ON THE COMPLETE INTEGRAL CLOSURE OF AN INTEGRAL DOMAIN

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We consider in this paper only commutative rings with identity. When R is considered as a subring of S it will always be assumed that R and S have the same identity. If R is a subring of S an element s of S is said to be *integral* over R if s is the root of a monic polynomial with coefficients in R. Following Krull [8], p. 102, we say s is *almost integral* over R provided all powers of s belong to a finite R-submodule of S.[1] If R_1 is the set of elements of S almost integral over R we say R_1 is the *complete integral closure of* R *in* S. If $R = R_1$ we say R is *completely integrally closed in* S. If $R_1 = S$ we say S is *almost integral* over R. If S is the total quotient ring of R, we call R_1 the *complete integral closure* of R, and in this case if $R = R_1$ we say simply R is *completely integrally closed.* The terms *integral closure of* R *in* S, R is *integrally closed in* S, S is *integral* over R, the *integral closure of* R, and R is *integrally closed* are similarly defined. For elementary results on these properties, see [14], Ch. 14, and [15], Ch. 5.

This paper is principally concerned with the complete integral closure D^* of an integral domain D and a determination of when D^* is completely integrally closed, though some results in more general cases are obtained.

Section 1 contains some results on the integral closure R^* of R in S as related to the integral closure of $R[X]$ in $S[X]$, X an indeterminate over S. Section 2 considers analogues of these results for the complete integral closure of R in S. Corollary 5 of section 2 implies that if the complete integral closure D^* of the domain D is completely integrally closed, then the complete integral closure of D in any extension of the quotient field of D is again completely integrally closed. Domains D for which D^* is completely integrally closed are investigated in section 3.

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[1] Krull actually considers the property of s being almost integral over R only in the case when R is an integral domain with identity and s belongs to the quotient field of R. In this case it is easily shown that s is almost integral over R if and only if there is a nonzero element r of R such that $rs^k \in R$ for each positive integer k ([14], p. 77). The natural generalization to the case when s is an element of the total quotient ring of the ring R is also valid.

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1

We begin with a lemma which is a standard result in elementary algebra texts for the case of an integral domain, but which seems not to be used extensively for commutative rings.

LEMMA 1. *Let X be an indeterminate over the ring R and suppose $f(X)$ is a nonconstant polynomial such that the leading coefficient of $f(X)$ is not a zero divisor in R. Then $(f(X)) \cap R = (0)$, so $R[X]/(f(X)) = S$ is a ring containing R (to within isomorphism) and $f(X)$ has a root in S. If the leading coefficient of $f(X)$ is a unit of R it follows that there exists a ring T containing R as a subring such that in $T[X]$, $f[X]$ splits into linear factors.*

PROOF. All assertions follow immediately from the fact that if $g(X)$ is a nonzero element of $R[X]$ then deg $(fg) = \deg f + \deg g$ since the leading coefficient of $f(X)$ is not a zero divisor in R.

THEOREM 1. *Suppose R is a subring of S and X is an indeterminate over S. If $f(X) \in R[X]$ and if in $S[X]$, $f(X) = g(X)h(X)$ where $h(X)$ is monic, then the coefficients of $g(X)$ are integral over R.*

PROOF. We suppose $f(X)$ has degree n; $f(X) = \sum_{i=0}^{n} a_i X^i$. For $h(X)$ of degree 0, $g(X) = f(X) \in R[X]$ and our conclusion follows. Suppose now $h(X) = X - s$ has degree one. We prove the coefficients of $g(X)$ are integral over R by induction on n. Thus if $g(X) = \sum_{i=0}^{n-1} b_i X^i$, then for $n = 1$ we have $b_0 = b_{n-1} = a_n \in R$ so the b_i's are integral over R. We assume the theorem is valid for $n \leqq k$. Then if $f(X) = (X-s)(\sum_{i=0}^{k} b_i X^i)$ we have:

$$(*)\qquad \begin{aligned} b_k &= a_{k+1} \\ b_{k-1} - sb_k &= a_k \\ &\vdots \\ b_0 - sb_1 &= a_1 \\ -sb_0 &= a_0. \end{aligned}$$

It follows that

$$\begin{aligned} f(s) = 0 = \sum_{i=0}^{k+1} a_i s^i &= a_{k+1}^k \left(\sum_{i=0}^{k+1} a_i s^i\right) \\ &= (a_{k+1}s)^{k+1} + a_k (a_{k+1}s)^k + \cdots + a_1 a_{k+1}^{k-1}(a_{k+1}s) + a_{k+1}^k a_0 = 0. \end{aligned}$$

Hence $a_{k+1}s = b_k s$ is integral with respect to R. Thus $b_{k-1} = a_k + sb_k$ is integral over R. The last k equations in the system (*) then show that $(X-s)(b_{k-1}X^{k-1} + \cdots + b_1 X + b_0)$ is a factorization of the polynomial $b_{k-1}X^k + a_{k-1}X^{k-1} + \cdots + a_0 \in R[b_{k-1}][X]$ in $S[X]$. The induction hypothesis implies $\{b_{k-1}, b_{k-2}, \cdots, b_0\}$ is a set of elements integral over $R[b_{k-1}]$, and hence integral over R. By the principle of mathematical induction, the coefficients of $g(X)$ are integral over R when $h(X)$ has degree 1.

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We suppose the theorem holds for $h(X)$ of degree k and we consider the case $f(X) = g_1(X)h_1(X)$, $h_1(X)$ of degree $k+1$. By Lemma 1, there exists a ring T containing S such that $h_1(X)$ has a root t in T. Hence in $T[X]$, $h_1(X) = (X-t)h_2(X)$ where $h_2(X)$ is monic of degree k, so that $f(X) = [g_1(X)(X-t)]h_2(X)$ in $T[X]$. By the induction hypothesis the coefficients of $g_1(X)(X-t)$ are integral over R. Let R_0 be the ring obtained by adjoining to R the coefficients of $g_1(X)(X-t)$. By assumption R_0 is integral over R. And by the case already proved when the monic factor is linear, the coefficients of $g_1(X)$ are integral over R_0, and hence integral over R. This completes the proof.

From Theorem 1 we obtain two corollaries which relate the integral closure of R in S with the integral closure of $R[X]$ in $S[X]$, X an indeterminate. Between the time we obtained these results and the time we received a referee's report, Corollaries 1—2 appeared in Bourbaki [1] pp. 18—19. Our proof, based on Theorem 1, is different from that presented in Bourbaki.

COROLLARY 1. *Let R be a subring of the ring S and let X be an indeterminate over S. R is integrally closed in S if and only if $R[X]$ is integrally closed in $S[X]$.*

PROOF. It is immediate that if $R[X]$ is integrally closed in $S[X]$ then R is integrally closed in S.

We suppose, conversely, that R is integrally closed in S. If $f(X) = \sum_{i=0}^{k} a_i X^i \in S[X]$ is integral over $R[X]$, then $X^{k+1}+f(X)$ is also integral over $R[X]$. Hence in showing that the coefficients of $f(X)$ are integral over R, there is no loss of generality in assuming $f(X)$ is monic.

Let $f(X)$ satisfy a monic polynomial $t(Y)$ in $R[X][Y]$ of degree $k \geqq 1$. To show $f(X) \in R[X]$ we use induction on k. For $k = 1$ it is clear that $f(X) \in R[X]$. Assuming the theorem true for $k \leqq n$ we then assume $\sum_{i=0}^{n+1} d_i(X)f^i(X) = 0$ where $d_i(X) \in R[X]$ and $d_{n+1}(X) = 1$. Then in $S[X]$ we have $d_0(X) = f(X)[-\sum_{i=0}^{n} d_{i+1}(X)f^i(X)]$ where $f(X)$ is monic. By Theorem 1 the coefficients of $-\sum_{i=0}^{n} d_{i+1}(X)f^i(X) = q_0(X)$ are integral over R and $q_0(X) \in S[X]$. Hence $q_0(X) \in R[X]$ and we have

$$f^n(X)+ \cdots +d_2(X)f(X)+d_1(X)+q_0(X) = 0.$$

The induction hypothesis then yields the desired conclusion that $f(X) \in R[X]$. Hence $R[X]$ is integrally closed in $S[X]$.

COROLLARY 2. *Let R be a subring of S, let X be an indeterminate over S, and let R^* be the integral closure of R in S. Then $R^*[X]$ is the integral closure of $R[X]$ in $S[X]$.*

2

We now turn our attention to the concept of almost integrity. Let R be a subring of S and let S be a subring of T. The differences between the results obtained for integrity and almost integrity are due mainly to the following two facts.

1. Almost integrity is not transitive; T almost integral over S and S almost integral over R do not imply T is almost integral over R.

2. If $s \in S$ then s *is almost integral over* R may depend upon whether we consider s as an element of S or as an element of T; s may be almost integral over R in the latter consideration but not in the former. This is illustrated for integral domains by Example 2. Concerning 1, it is in fact true that the complete integral closure of R in S need not be completely integrally closed in S, a point we illustrate in Example 1 for the case when R is an integral domain and S is its quotient field. However, the complete integral closure of R in S is integrally closed in S ([14], p. 76).

Observations 1, 2 mean that many of the classical results on integrity for integral domains will not carry over for integrity replaced by almost integrity. For instance, a valuation ring is completely integrally closed if and only if it has rank $\leqq 1$. ([9], p. 170, thm. 8). This led Krull to conjecture in [9] that a completely integrally closed domain is an intersection of valuation rings of rank $\leqq 1$. However, in [12], Nakayama gave an example of a completely integrally closed domain D which is contained in no rank one valuation ring having the same quotient field as that of D, thereby proving Krull's conjecture false. Also, it is easily seen that the "lying over" theorem for prime ideals of Cohen and Seidenberg ([3], p. 253, thm. 2) is no longer valid for S almost integral over R. And, as noted already, the complete integral closure of R in S need not be completely integrally closed in S.

We begin by giving an example of an integral domain D such that the complete integral closure of D is not completely integrally closed.

Example 1. Suppose K is a field and X and Y are indeterminates over K. We let $D = K[\{X^{2n+1}Y^{n(2n+1)}\}_{n=0}^{\infty}]$. The quotient field of D is $K(X, Y)$. Further $D' = K[\{XY^n\}_{n=0}^{\infty}]$ is integral over D so that $D \subset D' \subseteqq D^* \subseteqq K[X, Y]$ where D^* is the complete integral closure of D; $D^* \subseteqq K[X, Y]$ since $K[X, Y]$ is a unique factorization domain and is therefore completely integrally closed. ([14], p. 77). Now Y is almost integral over D', hence almost integral over D^*. We show, however, that $Y \notin D^*$ — that is, Y is not almost integral over D. For this purpose we observe that if $\sum a_\alpha X^{i_1}(X^3Y^3)^{i_2} \cdots (X^{2r+1}Y^{r(2r+1)})^{i_r}$ is in D, then the exponent $3i_2 + \cdots + r(2r+1)i_r$ of Y in any monomial in this sum is $\leqq (i_1 + \cdots + (2r+1)i_r)^2$, the square of the exponent of X in the same monomial. This shows that no nonzero element d of D is such that $dY^h \in D$ for all positive integers h.

REMARK 1. While the complete integral closure R^* of R in S need not be completely integrally closed in S, this does not occur if R^* is contained in a finite R-module contained in S. That this is true follows from Lemma 2.

LEMMA 2. *If R, R_1, and S are rings with $R \subseteqq R_1 \subseteqq S$ and if R_1 is contained in a finite R-module contained in S, then R and R_1 have the same complete integral closure in S.*

PROOF. Suppose R_1 is contained in the R-submodule of S generated by $\{s_i\}_{i=1}^n$. To prove the lemma it suffices to show that an element s of S almost integral over R_1 is almost integral over R. Thus if all powers of s belong to the R_1-submodule of S generated by $\{u_j\}_{j=1}^m$, it is patent that all powers of s belong to the R-submodule of S generated by all products $s_i u_j$ so that s is almost integral over R.

We now establish some elementary properties of almost integrity. In particular we show that the analogues of Corollaries 1, 2 hold for "integral" replaced by "complete integral" throughout.

PROPOSITION 1. *Suppose R is a subring of S, X is an indeterminate over S and R^* is the complete integral closure of R in S. Then $R^*[X]$ is the complete integral closure of $R[X]$ in $S[X]$.*

PROOF. Evidently $R^*[X]$ is almost integral over $R[X]$. Let $f(X) = a_n X^n + \cdots + a_0 \in S[X]$, $f(X)$ almost integral over $R[X]$ — say all powers of $f(X)$ belong to the finite $R[X]$-module generated by $\{d_i(X)\}_{i=1}^m$. Then for k a positive integer $f^k(X) = a_n^k X^{nk} + \cdots + a_0^k = \sum_{i=1}^m s_{ki}(X) d_i(X)$ for some $s_{ki}(X) \in R[X]$. By equating coefficients of X^{nk} over all k we see that all powers of a_n belong to the finite R-submodule of S generated by the coefficients of the $d_i(X)$'s. Hence $a_n \in R^*$. It follows that $f(X) - a_n X^n$ is also almost integral over $R[X]$. By an inductive argument we conclude that $f(X) - a_n X^n \in R^*[X]$, whence $f(X) \in R^*[X]$ as we wished to show.

COROLLARY 3. *If R is a subring of S and if X is an indeterminate over S, then R is completely integrally closed in S if and only if $R[X]$ is completely integrally closed in $S[X]$.*

PROOF. This follows immediately from Proposition 1.

We now investigate some problems inherent with our second observation at the beginning of this section.

PROPOSITION 2. *Suppose R is a subring of S_1 and S_1 is a subring of S_2. If R_i is the complete integral closure of R in S_i, then $R_1 \subseteqq R_2 \cap S_1$ and if either*

(a) *S_2 is a submodule of some S_1-module S_3 such that S_1 is a direct summand of S_3,* *or* (b) *each finite S_1-module contained in S_2 and containing S_1 is a submodule of an S_1-module of which S_1 is a direct summand, then $R_1 = R_2 \cap S_1$.*

Proof. That $R_1 \subseteqq R_2 \cap S_1$ is clear. Let $u \in R_2 \cap S_1$. Then all powers of u belong to a finite R-module M contained in S_2; say $\{m_i\}_{i=1}^n$ generates M. Under (a) or (b) above, the S_1-module generated by the m_i's and the identity of S_1 is contained in an S_1-module of the form $S_1 \oplus T$. For $1 \leqq i \leqq n$ we let $m_i = s_i + t_i$ where $s_i \in S_1$ and $t_i \in T$. If k is a positive integer, there exist $\{r_i\}_{i=1}^n \subseteqq R$ such that

$$u^k = \sum_{i=1}^n r_i m_i = \sum_{i=1}^n r_i(s_i+t_i) = \sum_{i=1}^n r_i s_i + \sum_{i=1}^n r_i t_i.$$

Because the sum $S_1 + T$ is direct, it follows that $u^k = \sum_{i=1}^n r_i s_i$. Hence all powers of u belong to the finitely generated R-submodule of S_1 generated by $\{s_i\}_{i=1}^n$, showing $u \in R_1$ and therefore that our conclusion holds.

Remark 2. If S_1 is a principal ideal domain, (b) holds for S_1, for in this case every finite S_1-module containing S_1 has a linearly independent module basis containing the identity of S_1. In particular $R_1 = R_2 \cap S_1$ if S_1 is a field. We shall use this particular case of Proposition 2 later. We first consider an example in which $R_1 \subset R_2 \cap S_1$.

Example 2. Let F be a field and let X and Y be indeterminates over F. We set $R = F[XY, XY^2, XY^3, \cdots]$, $S_1 = R[Y]$, and $S_2 = S_1[1/X]$. A straightforward computation shows $Y \in (R_2 \cap S_1) - R_1$. In this particular case, R, S_1, and S_2 are even domains with a common quotient field $F(X, Y)$.

In [10], p. 677, Krull establishes the following result:

If J is a completely integrally closed domain with quotient field K, if L is an algebraic extension field of K, and if $\bar{J}$ is the integral closure of J in L, then $\bar{J}$ is completely integrally closed.

From this it follows that $\bar{J}$ is equal to the complete integral closure J^* of J in L. That $\bar{J} \subseteqq J^*$ is clear. And if $y \in J^*$, y is almost integral over J, hence almost integral over $\bar{J}$, and therefore is in $\bar{J}$ by Krull's theorem. We next note that if L is any extension field of the quotient field K of a completely integrally closed domain J, if L_0 is the subfield of L consisting of all elements of L which are algebraic over K, and if J^* and J_0^*, respectively, denote the complete integral closure of J in L and L_0, respectively, then $J^* = J_0^*$. To prove this we observe that any element t of J^* is such that all powers of t belong to a finite J-submodule of L, hence to a finite K-submodule of L, and hence any such t is algebraic over the field K. Thus $J^* \subseteqq L_0$. Then by Remark 2, $J_0^* = J^* \cap L_0 = J^*$. Because L_0 is algebraically closed in L, it then follows that J^* is completely integrally closed in L. In summary we state

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THEOREM 2. *If J is a completely integrally closed domain with quotient field K, if L is an extension field of K, and if J^* is the complete integral closure of J in L, J^* is a completely integrally closed domain, J^* is the integral closure of J in L, and J^* is completely integrally closed in L.*

We can obtain a slightly more general result than Theorem 2.

COROLLARY 4. *If the complete integral closure D_1 of the domain D is completely integrally closed, then the complete integral closure D^* of D in any domain L_1 containing the quotient field of D is again completely integrally closed in L_1.*

PROOF. Let D_1^* denote the complete integral closure of D_1 in L_1, let L be the quotient field of L_1, and let D_1^{**} be the complete integral closure of D_1 in L. By Theorem 2, D_1^{**} is the integral closure of D_1 in L and is completely integrally closed in L. Now $D_1 \subseteqq D^*$ so that the integral closure of D_1 in L_1 is contained in the integral closure of D^* in L_1. But $D_1^* \subseteqq D_1^{**}$ so that D_1^* is the integral closure of D_1 in L_1. And D^* is integrally closed in L_1 ([14], p. 76). Hence $D_1^* \subseteqq D^*$ and therefore $D_1^* = D^*$. To complete the proof it suffices to show that D_1^* is completely integrally closed in L_1. Thus if $t \in L$ and t is almost integral over D_1^*, $t \in D_1^{**}$ by Theorem 2. Hence t is integral over D_1 and therefore $t \in D_1^*$.

3

Corollary 4 gives rise to the following interesting question: What domains D have the property that the complete integral closure D^* of D is completely integrally closed? Our results on this question are fragmentary. For D Noetherian, D^* is a Krull domain and hence is completely integrally closed ([11], thm. 33.10 and [16]). D^* is also completely integrally closed if D is a Prüfer domain in which each principal ideal has only finitely many minimal prime ideals. But for Prüfer domains in general or for more general D's which are not completely integrally closed, we are not able to answer the question referred to. This section presents some results which should be helpful in further consideration of this problem, however.

LEMMA 3. *Let D be an integral domain with quotient field K. If d is a nonzero element of D, $1/d$ is almost integral over D if and only if $\bigcap_{i=1}^{\infty}(d^i) \neq (0)$.*

PROOF. If $x \in D$ and k is a positive integer, $x[1/d]^k \in D$ if and only if $x \in (d^k)$.

COROLLARY 5. *If the domain D is completely integrally closed, then for each nonunit d of D, $\bigcap_{k=1}^{\infty}(d^k) = (0)$.*

The conditions of Corollary 5 are not sufficient in order that D be completely integrally closed; D could be any Noetherian domain which is not integrally closed. But for a class of domains considered by Gilmer and Ohm in [6] they are equivalent as Proposition 3 shows. We say, following [6], that a domain D has the *QR-property* if each domain between D and its quotient field is of the form D_N for some multiplicative system N.

PROPOSITION 3. *In D, a domain satisfying the QR-property, these statements are equivalent:*

(a) *D is completely integrally closed.*

(b) *if d is a nonunit of D, $\bigcap_{i=1}^{\infty} (d^i) = (0)$.*

(c) *if A is a finitely generated proper ideal of D, $\bigcap_{k=1}^{\infty} A^k = (0)$.*

PROOF. (a) $\leftrightarrow$ (b): (b) follows from (a) by Corollary 5. If (b) holds and if D^* is the complete integral closure of D, then $D^* = D_N$ where N is the set of elements of D which are units of D^* ([6], p. 97, prop. 1.1). If $n \in N$, $1/n \in D^*$ so by Lemma 3, $\bigcap_{i=1}^{\infty} (n^i) \neq (0)$. Since (b) holds, n is a unit of D. Thus $D^* = D_N \subseteqq D$ and (a) is valid.

(b) $\leftrightarrow$ (c): Obviously (c) implies (b). If (b) holds and A is finitely generated, then for some integer k and for some nonunit x of D, $A^k \subseteqq (x)$ by ([6], p. 99, thm. 2.5). Hence $\bigcap_{i=1}^{\infty} A^i = \bigcap_{i=1}^{\infty} A^{ki} \subseteqq \bigcap_{i=1}^{\infty} (x^i) = (0)$ and (c) is valid.

Our next results relate the complete integral closure of a domain to the concept of the conductor, where for R a subring of the ring S, the *conductor of R in S* is the set C of elements x of R such that $xS \subseteqq R$. C is characterized as the largest ideal of R which is also an ideal of S. For R a domain it is easily seen that necessary and sufficient conditions in order that the conductor of R in S be nonzero is that S be contained in a finite R-submodule of the quotient field of R. In view of these observations and Lemma 2, the proofs of Lemmas 4, 5 are immediate and will be omitted.

LEMMA 4. *If D is a domain with quotient field K and if ξ is in K, ξ is almost integral over D if and only if the conductor of D in $D[\xi]$ is nonzero.*

LEMMA 5. *If D_1 and D_2 are domains having a common quotient field, if $D_1 \subseteqq D_2$, and if the conductor of D_1 in D_2 is nonzero, then D_1 and D_2 have the same complete integral closure.*

COROLLARY 6. *If D^*, the complete integral closure of D, is such that the conductor of D in D^* is nonzero, then D^* is completely integrally closed. In particular if $D^* = D[t_1, \cdots, t_n]$ is a finite ring extension of D, then D^* is completely integrally closed.*

PROOF. By Lemma 5, D and D^* have the same complete integral closure. Hence D^* is the complete integral closure of D^*.

If $D^* = D[t_1, \cdots, t_n]$ and if $d_i \in D - \{0\}$ is such that $d_i t_i^k \in D$ for each positive integer k, then $d = d_1 d_2 \cdots d_k$ is a nonzero element of the conductor of D in D^*. Thus D^* is completely integrally closed by the first part of Corollary 6.

Following [5], we define the *pseudo-radical* of a domain D to be the intersection of all nonzero prime ideals of D. We shall use the fact that if D has nonzero pseudo-radical, then each proper prime ideal of D contains a proper minimal prime of D ([5], rmk. 1). Our next result concerns the complete integral closure of a domain with nonzero pseudo-radical.

PROPOSITION 4. *Let D be an integrally closed domain with quotient field K and nonzero pseudo-radical Q. Then D^*, the complete integral closure of D, is the intersection of the rank one valuation rings which contain D, and hence is completely integrally closed.*

PROOF. Let $\{V_\alpha\}$ be the collection of all nontrivial valuation rings between D and K. For any α, V_α has nonzero pseudo-radical since D does. Hence V_α contains a minimal prime ideal P_α and V_α is contained in the rank one valuation ring $W_\alpha = (V_\alpha)_{P_\alpha}$ with maximal ideal P_α ([16], p. 8, thm. 3). $P_\alpha \cap D$ is a proper prime ideal of D so that $Q \subseteqq P_\alpha \cap D$. That $D^* \subseteqq \cap W_\alpha$ is clear; to see the converse let $\xi \in \cap W_\alpha$ and let q be a nonzero element of Q. Then if k is a positive integer, $q\xi^k \in QW_\alpha \subseteqq P_\alpha W_\alpha = P_\alpha$. Therefore $q\xi^k \in \cap P_\alpha \subseteqq \cap V_\alpha = D$ so ξ is almost integral over D — that is, $\xi \in D^*$.

Except for Proposition 5 the remaining results of this paper concern Prüfer domains. They show that by imposing certain finiteness conditions on a Prüfer domain, we can conclude that its complete integral closure is an intersection of valuation rings of rank $\leqq 1$, and hence is completely integrally closed.

PROPOSITION 5.[2] *Let K be a field, let $\{V_\lambda\}$ be a family of valuation rings with K as quotient field, and let $V = \cap V_\lambda$. Suppose for each λ, v_λ is a valuation associated with the valuation ring V_λ. If V has quotient field K and if the family $\{v_\lambda\}$ has finite character in the sense that for any nonzero element x of K, $v_\lambda(x) \neq 0$ for only finitely many λ's, then the complete integral closure of V is $\cap V'_\lambda$ where for any λ, V'_λ is K if there is no rank one valuation ring between V_λ and K and V'_λ is the unique rank one valuation ring between V_λ and K otherwise.*

PROOF. It is sufficient to show that under the given hypothesis, $\cap V'_\lambda$ is almost integral over V. Thus let $\xi \in \cap V'_\lambda$ and let $v_1, \cdots, v_n$ be the finite number of valuations in the family $\{v_\lambda\}$ which have negative value on ξ. Then for any i, ξ^{-1} is a nonunit of V_i so that $\bigcap_{j=1}^{\infty}(\xi^{-1})^j V_i = P_i$ is a prime ideal of V_i ([7], p. 240, Lemma 2.10). If $P_i = (0)$, then V_i has a minimal prime ideal Q_i, and $\xi^{-1} \in Q_i$. But then ξ^{-1} is a nonunit of $(V_i)_{Q_i} = V'_i$, contrary to hypothesis. Hence no $P_i = (0)$ so that $A = P_1 \cap \cdots \cap P_n \cap V$ is nonzero. If a is a nonzero element of A, we have, for any positive integer k, $a \in (\xi^{-1})^k V_i$ so $a\xi^k \in V_i$ and hence

[2] A result equivalent to Proposition 5 has been proved independently by Butts and Smith and appears as Theorem 5 in the paper [2] which they have submitted for publication.

$a\xi^k \in \bigcap_{i=1}^n V_i$. But for $v_\lambda \notin \{v_1, \cdots, v_n\}$, $\xi \in V_\lambda$ so $a\xi^k \in V_\lambda$ also. Consequently, $a\xi^k \in \cap V_\lambda = V$ for any positive integer k and our proof is complete.

If D is a Prüfer domain with quotient field K, it is known that every valuation ring between D and K is of the form D_P for some prime ideal P of D. This allows us to restate Proposition 5 in terms of the ideal theory of V in case V is Prüfer.

Corollary 7. *Suppose D is a Prüfer domain containing a family $\{P_\lambda\}$ of prime ideals such that $D = \cap D_{P_\lambda}$ and such that each nonzero element of D belongs to only finitely many P_λ's. Then the complete integral closure of D is the intersection of all valuation rings of rank $\leqq 1$ lying between D and its quotient field, and is therefore completely integrally closed.*

We note that any Prüfer domain in which each nonzero element belongs to only finitely many maximal ideals is a domain satisfying the hypothesis of Corollary 7. The converse also holds, but we shall not establish it here. The complete integral closure of a Prüfer domain satisfying the hypothesis of Corollary 7 must have dimension $\leqq 1$. This will follow from Corollary 9.

Proposition 6. *Let P be a nonminimal prime ideal of the Prüfer domain D satisfying this condition: there exists a nonzero prime ideal Q contained in P and an element x of $P-Q$ such that (x) has only finitely many minimal prime ideals. Then D_P does not contain the complete integral closure of D.*

Proof. Let M be the minimal prime of (x) contained in P and let $Q_1, Q_2, \cdots, Q_n$ be the minimal primes of (x) distinct from M. We let v be a valuation associated with D_M and v_i be a valuation associated with D_{Q_i} for each i. If $y \in (Q_1 \cap \cdots \cap Q_n) - M$, then because $\sqrt{xD_{Q_i}}$ is $Q_i D_{Q_i}$ for each i, there exists a fixed power $u = y^m$ of y such that $v_i(u) > v_i(x)$ for each i. If then $\xi = u/x$, $v_i(\xi) > 0$ for each i and $v(\xi) < 0$. We show that ξ is almost integral over D. For this purpose, choose $q \in Q$, $q \neq 0$. We show that if k is a positive integer and if N is a maximal ideal of D, then $q\xi^k \in D_N$. It will then follow that $q\xi^k \in D$, ξ is almost integral over D, $\xi \notin D_P$.

If $x \notin N$, clearly $q\xi^k \in D_N$. If $x \in N$, N contains some minimal prime of (x). If $Q_i \subseteqq N$ for some i, then $q\xi^k \in Q_i D_{Q_i} \subseteqq ND_N \subseteqq D_N$. And if $M \subseteqq N$ then we have $qD_M \subseteqq QD_M \subset x^k D_M$, implying that $q/x^k \in MD_M$. Hence $q\xi^k = u^k(q/x^k) \in MD_M \subseteqq ND_N \subseteqq D_N$. In any case our proof is then complete.

Corollary 8. *Let D be a Prüfer domain such that each principal ideal of D has only finitely many minimal prime ideals. Then the complete integral closure D^* of D has dimension $\leqq 1$ and is completely integrally closed.*

Proof. D^* is Prüfer and if M^* is a maximal ideal of D^*, $D^*_{M^*} = D_P$ where $P = M^* \cap D$ ([4], thm. 1). By Proposition 6, P properly contains no nonzero prime ideal. Hence D_P has rank $\leqq 1$, M^* properly contains no nonzero prime of D^*, and D^* has dimension $\leqq 1$. Since D^* is an intersection of valuation rings of rank $\leqq 1$, D^* is completely integrally closed.

COROLLARY 9. *If D is a Prüfer domain in which each nonzero element belongs to only finitely many maximal ideals, the complete integral closure of D has dimension $\leqq 1$ and is completely integrally closed.*

In view of the preceding results we might hope to prove for a Prüfer domain D that the complete integral closure of D is an intersection of valuation rings of rank $\leqq 1$. That this is not the case is shown by Nakayama's example. The question remains open as to whetcher the complete integral closure of an arbitrary Prüfer domain is completely integrally closed. We are, in fact, unable to answer this question in more restrictive cases — for example, in the case when D has the QR-property.

Finally, we remark that J. Ohm has transmitted to us a copy of his paper [13], which concerns some interesting results related to the complete integral closure of a domain.

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