A}^d_S$ is a closed immersion, and we can take $y_i = 0$. Suppose that $n > d$.

Let $a \in V[x_1, \ldots, x_n]$ be an element of I with nonzero image in $k[x_1, \ldots, x_n]$. Such an element exists, for otherwise X_s is equal to $\mathbb{A}^n_s$, and hence cannot be of dimension $\leqslant d$ as $n > d$.

For an integer $m \geqslant 1$, consider the elements

$$z_i = x_i + x_n^{(Nm)^i}, \quad i = 1, \ldots, n-1,$$

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and let $R' \subset R$ be the V-subalgebra generated by $z_1, \dots, z_{n-1}$. By Lemma 5.2 applied to the image of a in $K[x_1, \dots, x_n]$ (with $F = K$) respectively $k[x_1, \dots, x_n]$ (with $F = k$), there exists an m such that the images of x_n in $R \otimes K$, respectively $R \otimes k$ will be integral over $R' \otimes K$, respectively $R' \otimes k$. As $x_i = z_i - x_n^{(Nm)^i}$, the other x_i will have the same property, which is to say, $\operatorname{Spec} R \longrightarrow \operatorname{Spec} R'$ is fiberwise finite over S.

We check that it is possible to apply the induction assumption to $X' = \operatorname{Spec} R'$ and $z_1, \dots, z_d \in R'$. Since R' is a subring of R and R is torsion-free, R' is torsion-free as well, and hence flat over V. As $R' \otimes_V K \longrightarrow R \otimes_V K$ is finite and injective, we have $\dim X'_\eta = \dim X_\eta$. Since R' is flat over V, we have $\dim X'_s \leqslant \dim X'_\eta$, so $\dim X'_s \leqslant d$ as well. Finally, R' is generated as a V-algebra by $n-1$ elements with $z_1, \dots, z_d$ among them.

By the induction assumption applied to $X' = \operatorname{Spec} R'$ and $z_1, \dots, z_d \in R$, there exists a fiberwise finite map $f' = (f_1, \dots, f_d) : \operatorname{Spec} R' \longrightarrow \mathbb{A}^d_S$ with $f_i = z_i + y'_i$, where the y'_i belong to the subring of R' generated by Nth powers of elements of R'. As the composition of fiberwise finite maps is clearly fiberwise finite, the composition $f = (f_1, \dots, f_n) : X = \operatorname{Spec} R \longrightarrow \mathbb{A}^d_S$ is fiberwise finite, proving part (i). We have $f_i = x_i + y_i$, $y_i = y'_i + (x_n^{N^{i-1}m^i})^N$, so part (ii) is satisfied as well. □

It would be interesting to have a generalization of this result to a general noetherian local base ring V.

5.5 η-étale maps

We now assume that $\operatorname{char} k = p > 0$. Let $f : X \longrightarrow Y$ be a map of S-schemes of finite type.

DEFINITION 5.6. We call f *η-étale at a point* $x \in X_s$ if there is an open neighborhood U of x in X such that $f_\eta : U_\eta \longrightarrow Y_\eta$ is étale. We call f *η-étale* if it is η-étale at all points $x \in X_s$, or equivalently, if there is an open neighborhood U of X_s in X such that $f_\eta : U_\eta \longrightarrow Y_\eta$ is étale.

We warn the reader not to confuse 'f is η-étale' with 'f_η is étale' (the latter is a stronger condition).

LEMMA 5.7. *Consider the following properties.*

- (i) *The map f is η-étale.*
- (ii) *There exists an $n \geqslant 0$ such that $(p^n \Omega^1_{X/Y})|_{X_s} = 0$ (pullback as an abelian sheaf).*
- (iii) *There exists an $n \geqslant 0$ such that $(p^n \Omega^1_{X/Y}) \otimes_V k = 0$.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii), *and the three properties are equivalent if X_η and Y_η are smooth of the same relative dimension d over S.*

Proof. The equivalence of (ii) and (iii) follows from Nakayama's lemma.

Suppose that f is η-étale, and let $U \subseteq X$ be an open subset containing X_s such that $f|_{U_\eta}$ is étale. In particular, $f|_{U_\eta}$ is unramified, and hence $\Omega^1_{X/Y}|_{U_\eta} = 0$.

Recall that if $\mathscr{F}$ is a coherent sheaf on a noetherian scheme U and $a \in \Gamma(U, \mathscr{O}_U)$, then $\mathscr{F}|_{D(a)} = 0$ if and only if $a^n \mathscr{F} = 0$ for some $n \geqslant 0$. Applying this to $\mathscr{F} = \Omega^1_{X/Y}|_U$ and $a = p$ (noting that $U_\eta = D(p)$), we get that $(p^n \Omega^1_{X/Y})|_U = 0$, and hence, in particular, $(p^n \Omega^1_{X/Y})|_{X_s} = 0$.

Suppose now that $(p^n \Omega^1_{X/Y})|_{X_s} = 0$ for some $n \geqslant 0$. By Nakayama's lemma, $(p^n \Omega^1_{X/Y})_x = 0$ for every $x \in X_s$, and hence there is an open subset U containing X_s such that $(p^n \Omega^1_{X/Y})|_U = 0$. As p is invertible on U_η, we get that $\Omega^1_{X/Y}|_{U_\eta} = 0$; that is, $f|_{U_\eta}$ is unramified. If X_η and Y_η are smooth of the same dimension over S, this is enough to guarantee that $f|_{U_\eta}$ is étale. □

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Lemma 5.8. *Suppose that f is closed and η-étale. There exists an open neighborhood W of Y_s in Y such that $f : f^{-1}(W)_\eta \longrightarrow W_\eta$ is étale.*

Proof. Let $Z \subseteq X_\eta$ be the locus where f_η is not étale. Then Z is closed in X. Since f is a closed, $f(Z)$ is closed in Y, and of course $f(Z) \cap Y_s = \emptyset$. Take $W = Y \backslash f(Z)$. $\square$

We now consider the case $Y = \mathbb{A}^d_S$.

Lemma 5.9. *Suppose that $f' : X \longrightarrow \mathbb{A}^d_S$ is such that $(p^n\Omega^1_{f'})|_{X_s} = 0$ and that $y_1, \ldots, y_d \in \Gamma(X, \mathscr{O}_X)$ are polynomials in p^{n+1}-powers of elements of $\Gamma(X, \mathscr{O}_X)$. If $f = f' + (y_1, \ldots, y_d)$, then $(p^n\Omega^1_f)|_{X_s} = 0$ as well.*

Proof. Let $S_n = \operatorname{Spec} V/p^{n+1}V$, $X_n = X \times_S S_n$. The presentations

$$\mathscr{O}^d_X \xrightarrow{df'_i} \Omega^1_{X/S} \longrightarrow \Omega^1_{f'} \longrightarrow 0, \quad \mathscr{O}^d_X \xrightarrow{df_i} \Omega^1_{X/S} \longrightarrow \Omega^1_f \longrightarrow 0$$

give after base change to S_n the short exact sequences

$$\mathscr{O}^d_{X_n} \xrightarrow{df'_i} \Omega^1_{X_n/S_n} \longrightarrow \Omega^1_{f'}/p^{n+1} \longrightarrow 0, \quad \mathscr{O}^d_{X_n} \xrightarrow{df_i} \Omega^1_{X_n/S_n} \longrightarrow \Omega^1_f/p^{n+1} \longrightarrow 0.$$

By the assumption on the y_i, we have $dy_i \in p^{n+1}\Omega^1_{X/S}$, and therefore the two maps $\mathscr{O}^d_{X_n} \longrightarrow \Omega^1_{X_n/S_n}$ above are the same. It follows that $\Omega^1_f/p^{n+1} \simeq \Omega^1_{f'}/p^{n+1}$. The assumption, that $(p^n\Omega^1_{f'})|_{X_s} = 0$, means that $p^n\Omega^1_{f'}/\pi p^n\Omega^1_{f'} = 0$ for a uniformizer π of V. As $p = u\pi^e$ for a unit $u \in V$ and $e \geqslant 1$ an integer, we have $p^n\Omega^1_{f'}/p^{n+1}\Omega^1_{f'} = 0$. Since $\Omega^1_f/p^{n+1} \simeq \Omega^1_{f'}/p^{n+1}$, the same holds for Ω^1_f. We thus have $(p^n\Omega^1_f)|_{X_n} = 0$, and hence $(p^n\Omega^1_f)|_{X_s} = 0$. $\square$

Proposition 5.10. *Assume that X affine and flat over S, that X_η is smooth of relative dimension d over S, and that $f' : X \longrightarrow \mathbb{A}^d_S$ is η-étale. There exists an $f : X \longrightarrow \mathbb{A}^d_S$ which is η-étale and fiberwise finite over S.*

Proof. By Lemma 5.7, there exists an n such that $(p^n\Omega^1_{f'})|_{X_s} = 0$. Apply Proposition 5.4 to $x_i = f'_i$ and $N = p^{n+1}$, obtaining a fiberwise finite $f : X \longrightarrow \mathbb{A}^d_S$ which differs from f' by some polynomials in p^{n+1}-powers. Then f is η-étale by Lemmas 5.9 and 5.7. $\square$

6. Existence of $K(\pi, 1)$-neighborhoods

Theorem 6.1. *Assume that $\operatorname{char} k = p > 0$. Let $(X, \mathscr{M}_X)$ be a log smooth log scheme over $(S, \mathscr{M}_S)$ such that X_η is smooth over η, and let $\overline{x} \longrightarrow X$ be a geometric point. There exists an étale neighborhood U of $\overline{x}$ such that U_η is a $K(\pi, 1)$.*

Proof. The proof is presented in §§ 6.1.1–6.1.7.

6.1.1. If $\overline{x}$ is contained in X_η, the existence of such a neighborhood follows from Theorem 3.3. We are therefore going to restrict ourselves to the case where $\overline{x}$ is a geometric point of the closed fiber X_s. The question being étale local around $\overline{x}$, we are allowed to shrink X around $\overline{x}$ if needed.

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6.1.2. Let π be a uniformizer of V, inducing a chart $(S, \mathscr{M}_S) \longrightarrow \mathbb{A}_{\mathbb{N}}$. By the discussion of § 4.4, in an étale neighborhood of $\overline{x}$ there exists a fine monoid P, an element $\rho \in P$ such that $\overline{P/\rho}$ is a free monoid, and an étale map

$$g : X \longrightarrow \mathbb{A}_{P,\rho,\pi} = \operatorname{Spec}\left(P \longrightarrow \frac{V[P]}{(\pi - \rho)}\right)$$

over S.

We can replace X by an étale neighborhood of $\overline{x}$ for which the above data exist. Shrinking X further, we can also assume that X is affine.

6.1.3. Let us denote by $P[\rho^{-1}]$ the submonoid of P^{gp} generated by P and ρ^{-1}, and by P/ρ the quotient of $P[\rho^{-1}]$ by the subgroup generated by ρ. Since $\overline{P[\rho^{-1}]} = \overline{P/\rho}$ is free, there is an isomorphism $P[\rho^{-1}] \simeq P[\rho^{-1}]^* \oplus \overline{P[\rho^{-1}]}$. Picking an isomorphism $\overline{P[\rho^{-1}]} \simeq \mathbb{N}^b$ and a decomposition of $P[\rho^{-1}]^*$, we can write $P[\rho^{-1}] \simeq T \oplus \mathbb{Z} \oplus \mathbb{Z}^a \oplus \mathbb{N}^b$ where T is a finite abelian group and ρ corresponds to an element of the $\mathbb{Z}$ summand. Dividing by ρ, we obtain an isomorphism $P/\rho \simeq T \oplus \mathbb{Z}^a \oplus \mathbb{N}^b$. Let $d = a + b$, and let $\chi_0 : \mathbb{N}^d = \mathbb{N}^a \oplus \mathbb{N}^b \longrightarrow T \oplus \mathbb{Z}^a \oplus \mathbb{N}^b \simeq P/\rho$ be the map implied by the notation. As the source of χ_0 is free and $P \longrightarrow P/\rho$ is surjective, we can choose a lift $\chi : \mathbb{N}^d \longrightarrow P$ of χ_0.

$$\begin{array}{ccc} & & \mathbb{N}^d \\ & \overset{\chi}{\swarrow} & \downarrow{\scriptstyle \chi_0} \\ P & \twoheadrightarrow & P/\rho \end{array}$$

Then χ induces a map $h : \mathbb{A}_{P,\rho,\pi} \longrightarrow \mathbb{A}^d_S$ over S.

6.1.4. I claim that h_η is étale. Note first that h_η is the pullback under $\pi : \eta \longrightarrow \mathbb{A}_{\mathbb{Z}}$ of the horizontal map in the following diagram.

$$\begin{array}{ccc} \mathbb{A}_{P[\rho^{-1}]} & \xrightarrow{\mathbb{A}_{\rho \oplus \chi}} & \mathbb{A}_{\mathbb{Z}} \times \mathbb{A}_{\mathbb{N}^d} \\ & \searrow \quad \swarrow & \\ & \mathbb{A}_{\mathbb{Z}} & \end{array}$$

We want to check that the horizontal map becomes étale after base change to $\mathbb{Q}$. Since the base is $\mathbb{A}_{\mathbb{Z}} = \mathbb{G}_m$ and the map is $\mathbb{G}_m$-equivariant, it suffices to check this on one fiber. If we set $\rho = 1$, the resulting map is none other than the map induced by

$$\mathbb{N}^d = \mathbb{N}^a \oplus \mathbb{N}^b \hookrightarrow \mathbb{Z}^a \oplus \mathbb{N}^b \hookrightarrow T \oplus \mathbb{Z}^a \oplus \mathbb{N}^b \simeq P/\rho = P[\rho^{-1}]/\rho,$$

which is étale after adjoining $1/\#T$.

6.1.5. Let $f' = h \circ g : X \longrightarrow \mathbb{A}^d_S$. This map is η-étale, and therefore, by Proposition 5.10, there exists a map $f : X \longrightarrow \mathbb{A}^d_S$ which is η-étale and fiberwise finite over S (hence quasi-finite). Let $\overline{y} \longrightarrow \mathbb{A}^d_S$ be the image of $\overline{x}$ under f. As f is quasi-finite, we can perform an étale localization at $\overline{x}$ and $\overline{y}$ which will make it finite. More precisely, we apply [EGAIV, Théorème 18.12.1] (or [Sta14, Tag 02LK]) and conclude that there exists a commutative diagram

$$\begin{array}{ccccc} \overline{x} & \longrightarrow & U' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \overline{y} & \longrightarrow & W' & \longrightarrow & \mathbb{A}^d_S \end{array}$$

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with $U' \longrightarrow X'$ and $W' \longrightarrow \mathbb{A}^d_S$ étale and $f : U' \longrightarrow W'$ finite. It follows that $f : U' \longrightarrow W'$ is also η-étale.

6.1.6. By Lemma 5.8 applied to $f : U' \longrightarrow W'$, we can shrink W' around $\overline{y}$ (and shrink U' accordingly to be the preimage of the new W') so that $U'_\eta \longrightarrow W'_\eta$ is finite étale.

6.1.7. Since W' is smooth over S, by Faltings' theorem (Theorem 3.4) there is an open neighborhood W of $\overline{y}$ in W' such that W_η is a $K(\pi, 1)$. Let U be the preimage of W in U' under $f : U' \longrightarrow W'$. The induced map $f_\eta : U_\eta \longrightarrow W_\eta$ is finite étale, and hence U_η is a $K(\pi, 1)$ as well by Proposition 3.2(b). □

6.2 Relatively smooth log structures

A reader familiar with the notion of a relatively smooth log structure (cf. [NO10, Definition 3.6], [Ogu09]) might appreciate the fact that the above proof applies to relatively smooth X/S as well. Recall that we call $(X, \mathscr{F})/(S, \mathscr{M}_S)$ *relatively log smooth* if, étale locally on X, there exists a log smooth log structure $(X, \mathscr{M})/(S, \mathscr{M}_S)$ and an inclusion $\mathscr{F} \subseteq \mathscr{M}$ as a finitely generated sheaf of faces, for which the stalks of $\mathscr{M}/\mathscr{F}$ are free monoids. We can then apply Theorem 6.1 to $(X, \mathscr{M})$ instead of $(X, \mathscr{F})$.

Important examples of relatively log smooth X/S appear in the Gross–Siebert program in mirror symmetry [GS06, GS10, GS11] as so-called *toric degenerations*. Degenerations of Calabi–Yau hypersurfaces in toric varieties are instances of such. For example, the *Dwork families*

$$X = \operatorname{Proj} V[x_0, \ldots, x_n] \Big/ \left((n+1) x_0 \cdot \ldots \cdot x_n - \pi \cdot \sum_{i=0}^n x_i^{n+1} \right)$$

(with the standard compactifying log structure) are relatively log smooth over S if $n + 1$ is invertible on S, but not log smooth for $n > 2$ [Ogu09, Proposition 2.2].

6.3 Obstacles in characteristic zero

The need for the positive residue characteristic assumption in our proof of Theorem 6.1 can be traced down to the application of Proposition 5.10: one can perform relative Noether normalization on an η-étale map $f' : X \longrightarrow \mathbb{A}^d_S$ without sacrificing η-étaleness. One might think that this is too crude and that one could replace that part with a Bertini-type argument. After all, we only need η-étaleness at one point! Unfortunately, this is bound to fail in characteristic zero even in the simplest example, that of a semistable curve.

Proposition 6.4. *Let X be an open subset of $\operatorname{Spec} V[x, y]/(xy - \pi) \subseteq \mathbb{A}^2_S$ containing the point $P = (0, 0) \in \mathbb{A}^2_k$ and let $f : X \longrightarrow \mathbb{A}^1_S$ be an S-morphism. If $f_\eta : X_\eta \longrightarrow \mathbb{A}^1_\eta$ is étale, then df is identically zero on one of the components of X_s. In particular, if $\operatorname{char} k = 0$, then f has to contract one of the components of X_s, and hence is not quasi-finite.*

Proof. Let Z be the support of $\Omega^1_{X/\mathbb{A}^1_S}$. As f_η is étale, $Z \subseteq X_s$. On the other hand, the short exact sequence

$$\mathscr{O}_X^2 \xrightarrow{\begin{bmatrix} f_x & y \\ f_y & x \end{bmatrix}} \mathscr{O}_X^2 \xrightarrow{\begin{bmatrix} dx & dy \end{bmatrix}} \Omega^1_{X/\mathbb{A}^1_S} \longrightarrow 0$$

shows that Z is the set-theoretic intersection of two divisors in $\mathbb{A}^2_S$ (given by the equations $xy = \pi$ and $xf_x - yf_y = 0$), each of them passing through P, for if $g \cdot \Omega^1_{X/\mathbb{A}^1_S} = 0$, then

$$\begin{bmatrix} f_x & y \\ f_y & x \end{bmatrix} \cdot C = \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}$$

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for some matrix C, and hence $g^2 = (xf_x - yf_y) \cdot \det(C) \in (xf_x - yf_y)$. Since $\mathbb{A}^2_S$ is regular, by the dimension theorem we know that each irreducible component of Z passing through P has to be of positive dimension. Therefore Z has to contain one of the components of X_s. □

7. A counterexample

The following is an example of an X/S where X is *regular*, but for which $K(\pi, 1)$ neighborhoods do not exist.

PROPOSITION 7.1. *Let $k = \mathbb{C}$, $S_0 = \mathbb{A}^1_k$ with coordinate π, $X_0 = \operatorname{Spec} k[\pi, x_0, \dots, x_n]/(\pi - f)$ where $f = x_0^2 + \cdots + x_n^2$ for some $n > 1$. Let S be the henselization of S_0 at 0, η its generic point, and $\overline{\eta}$ a geometric point above η. Finally let $X = X_0 \times_{S_0} S$ and $\overline{x} = (0, 0, \dots, 0) \in X$. Then $(X_{(\overline{x})})_{\overline{\eta}}$ is not a $K(\pi, 1)$. In particular, there does not exist a basis of étale neighborhoods of $\overline{x}$ in X whose generic fibers are $K(\pi, 1)$. However, X is regular.*

Proof. Note that $X_{(\overline{x})} = (X_0)_{(\overline{x})}$, $(X_{(\overline{x})})_\eta = (X_0)_{(\overline{x})} \backslash \{\pi = 0\}$. It is enough to show that

(1) $H^n((X_{(\overline{x})})_{\overline{\eta}}, \mathbb{Z}/\ell) \simeq \mathbb{Z}/\ell$;

(2) the scheme $(X_{(\overline{x})})_{\overline{\eta}}$ is simply connected.

Fact (1) follows from the computation of vanishing cycles [SGA7, XV 2.2.5]. For fact (2), it suffices to prove that $(X_{(\overline{x})})_\eta \to \eta$ induces an isomorphism on fundamental groups, or equivalently on $H^1(-, G)$ for every finite group G. By the comparison theorem [SGA4, XVI 4.1] applied to the inclusion $j : X_0 \backslash \{\pi = 0\} \hookrightarrow X_0$, $q = 1$ and a finite group G, we have

$$H^1((X_{(\overline{x})})_\eta, G) \simeq (R^1 j_{\text{ét}*} G)_{\overline{x}} \simeq (R^1 j_{cl*} G)_{\overline{x}} \simeq \operatorname*{colim}_{\varepsilon \to 0} H^1(B(\varepsilon) \backslash f^{-1}(0), G)$$

where $B(\varepsilon) = \{(x_0, \dots, x_n) \in \mathbb{C}^n : \sum |x_i|^2 < \varepsilon\}$. But by the Milnor fibration and bouquet theorems [Mil68], the homotopy fiber of $f : B(\varepsilon) \longrightarrow D(\varepsilon) := \{z \in \mathbb{C} : |z| < \varepsilon\}$ has type S^n and hence is simply connected as $n > 1$. The long exact sequence of homotopy groups of that fibration shows that $\pi_1(B(\varepsilon)) \simeq \pi_1(S^1)$, and hence $H^1(B(\varepsilon) \backslash f^{-1}(0), G) \simeq H^1(S^1, G) \simeq H^1(D(\varepsilon) \backslash \{0\}, G)$. Because the diagram

$$\begin{array}{ccc} H^1((X_{(\overline{x})})_\eta, G) & \longrightarrow & \operatorname*{colim}_{\varepsilon \to 0} H^1(B(\varepsilon) \backslash f^{-1}(0), G) \\ \uparrow & & \uparrow \\ H^1(\eta, G) & \longrightarrow & \operatorname*{colim}_{\varepsilon \to 0} H^1(D(\varepsilon) \backslash \{0\}, G) \end{array}$$

commutes, we conclude that $H^1(\eta, G) \longrightarrow H^1((X_{(\overline{x})})_\eta, G)$ is an isomorphism as claimed. □

Remark. Note that, on the other hand, for $f : \mathbb{A}^n \longrightarrow \mathbb{A}^1$ given by a monomial $x_1 \dots x_r$, the Milnor fiber at 0 is homotopy equivalent to a torus $(S^1)^r$, which is a $K(\pi, 1)$. This explains why one should expect Theorems 6.1 and 8.1 to be true.

8. The equicharacteristic zero case

THEOREM 8.1. *Let $(X, \mathscr{M}_X)$ be a regular (cf. [Kat94, Definition 2.1]) log scheme over $\mathbb{Q}$ such that X is locally noetherian, and let $\overline{x} \longrightarrow X$ be a geometric point. Let $\mathscr{F}$ be a locally constant constructible abelian sheaf on $X^\circ := (X, \mathscr{M}_X)_{\text{tr}}$, the biggest open subset on which $\mathscr{M}_X$ is trivial, and let $\zeta \in H^i(X^\circ, \mathscr{F})$ for some $i > 0$. There exists an étale neighborhood U of $\overline{x}$ and a finite étale surjective map $V \longrightarrow U^\circ$ such that ζ maps to zero in $H^i(V, \mathscr{F})$.*

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Proof. The proof is presented in §§ 8.1.1–8.1.3. □

8.1.1. In proving the assertion, I claim that we can assume that $\mathscr{F}$ is constant. Let $Y \longrightarrow X^\circ$ be a finite étale surjective map such that the pullback of $\mathscr{F}$ to Y is constant. By the logarithmic version of Abhyankar's lemma [GR14, Theorem 10.3.43], $Y = X'^\circ$ for a finite and log étale $f : (X', \mathscr{M}_{X'}) \longrightarrow (X, \mathscr{M}_X)$. Then $(X', \mathscr{M}_{X'})$ is also log regular (by [Kat94, Theorem 8.2]). Choose a geometric point $\overline{x}' \longrightarrow X'$ mapping to $\overline{x}$, and let $\mathscr{F}' = f^{\circ *}\mathscr{F}$, which is a constant sheaf on X'°.

Suppose that we found an étale neighborhood U' of x' and a finite étale surjective map $V' \longrightarrow U'^\circ$ killing $\zeta' := f^{\circ *}(\zeta) \in H^i(X'^\circ, \mathscr{F}')$. Let X'' be the normalization of U' in V', and choose a geometric point $\overline{x}''$ mapping to $\overline{x}'$. By [EGAIV, Théorème 18.12.1] (or [Sta14, Tag 02LK]); then there exists a diagram

$$\begin{CD} \overline{x}'' @>>> V @>>> X'' \\ @VVV @VVV @VVV \\ \overline{x} @>>> U @>>> X \end{CD}$$

with $U \longrightarrow X$ and $V \longrightarrow X''$ étale and $V \longrightarrow U$ finite. It follows that $V^\circ \longrightarrow U^\circ$ is also étale, and that the pullback of ζ to V° is zero.

In proving the theorem, we can therefore assume that $\mathscr{F} \simeq \mathbb{Z}/n\mathbb{Z}$ for some integer n, by considering the direct summands.

8.1.2. The question being étale local around $\overline{x}$, we can assume that there exists a chart $g : (X, \mathscr{M}_X) \longrightarrow \mathbb{A}_P$ for a fine saturated monoid P, which we use to form a cartesian diagram.

$$\begin{CD} (X', \mathscr{M}_{X'}) @>{f}>> (X, \mathscr{M}_X) \\ @V{g'}VV @VV{g}V \\ \mathbb{A}_P @>>{\mathbb{A}_{\cdot n}}> \mathbb{A}_P \end{CD} \tag{8.1.1}$$

Then $(X', \mathscr{M}_{X'})$ is log regular, f is finite, and $f : X'^\circ \longrightarrow X^\circ$ is étale. Choose a geometric point $\overline{x}' \longrightarrow X'$ mapping to $\overline{x}$. We have a commutative diagram

$$\begin{CD} \overline{\mathscr{M}}^{gp}_{X,\overline{x}} @>{f^*}>> \overline{\mathscr{M}}^{gp}_{X',\overline{x}'} \\ @AAA @AAA \\ \overline{P} @>>{\cdot n}> \overline{P} \end{CD}$$

where the vertical maps are surjections induced by the strict morphisms g and g'. We conclude that the map

$$f^\flat \otimes \mathbb{Z}/n\mathbb{Z} : \overline{\mathscr{M}}^{gp}_{X,\overline{x}} \otimes \mathbb{Z}/n\mathbb{Z} \longrightarrow \overline{\mathscr{M}}^{gp}_{X'\overline{x}'} \otimes \mathbb{Z}/n\mathbb{Z} \tag{8.1.2}$$

is zero.

8.1.3. Denote the inclusion $X^\circ \hookrightarrow X$ (respectively $X'^\circ \hookrightarrow X'$) by j (respectively j'). By log absolute cohomological purity (4.5.1), there is a functorial isomorphism

$$R^i j_*(\mathbb{Z}/n\mathbb{Z}) \simeq \bigwedge^i (\overline{\mathscr{M}}^{gp}_X \otimes \mathbb{Z}/n\mathbb{Z}(-1)).$$

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In our situation, this means that there is a commutative diagram

$$\begin{array}{ccccc} H^i(X'^{\circ},\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{sp_{\overline{x}'}} & R^i j'_*(\mathbb{Z}/n\mathbb{Z})_{\overline{x}'} & \xrightarrow{\sim} & \bigwedge^i(\overline{\mathscr{M}}^{gp}_{X',\overline{x}'}\otimes\mathbb{Z}/n\mathbb{Z}) \\ \uparrow f^* & & \uparrow f^* & & \uparrow \bigwedge^i(8.1.2) \\ H^i(X^{\circ},\mathbb{Z}/n\mathbb{Z}) & \xrightarrow[sp_{\overline{x}}]{} & R^i j_*(\mathbb{Z}/n\mathbb{Z})_{\overline{x}} & \xrightarrow{\sim} & \bigwedge^i(\overline{\mathscr{M}}^{gp}_{X,\overline{x}}\otimes\mathbb{Z}/n\mathbb{Z}) \end{array}$$

where the rightmost map is zero for $i>0$ because (8.1.2) is zero.

It follows that ζ maps to zero in $R^i j'_*(\mathbb{Z}/n\mathbb{Z})_{\overline{x}'}$, and hence there exists an étale neighborhood U' of $\overline{x}'$ such that ζ maps to zero in $H^i(U'^{\circ},\mathbb{Z}/n\mathbb{Z})$. Applying once again the argument of the second paragraph of § 8.1.1 yields an étale neighborhood U of $\overline{x}$ and a finite étale map $V \longrightarrow U$ killing ζ, as desired. □

9. The comparison theorem

In [AG15], Abbes and Gros have developed a theory of generalized co-vanishing topoi, of which the Faltings' topos is a special case. This topos has first been introduced in [Fal02], though the definition of [AG15] is different. For the reader's convenience, let us recall the definitions, adapting them to our setup.

DEFINITION 9.1. Let $f : Y \longrightarrow X$ be a morphism of schemes.

(a) The *Faltings' site* E associated to f is the site with

- OBJECTS morphisms $V \longrightarrow U$ over $f : Y \longrightarrow X$ with $U \longrightarrow X$ étale and $V \longrightarrow U \times_X Y$ finite étale,
- MORPHISMS commutative squares over $f : Y \longrightarrow X$,
- TOPOLOGY generated by coverings of the following form:
  - (V, for vertical) $\{(V_i \longrightarrow U) \longrightarrow (V \longrightarrow U)\}$ with $\{V_i \longrightarrow V\}$ a covering,
  - (C, for cartesian) $\{(V \times_U U_i \longrightarrow U_i) \longrightarrow (V \longrightarrow U)\}$ with $\{U_i \longrightarrow U\}$ a covering.

(b) The *Faltings' topos* $\widetilde{E}$ is the topos associated to E.

(c) We denote by $\Psi : Y_{\text{ét}} \longrightarrow \widetilde{E}$ the morphism of topoi induced by the continuous map of sites $(V \longrightarrow U) \mapsto V : E \longrightarrow \acute{E}t_{/Y}$.

LEMMA 9.2. *If* $\dim H^0(X_{\text{ét}},\mathbb{F}_\ell)$ *is finite then* X *has a finite number of connected components.*

Proof. The projection $\alpha : X_{\text{ét}} \to X$ induces an injection $\mathbb{F}_\ell \to \alpha_*\mathbb{F}_\ell$, so $H^0(X,\mathbb{F}_\ell)$ is finite as well. Since the map sending a clopen subset to its characteristic function is injective, the set of clopen subsets of X must be finite. If x is a point in X, let U_x be the intersection of all the clopen sets containing x. Since there are only finitely many such sets, U_x is open and closed. Furthermore, it is connected, so it must be the connected component of X containing x. Hence every connected component of X is clopen, and X has finitely many connected components. □

PROPOSITION 9.3. *In the notation of § 1.2, let* X *be a scheme of finite type over* S*. Let* $X^{\circ} \subseteq X$ *be an open subset such that the inclusion* $u : X^{\circ} \hookrightarrow X$ *is an affine morphism, and let* $Y = X^{\circ}_{\overline{\eta}}$*. Finally, let* $\wp$ *be a set of prime numbers. The following conditions are equivalent.*

(a) *For every étale* U *over* X *and every* $\wp$*-torsion locally constant constructible abelian sheaf* $\mathscr{F}$ *on* $U \times_X Y$*, we have* $R^i\Psi_{U*}\mathscr{F} = 0$ *for* $i>0$*, where* $\Psi_U : U \times_X Y \longrightarrow \widetilde{E}_U$ *is the morphism of Definition 9.1(c) for* $U \times_X Y \longrightarrow U$*.*

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(b) *For every étale U over X, every $\wp$-torsion locally constant constructible abelian sheaf $\mathscr{F}$ on $U \times_X Y$, every class $\zeta \in H^i(U \times_X Y, \mathscr{F})$ with $i > 0$, and every geometric point $\overline{x} \longrightarrow U$, there exists an étale neighborhood U' of $\overline{x}$ in U and a finite étale surjective map $V \longrightarrow U \times_X Y$ such that the image of ζ in $H^i(V, \mathscr{F})$ is zero.*

(c) *For every geometric point $\overline{x} \longrightarrow X$, $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is a $K(\pi, 1)$ for $\wp$-adic coefficients.*

Proof. The equivalence of (a) and (b) follows from the fact that $R^i\Psi_*\mathscr{F}$ is the sheaf associated to the presheaf $(V \longrightarrow U) \mapsto H^i(V, \mathscr{F})$ on E and the argument in Proposition 3.2(b). Note that $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is affine, therefore coherent. In case $\overline{x} \in X_s$, the finiteness of the number connected components follows from Lemma 9.2 and the finiteness of $\Gamma((X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ, \mathbb{F}_\ell)$, which is the stalk at $\overline{x}$ of the 0th nearby cycle functor of $u_*\mathbb{F}_\ell$ [SGA4$\frac{1}{2}$, Th. finitude 3.2]. If $\overline{x} \in X_\eta$, this follows similarly from the constructibility of $u_*\mathbb{F}_\ell$. The equivalence of (b) and (c) is then clear in the view of Proposition 3.2. $\square$

Corollary 9.4. *Suppose that X has a basis for the étale topology consisting of U for which $U \times_X Y$ is a $K(\pi, 1)$ for $\wp$-adic coefficients. Then the conditions of Proposition 9.3 are satisfied.*

Theorem 9.5. *Let $(X, \mathscr{M}_X)$ be a log smooth log scheme over $(S, \mathscr{M}_S)$ such that X_η is smooth over η, and let $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$. If $\operatorname{char} k = 0$, assume moreover that $(X, \mathscr{M}_X)$ is saturated. Then for every geometric point $\overline{x}$ of X, $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is a $K(\pi, 1)$.*

Proof. We should first note that $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ satisfies condition § 3.0.1 by the argument used in the proof of Proposition 9.3.

In case $\operatorname{char} k = 0$, as $(X, \mathscr{M}_X)$ is regular by [Kat94, Theorem 8.2], Theorem 8.1 implies condition (b) of Proposition 9.3, and hence $X_{(\overline{x})} \times_X X^\circ_\eta$ is a $K(\pi, 1)$ (note that $X^\circ \subseteq X_\eta$). As $(X_{(\overline{x})} \times_{S_{(f(\overline{x}))}} \overline{\eta}) \times_X X^\circ$ is a limit of finite étale covers of $X_{(\overline{x})} \times_X X^\circ_\eta$, it is a $K(\pi, 1)$ as well.

We will now assume that $\operatorname{char} k = p > 0$ and follow [Fal88, Lemma II 2.3] (see also [Ols09, 5.10–5.11]). As before, if $\overline{x} \in X_\eta$, this follows from Theorem 8.1 applied to $X_{\overline{\eta}}$, so let us assume that $\overline{x} \in X_s$. For simplicity, we can also replace S by $S_{(f(\overline{x}))}$ and X by the suitable base change. By Theorem 6.1 and Corollary 9.4, we know that $Z := (X_{(\overline{x})})_{\overline{\eta}}$ is a $K(\pi, 1)$. Since X_η is smooth, Z is regular and $Z^\circ = X_{(\overline{x})} \times_X X^\circ_{\overline{\eta}}$ is obtained from Z by removing divisor with strictly normal crossings $D = D_1 \cup \cdots \cup D_r$. Let $\mathscr{F}$ be a locally constant constructible abelian sheaf on Z°, and pick a $\zeta \in H^i(Z^\circ, \eta)$ $(i > 0)$. We want to construct a finite étale cover of Z° killing ζ.

By Abhyankar's lemma [SGA1, Example XIII, Appendice I, Proposition 5.2], there is an integer n such that if $f : Z' \longrightarrow Z$ is a finite cover with ramification indices along the D_i nonzero and divisible by n, then $f^{\circ *}\mathscr{F}$ extends to a locally constant constructible sheaf on Z'. I claim that we can choose Z' which is a $K(\pi, 1)$. By the previous considerations, it suffices to find Z' equal to $(X'_{(\overline{x}')})_{\overline{\eta}}$ for some X'/S' satisfying the same assumptions as X. We can achieve this by choosing a chart $X \longrightarrow \mathbb{A}_P$ around $\overline{x}$ as before and taking a fiber product as in (8.1.1) (and $S' = \operatorname{Spec} V[\pi']/(\pi'^n - \pi)$).

Now that we can assume that $\mathscr{F} = j^*\mathscr{F}'$ where $\mathscr{F}'$ is locally constant constructible on Z and $j : Z^\circ \hookrightarrow Z$ is the inclusion, we choose a finite étale cover $g : Y \longrightarrow Z$, Galois with group G, for which $g^*\mathscr{F}'$ is constant.

Let $f : Z' \longrightarrow Z$ be a finite cover with ramification indices along the D_i nonzero and divisible by some integer n. I claim that for any $b \geqslant 0$, the base change map

$$f^* R^b j_* \mathscr{F} \longrightarrow R^b j'_*(f^{\circ *}\mathscr{F}) \tag{9.5.1}$$

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is divisible by n^b. In case $\mathscr{F}$ is constant, this follows once again from logarithmic absolute cohomological purity (4.5.1), and in general can be checked étale locally, e.g. after pulling back to Y, where $\mathscr{F}$ becomes constant. Consider the Leray spectral sequence for j:

$$E_2^{a,b} = H^a(Z, R^b j_*\mathscr{F}) \quad \Rightarrow \quad H^{a+b}(Z^\circ, \mathscr{F}),$$

inducing an increasing filtration F^b on $H^i(Z^\circ, \mathscr{F})$. Let $b(\zeta)$ be the smallest $b \geqslant 0$ for which $\zeta \in F^b$. We prove the assertion by induction on $b(\zeta)$. If $b(\zeta) = 0$, then ζ is in the image of a $\zeta' \in H^i(Z, j_*\mathscr{F})$, and since $j_*\mathscr{F}$ is locally constant and Z is a $K(\pi, 1)$, we can kill ζ' by a finite étale cover of Z. For the induction step, let n be an integer annihilating $\mathscr{F}$, and pick a ramified cover $f : Z' \longrightarrow Z$ as in the previous paragraph, such that again $Z' = (X'_{(\overline{x}')})_{\overline{\eta}}$ for some X'/S' satisfying the assumptions of the theorem. Note that since (9.5.1) is divisible by n, it induces the zero map on $E_2^{a,b}$ for $b > 0$, hence $b(f^*\zeta) < b(\zeta)$, and we finish the proof by induction. □

COROLLARY 9.6. *Let $(X, \mathscr{M}_X)$ be as in Theorem 9.5, and let $X^\circ = (X, \mathscr{M}_X)_{\mathrm{tr}}$. Consider the Faltings' topos $\widetilde{E}$ of $X^\circ_{\overline{\eta}} \longrightarrow X$ and the morphism of topoi*

$$\Psi : X^\circ_{\overline{\eta},\text{ét}} \longrightarrow \widetilde{E}.$$

Let $\mathscr{F}$ be a locally constant constructible abelian sheaf on $X^\circ_{\overline{\eta}}$. Then $R^i\Psi_*\mathscr{F} = 0$ for $i > 0$, and the natural maps (2.2.1)

$$\mu : H^i(\widetilde{E}, \Psi_*(\mathscr{F})) \longrightarrow H^i(X^\circ_{\overline{\eta},\text{ét}}, \mathscr{F})$$

are isomorphisms.

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Piotr Achinger achinger@math.berkeley.edu
Department of Mathematics, University of California, Berkeley, CA 94720, USA

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