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TWO-SIDED LOCALIZATION IN SEMIPRIME FBN RINGS

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The main result of this paper is that the left- and right-quotient rings at a hereditary link closed set of prime ideals of a semiprime fully bounded Noetherian (FBN) ring coincide. This was a result already known for nonsemiprime FBN rigns, but a question left open in the semiprime case. A cornerstone of our approach is that the torsion theory determined by a link-closed hereditary set of prime ideals in an FBN ring is "nice", but not necessarily perfect. Some conditions which do produce perfect torsion theories are investigated.

Introduction

In [7], Müller showed that for an FBN ring with locally finite links, the left and right quotient rings at a hereditary link-closed set of prime ideals coincide. Such rings include all Noetherian PI rings and all FBN rings which are not semiprime [7, Theorem 7], but the question of two-sided quotient rings in semiprime FBN rings was left open (see [7], [8]). In this paper, we show that the left and right quotient rings at a hereditary link-closed set of prime ideals of a semiprime FBN ring do coincide (Theorem 8).

Throughout R denotes a fully bounded Noetherian (FBN) ring on both sides. Modules are unitary right modules unless a subscript indicates a

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left module. Maps are written opposite scalars. We assume familarity with the concept of a torsion theory determined by

(i) an injective module,

(ii) an incomparable (possibly infinite) set of prime ideals of R ($\mathcal{Q} \subseteq R$ is incomparable if $P, Q \in \mathcal{Q}$ and $P \subseteq Q \Rightarrow P = Q$).

$\tau(M)$ denotes the torsion submodule of a module M, $\Delta(M)$ denotes the quotient module of a right module M and $\Lambda(M)$ denotes the quotient module of a left module ${}_RM$. If the torsion theory is not clear from the context, suitable subscripts may be added. See [4], [11] for details. When the meaning is clear from the context we shall let Δ denote $\Delta(R_R)$ and Λ denote $\Lambda({}_RR)$. When R is FBN, there is a one-to-one correspondence between prime ideals of R and indecomposable injective modules (up to isomorphism). E_Q denotes the indecomposable injective with associated prime ideal Q. A set $\mathcal{P} \subseteq \operatorname{Spec} R$ is called hereditary if $P \subseteq Q \in \mathcal{P} \Rightarrow P \in \mathcal{P}$. The torsion theories over an RBN ring are in one-to-one correspondence with the hereditary sets of prime ideals, these being the closed prime ideals in the torsion theory. Every dense right ideal contains a product of dense prime ideals [1].

We also assume familiarity with the ideas of Krull dimension, critical module, basic series, and so on, developed in [3] as well as the definitions of a short right link $P \rightsquigarrow Q$ and a long right link $P \rightsquigarrow \rightsquigarrow Q$ between prime ideals P and Q. (See [6].) Call $\mathcal{P} \subseteq \operatorname{Spec} R$ link-closed if $P \rightsquigarrow Q$ and one of P, Q in $\mathcal{P}$ implies the other also belongs to $\mathcal{P}$.

We shall begin with a general torsion-theoretic result and then consider two-sided localization in certain factor rings. Out of this comes a useful connection between $\Lambda(R)$ and $\Delta(R)$ which will then be applied to the question of two-sided localization over a semiprime FBN ring R. Finally we offer some contributions to the study of perfect torsion theories and link-closed hereditary subsets of $\operatorname{Spec}(R)$. It is known that a link-closed hereditary subset of $\operatorname{Spec}(R)$ does not necessarily produce a perfect torsion theory. We present some sufficient conditions for this to happen.

We shall require the following result concerning a torsion theory over an RBN ring R. If τ is a torsion radical on Mod-R and A is a τ-closed right ideal of R, then $\tau(R) \subseteq A$. Hence $A/\tau(A) = A/\tau(R) \subseteq R/\tau(R)$ and $\Delta(A) \subseteq \Delta(R)$. Furthermore $\Delta(A) \cap \{R/\tau(R)\} = A/\tau(R)$.

PROPOSITION 1. *Let $\mathcal{Q}$ be an incomparable link closed subset of* Spec R *where* R *is* FBN. *Then*

(a) every $\mathcal{Q}$-torsion-free factor module of $\Delta_{\mathcal{Q}}(R)$ is divisible,

(b) $\Delta_{\mathcal{Q}}(R)/\Delta_{\mathcal{Q}}(Q) \approx \Delta_{\mathcal{Q}}(R/Q)$ for all $Q \in \mathcal{Q}$,

(c) $\Delta_{\mathcal{Q}}(Q)$ is a prime ideal of $\Delta_{\mathcal{Q}}(R)$ for all $Q \in \mathcal{Q}$.

Proof. Let Δ denote $\Delta_{\mathcal{Q}}(R)$.

(a) We first show that any basic submodule of a finitely generated $\mathcal{Q}$-torsion-free module X is closed in the divisible hull of X, $D(X)$. Let C be a basic submodule of some finitely generated, $\mathcal{Q}$-torsion-free module X (that is, C is maximal among the α-critical submodules of X where α is the least possible Krull dimension of a nonzero submodule of X). Suppose $t \in D(X)\backslash C$ and $tD \subseteq C$ where D is a dense ideal. C is uniform [3] and X is torsion-free. Hence $\mathrm{Ann}_R C$ is a prime ideal $P \subseteq Q$ for some $Q \in \mathcal{Q}$. If $tR \in tR\backslash C$, $trD = 0$ would imply $tr = 0$, contradiction. Hence $tr + C$ is an essential, hence uniform extension of C. If $(tR+C)/C$ is not critical, it contains a basic submodule C_1/C [3]. Let $P' = \mathrm{Ann}_R\{C_1/C\}$. Since $P' \supseteq D$, P' is $\mathcal{Q}$-dense. Now $0 \subsetneqq C \subsetneqq C_1$ is a basic series for C_1. So

$$K - \dim(R/P) = K - \dim(C) \leq K - \dim(R/P')$$

[3, Theorem 3.4]. On the other hand, the canonical epimorphism $C_1 \to C_1/C \to 0$ induces a nonzero homomorphism $E_P \to E_{P'}$. This implies P' contains some prime ideal B with $P \rightsquigarrow \ldots \rightsquigarrow B$ [6, Lemma 3]. Hence $K - \dim(R/P') \leq K - \dim(R/B) = K - \dim(R/P)$. In other words, we have a link between closed prime ideal P and dense prime ideal P' which contradicts [7, Corollary 3].

Now consider the module $X = \Delta/Y$ where Y_R is a $\mathcal{Q}$-closed submodule of Δ_R. Then Y_R is torsion-free divisible. Thus $Y\Delta = Y$ and we may

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assume $X = x\Delta$. We proceed by induction on the Krull-composition length of xR .

If xR is already basic, we have shown xR is closed in $D(xR)$, hence is divisible. Then $x\Delta \subseteq D(xR) = xR$. If xR has Krull-composition series of length n , say $0 = B_0 \subsetneqq B_1 \subsetneqq B_2 \subsetneqq \cdots \subsetneqq B_n = xR$ then, by induction assumption, $x\Delta/B_1$ is divisible. Also B_1 is divisible. Since divisible modules are closed under group extensions, $x\Delta$ is divisible.

(b) We first show that $\Delta(R)/\Delta(Q)$ is a torsion-free R-module. If for some $x \in \Delta(R)$ there exists a dense (right) ideal D such that $xD \subseteq \Delta(Q)$, since D is finitely generated, we can find a dense ideal D' such that $xDD' \subseteq Q/\tau(R)$. This implies $x \in \Delta(Q)$. By *(a)*, $\Delta(R)/\Delta(Q)$ is torsion-free divisible. The canonical homomorphism $m : R/Q \to \Delta(R)/\Delta(Q)$ is an essential monomorphism whose image is dense in $\Delta(R)/\Delta(Q)$. Hence $\Delta(R/Q) = \Delta(R)/\Delta(Q)$.

(c) Suppose $x\Delta(R)y \subseteq \Delta(Q)$ for elements $x, y \in \Delta(R)$. Let D_x, D_y be dense ideals such that xD_x and $yD_y \subseteq R/\tau(R)$. Then $xD_xyD_y \subseteq Q/\tau(R)$. Either $xD_x \subseteq Q/\Delta(R)$ and $x \in \Delta(Q)$ or $yD_y \subseteq Q/\tau(R)$ and $y \in \Delta(Q)$.

REMARK. Lambek [4] calls an injective module I "nice" if the localization at I satisfies property *(a)* of Proposition 1. It is clear that if the torsion theory determined by I is perfect, then I is certainly nice. The converse is not generally true. Indeed, if a ring R is right hereditary, every injective module is nice, [4, Example 1, p. 68]. However, if R is hereditary but non-Noetherian, then any torsion theory whose filter of dense right ideals contains a non-finitely generated right ideal yields a nice injective whose corresponding torsion theory is not perfect.

Two-sided localization in factor rings

In this section we assume that R is FBN, not necessarily semiprime. We show that under suitable conditions, left- and right-quotient rings coincide for certain factor rings of R .

Consider an incomparable, link-closed set $\mathcal{Q} \subseteq \operatorname{Spec} R$. Let π be any product of members of $\mathcal{Q}$. Then $N_\pi = \{P \in \mathcal{Q} : P \supseteq \pi\}$ is finite; in

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fact, it corresponds to the set of minimal prime ideals of R/π. Since Q is link-closed, $\{P/\pi : P \in \mathrm{N}_\pi\}$ is link-closed [6, Theorem 5]. Hence the semiprime ideal N_π/π (where $N_\pi = \cap\{P : P \in \mathrm{N}_\pi\}$) is localizable and the corresponding torsion theory is induced by the localizable multiplicative set of regular elements modulo N_π/π, $C_{R/\pi}(N_\pi/\pi)$. One consequence is that the left- and right-quotient rings of R/π at N_π/π coincide. Denote this quotient ring by Q_π. Furthermore N_π/π is a semiprime ideal of R/π such that some power of N_π/π is torsion (in fact, 0). Since N_π/π is localizable, by Lambek and Michler [5, 3.3 and 3.4], Q_π is right and left Artinian and every element of $C_R(N_\pi)$ is regular modulo π. Some immediate consequences are:

LEMMA 2. *Let Q, R and π be as described above. Then*

(a) $\tau_Q(R) \subseteq \pi$,

(b) π *is* Q*-closed,*

(c) R/π *is a subring of* Q_π.

PROPOSITION 3. *Suppose Q is an incomparable link-closed subset of* Spec R *(R FBN) and π is any product of elements of Q. Then* $\Delta_Q(R)/\Delta_Q(\pi) \cong Q_\pi \cong \Delta_Q(R/\pi)$ *(as rings)*

Proof. Let $\bar{R} = R/\tau_Q(R)$. We have $\Delta_Q(\pi) \cap \bar{R} = \bar{\pi}$, hence $R/\pi \hookrightarrow \Delta_Q(R)/\Delta_Q(\pi)$. Now $\Delta_Q(R)/\Delta_Q(\pi)$ is Q-torsion free, hence divisible by the argument of Proposition 1 *(b)*. Thus $\Delta_Q(R)/\Delta_Q(\pi) \cong \Delta_Q(R/\pi) \subseteq Q_\pi$ since every Q-dense ideal is certainly N_π-dense. For the reverse inclusion, given $t \in Q_\pi$, there exists a product of N_π/π-dense prime ideals of R/π - say $F = (P_1P_2 \ldots P_k+\pi)/\pi$ - such that $tF \subseteq R/\pi$. The incomparability of Q and the definition of N_π guarantee that $P_1 \ldots P_k$ is Q-dense. Hence $t \in \Delta_Q(R/\pi) \cong \Delta_Q(R)/\Delta_Q(\pi)$.

COROLLARY 4. *Under the same hypotheses,*

$$\Lambda_Q(R/\pi) \cong \Lambda_Q(R)/\Lambda_Q(\pi) \cong Q_\pi \cong \Delta_Q(R)/\Delta_Q(\pi) \cong \Delta_Q(R/\pi).$$

It should be noted that the isomorphisms in Corollary 4 leave the ring R/π fixed. Hence we obtain the following explicit rule for the isomorphism $\psi : \Delta_{\mathcal{Q}}(R/\pi) \to Q_\pi$. Given $q \in \Delta_{\mathcal{Q}}(R/\pi)$, there exists a $\mathcal{Q}$-dense ideal D such that $qD \subseteq R/\pi$. Since D is N_π-dense, there exists $c \in C(N_\pi) \cap D$ and $qc = [r]_\pi \in R/\pi$. Then $\psi(qc) = [r]_\pi$. But $\psi(qc) = \psi(q)c$. Hence $\psi(q) = [r]_\pi [c]_\pi^{-1}$.

PROPOSITION 5. *Let* $\mathcal{Q}$ *be an incomparable link-closed subset of* Spec R *(*R FBN*). Let* $\pi' \subseteq \pi$ *be products of members of* $\mathcal{Q}$ *. Then there is a commutative diagram*

$$\begin{array}{ccc} R/\pi' & \xrightarrow{\;p\;} & R/\pi \\ \big\downarrow{\scriptstyle i'} & & \big\downarrow{\scriptstyle i} \\ Q_{\pi'} & \xrightarrow{\;f\;} & Q_\pi \end{array}$$

(p *is the canonical epimorphism and* i, i' *are inclusions.) Furthermore* f *is onto.*

Proof. $N_{\pi'} = N_\pi \cap A$ where $A = R$ or A is an intersection of members of $\mathcal{Q}$. Consequently, R/π' is an N_π-dense submodule of $Q_{\pi'}$. Since Q_π is N_π-torsion free divisible, there exists a unique $f : Q_{\pi'} \to Q$ such that $fi' = ip$. To be explicit, given $[r]_{\pi'}[c]_{\pi'}^{-1} \in Q_\pi$ with $c \in C(N_{\pi'}) \subseteq C(N_\pi)$, we must have $f\left([r]_{\pi'}[c]_{\pi'}^{-1}\right) = [r]_\pi [c]_\pi^{-1}$.

To verify that f is onto, select any $q \in Q_\pi$. There exists a product D of prime ideals, each non-minimal among primes containing π , such that $qD \subseteq R/\pi$. Since $\mathcal{Q}$ is incomparable, D is $\mathcal{Q}$-dense, hence $N_{\pi'}$-dense. Then there exists $c \in D \cap C(N_{\pi'})$ and $r \in R$ such that $qc = [r]_\pi \in R/\pi$. It follows that $q = [r]_\pi [c]_\pi^{-1} = f\left([r]_{\pi'}[c]_{\pi'}^{-1}\right)$.

In what follows, E denotes $\oplus\{E_Q : Q \in \mathcal{Q}\}$ where $\mathcal{Q}$ is an incomparable link-closed subset of Spec R . Since E is $\mathcal{Q}$-torsion free, every R-homomorphism $\phi : \Delta_{\mathcal{Q}}(R) \to E$ is automatically a $\Delta_{\mathcal{Q}}(R)$-

homomorphism. It follows that E is $\Delta_{\mathcal{Q}}(R)$-injective. Also, for any $Q \in \mathcal{Q}$, since $\Delta_{\mathcal{Q}}(R)/\Delta_{\mathcal{Q}}(Q)$ is an essential extension of R/Q, it is embedded in a finite product of copies of E.

Consider the following topologies on $\Delta = \Delta_{\mathcal{Q}}(R)$ (*cf.* [5]):

(1) the E-adic, whose basic open neighborhoods of zero have the form $\ker(f)$ for some $f \in \mathrm{Hom}_{\Delta}(\Delta, E^n)$ $(n \in N)$;

(2) the $\Delta(\pi)$-adic, whose basic open neighborhoods of zero have the form $\Delta(\pi)$ where π is some product of members of $\mathcal{Q}$.

Recall that a torsion theory is called *stable* if the family of torsion modules is closed under injective hulls.

PROPOSITION 6. *Let* $\mathcal{Q}$ *be an incomparable, link-closed set of prime ideals of an* FBN *ring* R. *Then the* E*-adic and the* $\Delta(\pi)$*-adic topologies on* $\Delta = \Delta_{\mathcal{Q}}(R)$ *coincide.*

Proof. By Propositions 1 and 3, $\Delta(R)/\Delta(\pi)$ is right (and left) Artinian for all π and $\Delta(R)/\Delta(Q)$ is simple Artinian for all $Q \in \mathcal{Q}$. We claim that for any i, $\Delta(Q_1 \ldots Q_{i-1})\Delta(Q_i) \subseteq \Delta(Q_1 \ldots Q_i)$. Indeed, given $x \in \Delta(Q_1 \ldots Q_{i-1})$ and $y \in \Delta(Q_i)$, let D_x and D_y be dense ideals such that $xD_x \subseteq Q_1 \ldots Q_{i-1}/\tau(R)$ and $yD_y \subseteq Q_i/\tau(R)$. Since $\mathcal{Q}$ is closed under links, the $\mathcal{Q}$-torsion theory is stable [1], [11]. Hence, by [1, Theorem 1.2], there exists a dense ideal D' such that $\overline{D}_x(yD_y) \supseteq \overline{R}yD_y\overline{D}'$. Then $xyD_yD' \subseteq xD_xyD_y \subseteq Q_1 \ldots Q_i/\tau(R)$. This implies $xy \in \Delta(Q_1 \ldots Q_i)$ as desired. Given a product $\pi = Q_1 \ldots Q_k$ with all $Q_i \in \mathcal{Q}$, each factor of the chain

$$\Delta(R) \supseteq \Delta(Q_1) \supseteq \Delta(Q_1Q_2) \supseteq \ldots \supseteq \Delta(Q_1 \ldots Q_k)$$

is a finitely generated $\Delta(R)/\Delta(Q_i)$-module $(i = 1, 2, \ldots, k)$, hence a finite direct sum of simple $\Delta(R)/\Delta(Q_i)$-modules, which embeds in a finite direct product of copies of E. It follows that $\Delta(R)/\Delta(\pi) \subseteq E^m$ for some $m \in \mathbb{N}$.

Conversely, given $f : \Delta(R) \to E^n = \oplus E_Q^n$, let $f(1) = x_1 + \ldots + x_t$ where each x_j is in E_{Q_j} for some $Q_j \in \mathcal{Q}$. By considering a critical series for x_jR [3], one sees that x_jR is annihilated by a product of prime ideals in $\mathcal{Q}$. It follows that $f(\Delta(\pi)) = f(1)\Delta(\pi) = 0$ for some product π of members of $\mathcal{Q}$.

COROLLARY 7. *Under the same hypotheses, the bicommutator of* E *is isomorphic to* $S = \varprojlim \{\Delta(R)/\Delta(\pi) : \pi$ *is a product of members of* $\mathcal{Q}\}$.

Proof. The conditions of [4, Propositions 2 and 3] are satisfied.

REMARK. E is naturally an S-module under the following rule. Given $e \in E$ and $s = \langle [s_\pi]_{\Delta(\pi)} \rangle \in S$, where $s_\pi \in \Delta(R)$, choose any π such that $e\pi = 0$. Define $es = es_\pi$. It is tedious to verify that this is a well-defined S-action on E which extends the existing $\Delta(R)$-action. Both $\Delta(R)$ and $\Lambda(R)$ can be viewed as subrings of S containing $\overline{R} = R/\tau(R)$.

Two-sided quotient rings in semiprime FBN rings

The main result of this section is Theorem 8. If R is a semiprime FBN ring and P is a hereditary link-closed set of prime ideals of R, then the left- and right-quotient rings of R at P coincide. We begin with some observations on torsion theories in a semiprime FBN ring.

Suppose P is a hereditary link-closed set of prime ideals of semiprime FBN ring R. Let $S_1, S_2, \ldots, S_n$ be the minimal prime ideals of R (Minspec R). Suppose $S_1, S_2, \ldots, S_t \in P$ and $S_{t+1}, \ldots, S_n \notin P$.

Let $N_P = \bigcap_{i=1}^{t} S_i$. Since $\cap\{S_j : j = t+1, \ldots, n\}$ is P-dense and annihilates N_P, $N_P \subseteq \tau_P(R)$. On the other hand, $\tau_P(R) \subseteq P$ for all $P \in P$. Hence $\tau_P(R) = N_P$. Both $\Lambda(R)$ and $\Delta(R)$ may be viewed as subrings of $Q_{\max}(R/N_P) = Q_{\mathrm{cl}}(R/N_P)$.

THEOREM 8. *Suppose* R *is a semiprime* FBN *ring. Let* P *be a hereditary link-closed subset of* Spec R. *Then the left- and right-quotient rings of* R *coincide.*

Proof. Let $\overline{R} = R/\tau_P(R) = R/N_P$. Let

$$\mathcal{Q} = \text{Max } P = \{Q \in P : Q \text{ is maximal among elements of } P\} .$$

Then $\mathcal{Q}$ is link-closed and incomparable ([7], [11]). Let $E = \oplus\{E_Q : Q \in \mathcal{Q}\}$. We have $\overline{R} \subseteq E(\overline{R}) \subseteq I = \pi E$ for some product of copies of E . Now $\text{Bic}(I) \cong \text{Bic}(E) = S \cong \varprojlim \Delta(R)/\Delta(\pi)$ ([2, Proposition A2]). By [4, Lemma 3], iS/iR is $\mathcal{Q}$-torsion for all $i \in I$. In particular, if $i = \overline{1}$, we get that $(\Lambda/R)_R$ is torsion. Since Λ may be viewed as a subring of $Q_{\max}(\overline{R})$ and since $Q_{\max}(\overline{R}) \cong E(\overline{R})$, Λ is an essential extension of $\overline{R}$. Hence $\Lambda \subseteq \Delta$. By symmetry, $\Delta \subseteq \Lambda$.

Semiprime FBN rings and perfect torsion theories

A torsion theory with right localization functor Δ is perfect (on Mod-R) if every $\Delta(R)$ module, considered as an R-module, is torsion-free. This is one of many characterizations of perfect torsion theories. See Stenstrom [10] and Richards [9] for more details. Call a hereditary set $P \subseteq \text{Spec } R$ (or the corresponding incomparable set $\mathcal{Q} = \max P$) perfect if the P-torsion theories on Mod-R and on R-Mod are perfect.

A torsion theory with torsion radical τ_0 is said to decompose as a direct sum of torsion theories τ_1 and τ_2 if for all (right) R-modules M , $\tau_0(M) = \tau_1(M) \oplus \tau_2(M)$. Müller has established the following necessary and sufficient conditions for such a decomposition. Let $\mathcal{D}_i$ denote the set of τ_i-dense prime ideals for $i = 0, 1, 2$. Then $\tau_0 = \tau_1 \oplus \tau_2$ if $\mathcal{D}_0$ is the disjoint union of $\mathcal{D}_1$ and $\mathcal{D}_2$ and there are no links between prime ideals in $\mathcal{D}_1$ and prime ideals in $\mathcal{D}_2$ [8]. Now by Goldie's theorem, Minspec R is closed under links. In particular, the set of all maximal ideals which are also minimal prime ideals ("min-max" ideals) is finite and link-closed. Any link-closed set $\mathcal{Q}_1$ of min-max ideals induces a direct sum decomposition $R = N_1 \oplus B$ where N_1 is the $\mathcal{Q}_1$-torsion submodule of R [8, Theorem 3]. As noted above, N_1 is the intersection of the ideals in $\mathcal{Q}_1$ and B is the intersection of the minimal prime ideals not in $\mathcal{Q}_1$. Furthermore, the $\mathcal{Q}_1$-torsion theory is classical, hence perfect.

PROPOSITION 9. *Let $\mathcal{Q}$ be any link-closed set of maximal ideals in a semiprime* FBN *ring R. Let $\mathcal{Q}_1 = \mathcal{Q} \cap \text{Minspec } R$ and $\mathcal{Q}_2 = \mathcal{Q}\backslash\mathcal{Q}_1$. Let $N_1 = \tau_{\mathcal{Q}_1}(R)$, $N_2 = \tau_{\mathcal{Q}_2}(R)$ and let B denote the intersection of all minimal prime ideals not in $\mathcal{Q}_1$. Let Δ, Δ_1, and Δ_2 denote the localization functors associated with $\mathcal{Q}$, $\mathcal{Q}_1$, and $\mathcal{Q}_2$ respectively. Then*

$$\Delta(R) \cong \Delta_1(R/N_1) \oplus \Delta_2(R/B) = \Delta_1(R) \oplus \Delta_2(R) .$$

Proof. We have $R = N_1 \oplus B$, $N_1 \cong R/B$, $B \cong R/N_1$. It is clear that $\Delta(R) \cong \Delta(R/N_1) \oplus \Delta(R/B)$. We shall show that $\Delta(R/N_1) = \Delta_1(R) = \Delta_1(R/N_1)$ and $\Delta(R/B) = \Delta_2(R/B) = \Delta_2(R)$. Now $\tau_{\mathcal{Q}_1}(R) = N_1$. Hence $R/\tau_{\mathcal{Q}_1}(R) = R/N_1 \to \Delta_1(R)$. If $x \in \Delta_1(R)$, there exists a $\mathcal{Q}_1$-dense ideal D such that $xD \subseteq R/N_1$. Now R/N_1 is $\mathcal{Q}_2$-torsion and the $\mathcal{Q}_2$-torsion theory is stable. Hence there exists a $\mathcal{Q}_2$-dense ideal D' such that $xD' = 0$. The prime ideals containing $D + D'$ are both $\mathcal{Q}_1$- and $\mathcal{Q}_2$-dense, hence $\mathcal{Q}$-dense. It follows that $x \in \Delta(R/N_1) = \Delta(R)$. Trivially, $\Delta(R/N_1) \subseteq \Delta_1(R/N_1) = \Delta_1(R)$ since every $\mathcal{Q}$-dense ideal is $\mathcal{Q}_1$-dense and R/N_1 is $\mathcal{Q}$-torsion free. The proof that $\Delta(R/B) = \Delta_2(R/B)$ is similar after noting that $\tau_2(R/B) = N_2/B = \tau(R/B)$.

LEMMA 10. *Suppose R is a semiprime* FBN *ring with no min-max ideals. If the torsion theory associated with the hereditary set of non-maximal prime ideals is perfect, then the torsion theory determined by any link-closed set of maximal ideals is perfect. The converse also holds.*

Proof. Let P_0 denote the set of non-maximal ideals and let $\mathcal{Q}_1$ be any link-closed subset of Maxspec R. Let $\mathcal{Q}_2 = \text{maxspec } R\backslash\mathcal{Q}_1$. Denote the corresponding cohereditary sets of dense prime ideals by $\mathcal{D}_0$, $\mathcal{D}_1$ and $\mathcal{D}_2$ and the corresponding torsion radicals by τ_0, τ_1 and τ_2. Clearly,

$\mathcal{D}_0 = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$ and there are no links between primes in $\mathcal{D}_1$ and primes in $\mathcal{D}_2$. By [8, Lemma 2], $\tau_0 = \tau_1 \oplus \tau_2$. But

$$\tau_0(R) = \bigcap\{P : P \text{ is a minimal } \mathcal{Q}_0\text{-closed prime}\}$$
$$= \bigcap\{P : P \text{ is a minimal prime}\} = 0 .$$

Hence, by [9, 1.9], τ_1 and τ_2 are perfect if and only if τ_0 is.

REMARK. Obviously the conditions of Lemma 10 are satisfied in a semiprime ring of Krull dimension one with no min-max ideals, in particular in a bounded Noetherian prime ring of Krull dimension one.

PROPOSITION 11. *Suppose* R *is a semiprime* FBN *ring in which the torsion theory determined by the hereditary set of non-maximal ideals is perfect. Then every link-closed set of maximal ideals determines a perfect torsion theory.*

Proof. Let $\mathcal{Q}$ be a link-closed set of maximal ideals. Let $\mathcal{Q}_1 = \text{Minspec } R \cap \mathcal{Q}$ and $\mathcal{Q}_2 = \mathcal{Q}\backslash\mathcal{Q}_1$ as in Proposition 9. To show that the $\mathcal{Q}$-torsion theory is right perfect, it is sufficient to show that every $\Delta(R)$-module, considered as an R-module, is $\mathcal{Q}$-torsion free (where Δ and Δ_i represent the localization functors corresponding to $\mathcal{Q}$ and $\mathcal{Q}_i$, $i = 1, 2$). Since $\Delta(R) = \Delta_1(R) \oplus \Delta_2(R)$, any $\Delta(R)$-module M is both a $\Delta_1(R)$- and a $\Delta_2(R)$-module. If $\Delta_1(R) \neq 0$ then M is $\mathcal{Q}_1$-torsion free, hence $\mathcal{Q}$-torsion free.

If $\mathcal{Q}_1 = \emptyset$, let A denote the intersection of all min-max ideals and B the intersection of the remaining minimal primes. Then $R = A \oplus B$ where A is a semiprime ring containing no min-max ideals. $\Delta(R) = \Delta(R/B)$. Applying Lemma 10 to R/B we get the desired result.

PROPOSITION 12. *Every link-closed hereditary set of prime ideals in a semiprime* FBN *ring of Krull dimension one is perfect.*

Proof. Let $\mathcal{Q} = \text{Max } \mathcal{P}$ where $\mathcal{P}$ is a hereditary link-closed subset of Spec R . If $\mathcal{Q}$ consists entirely of maximal ideals, the conditions of Proposition 11 are satisfied. If not, let

$$\mathcal{Q}_1 = \{\text{min-max ideals in } \mathcal{Q}\} ,$$

$$\mathcal{Q}_2 = \{\text{non-maximal prime ideals in } \mathcal{Q}\} ,$$

$$\mathcal{Q}_3 = \{\text{non-minimal prime ideals in } \mathcal{Q}\} .$$

$\mathcal{Q}_1$ and $\mathcal{Q}_2$ are finite link-closed (hence perfect) sets of minimal primes of R . $\mathcal{Q}_3$ is perfect by Proposition 11. Let

$$N_i = \cap\{S : S \text{ is a minimal } \mathcal{Q}_i\text{-closed prime ideal}\} , \quad i = 1, 2, 3 .$$

There is an essential monomorphism (where $N = \tau_{\mathcal{Q}}(R)$)

$$R/N \xrightarrow{h} R/N_1 \oplus R/N_2 \oplus R/N_3 .$$

Let

$$0 \to R/N \to \bigoplus_{i=1}^{3} R/N_1 \to X \to 0$$

be exact. Applying the exact functor $\Delta = \Delta_{\mathcal{Q}_i}$ for $i = 1, 2, 3$, we have exact sequences

$$0 \to \Delta_i(R/N) \to \bigoplus_{j=1}^{3} \Delta_i(R/N_j) \to \Delta_i(X) \to 0 .$$

But $\Delta_i(R/N_j) = 0$ for $i \neq j$. Indeed, R/N_j is $\mathcal{Q}_i$-torsion for $i \neq j$, and the $\mathcal{Q}_i$-torsion theory is stable. Also $\Delta_i(R/N) = \Delta_i(R/N_i)$. Hence $\Delta_i(X) = 0$, for $i = 1, 2, 3$, from which we get that X is $\mathcal{Q}$-torsion. In other words, R/N is a $\mathcal{Q}$-dense submodule of $\bigoplus_{j=1}^{3} R/N_j$. It follows that $\Delta(R/N) \cong \Delta(R/N_1) \oplus \Delta(R/N_2) \oplus \Delta(R/N_3)$. By the proof of Proposition 9, $\Delta(R/N_i) = \Delta_i(R/N_i)$ for $i = 1, 2, 3$. Consequently, the perfectness of $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$, implies perfectness of $\mathcal{Q}$.

REMARK 1. Proposition 12 shows that at least in the case of a semiprime FBN ring, the assumption of zero socle in [7, Proposition 15] can be dropped. We do not know whether this is true for non-semiprime R .

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REMARK 2. It is an open question when the hypotheses of Lemma 10 and Proposition 11 are satisfied. If $R = k[x, y]$ for some field k, then R is a prime FBN ring of Krull dimension 2 in which the hypotheses of Lemma 10 are not satisfied (see [7]).

References

[1] J.A. Beachy, "Stable torsion radicals over FBN rings", *J. Pure Appl. Algebra* 24 (1982), 227-233.

[2] C. Faith, *Injective modules and injective quotient rings* (Lecture Notes in Pure and Applied Mathematics, 72. Marcel Dekker, New York, 1982).

[3] A.V. Jategaonkar, "Jacobson's conjecture and modules over fully bounded Noetherian prime rings", *J. Algebra* 30 (1974), 103-121.

[4] J. Lambek, "Bicommutators of nice injectives", *J. Algebra* 21 (1972), 60-73.

[5] J. Lambek and G. Michler, "Localization of right Noetherian rings at semiprime ideals", *Canad. J. Math.* 26 (1974), 1069-1085.

[6] B.J. Müller, "Localization in fully bounded Noetherian rings", *Pacific J. Math.* 67 (1976), 233-245.

[7] B.J. Müller, "Two-sided localization in Noetherian PI rings", *J. Algebra* 63 (1980), 359-373.

[8] B.J. Müller, "Links between maximal ideals in bounded Noetherian prime rings of Krull dimension one", preprint.

[9] R. Richards, "Noetherian prime rings of Krull dimension 1 " (PhD Thesis, McMaster University, Hamilton, Ontario, 1977).

[10] B. Stenstrom, *Rings and modules of quotients* (Lecture Notes in Mathematics, 237. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[11] M. Upham, "Right localizable multiplicative sets and prime ideals of FBN rings", *Comm. Algebra* 9 (1981), 1093-1103.

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