IDEAL DECOMPOSITIONS IN NOETHERIAN RINGS

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An interesting identity is obtained for ideals A and B in a Noetherian ring:

$$A = (A + B^n) \cap (A : B^n)$$

for sufficiently large n. This identity is applied to obtain Fuchs' quasi-primary decomposition of A in an improved form, and to obtain Krull's theorem on the intersection of the powers of A, both developments making no use of the Noetherian primary decomposition of A. Finally, the identity is used to obtain the primary decomposition without reference to irreducible ideals, in a largely constructive manner which yields the decomposition in an illuminating, automatically normal form, and which, subject to certain simple conditions, is unique.

Throughout the paper R is a (commutative) Noetherian ring (with unity).

LEMMA 1. *If A and B are ideals of R, then for a sufficiently large integer n, $A = (A + B^n) \cap (A : B^n)$.*

Proof. Let $B = (x_1, \ldots, x_q)$, and choose positive integers $r, n_1, \ldots, n_q$ such that

$$A : B^r = A : B^{r+1} = \ldots, \tag{1}$$

$$n_i > r \qquad \text{for } i = 1, \ldots, q, \tag{2}$$

$$A : (x_1^{n_1}) = A : (x_1^{n_1+1}) = \ldots \tag{3}$$

and

$$[A + (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}})] : (x_i^{n_i}) = [A + (x_1^{n_1}, \ldots, x_{i-1}^{n_{i-1}})] : (x_i^{n_i+1}) = \ldots \quad \text{for } i = 2, \ldots, q.$$

To simplify the notation, let $x_i^{n_i} = y_i$, $i = 1, \ldots, q$.

By a well-known identity (**4**, p. 22), $A = (A + (y_1)) \cap (A : (y_1))$, and if $A = (A + (y_1, \ldots, y_i)) \cap (A : (y_1, \ldots, y_i))$, then

$$\begin{aligned} A &= (A + (y_1, \ldots, y_{i+1})) \cap [(A + (y_1, \ldots, y_i)) : (y_{i+1})] \cap (A : (y_1, \ldots, y_i)) \\ &\supseteq (A + (y_1, \ldots, y_{i+1})) \cap (A : (y_{i+1})) \cap (A : (y_1, \ldots, y_i)) \\ &= (A + (y_1, \ldots, y_{i+1})) \cap (A : (y_1, \ldots, y_{i+1})). \end{aligned}$$

Thus, by induction,

$$A = (A + (y_1, \ldots, y_q)) \cap (A : (y_1, \ldots, y_q)).$$

Received October 16, 1963.

If $m = n_1 + \ldots + n_q - q + 1$, then $B^m \subseteq (y_1, \ldots, y_q) \subseteq B^r$, and so

$$A \subseteq (A + B^m) \cap (A:B^r) \subseteq (A + (y_1, \ldots, y_q)) \cap (A:(y_1, \ldots, y_q)) = A.$$

Thus, if $n = \max\{m, r\}$, then by assumption (1), $A = (A + B^n) \cap (A:B^n)$, and the proof is complete.

COROLLARY. *For ideals A and B of R and a sufficiently large integer n, $(A:B^n) \cap B^n \subseteq A$.*

Note that by Dedekind's modular law,

$$(A:B^n) \cap (A + B^n) = A + ((A:B^n) \cap B^n)$$

and so $A = (A:B^n) \cap (A + B^n)$ if and only if $(A:B^n) \cap B^n \subseteq A$. Also, if $A = (A + B^n) \cap (A:B^n)$, then

$$A:B^n = ((A + B^n):B^n) \cap (A:B^{2n}) = A:B^{2n},$$

and hence $A:B^n = A:B^{n+1} = \ldots$.

It follows directly from the maximum condition in R that any ideal A contains a finite product of prime over ideals (**5**, p. 200). The factors of such a product with the smallest number of distinct factors are precisely the minimal prime ideals of A. Thus, A is contained in a finite number of minimal prime ideals and contains a product of them.

THEOREM 1. *If A is any ideal of R and $P_1, \ldots, P_k$ are the minimal prime ideals of A, then for a sufficiently large integer n,*

$$A = (A + P_1^n) \cap ((A:P_1^n) + P_2^n) \cap \ldots \cap ((A:P_1^n \ldots P_{k-1}^n) + P_k^n).$$

Proof. By systematic application of Lemma 1, there exist $n_1, \ldots, n_k$ such that

$$\begin{aligned} A &= (A + P_1^{n_1}) \cap (A:P_1^{n_1}) \\ &= (A + P_1^{n_1}) \cap ((A:P_1^{n_1}) + P_2^{n_2}) \cap ((A:P_1^{n_1}):P_2^{n_2}) \\ &= (A + P_1^{n_1}) \cap ((A:P_1^{n_1}) + P_2^{n_2}) \cap (A:P_1^{n_1}P_2^{n_2}) = \ldots \\ &= (A + P_1^{n_1}) \cap ((A:P_1^{n_1}) + P_2^{n_2}) \cap \ldots \cap ((A:P_1^{n_1} \ldots P_{k-1}^{n_{k-1}}) + P_k^{n_k}) \\ &\qquad \cap (A:P_1^{n_1} \ldots P_k^{n_k}). \end{aligned}$$

There exists an $n \geqslant n_i$, $i = 1, \ldots, k$, such that $(P_1 \ldots P_k)^n \subseteq A$. Then $A:P_1^n \ldots P_k^n = R$, and since by an earlier remark

$$\begin{aligned} A:P_1^{n_1} \ldots P_i^{n_i} &= (A:P_1^{n_1}):P_2^{n_2} \ldots P_i^{n_i} = (A:P_1^n):P_2^{n_2} \ldots P_i^{n_i} = \ldots \\ &= (\ldots (A:P_1^n):P_2^n):\ldots):P_i^n = A:P_1^n \ldots P_i^n, \end{aligned}$$

we have that

$$A = (A + P_1^n) \cap ((A:P_1^n) + P_2^n) \cap \ldots \cap ((A:P_1^n \ldots P_{k-1}^n) + P_k^n),$$

as required.

https://doi.org/10.4153/CJM-1965-017-x Published online by Cambridge University Press

COROLLARY. *If A is any ideal of R and $P_1, \ldots, P_k$ are the minimal prime ideals of A, then for a sufficiently large integer n,*

$$A = (A + P_1^n) \cap (A + P_2^n) \cap \ldots \cap (A + P_k^n).$$

Recall that, following Fuchs **(1)**, an ideal A is quasi-primary if and only if the radical of A is prime. A representation of an ideal A as an intersection of quasi-primary ideals is in shortest form if none of the quasi-primary components can be omitted and no intersection of two or more of the components is itself quasi-primary.

If A is an ideal contained in the prime ideal P, then the radical of $A + P^n$ is P and so $A + P^n$ is quasi-primary. Thus, the decomposition above is a representation of A as an intersection of quasi-primary ideals. Moreover, this is a shortest representation, since the prime ideals $P_1, \ldots, P_k$ are the minimal prime ideals of A, and hence none of the quasi-primary components $A + P_i^n$ can be omitted and no intersection of two or more is quasi-primary.

If the index of a quasi-primary ideal Q is defined as the smallest power of the radical of Q contained in Q, then given any shortest representation $A = Q_1 \cap \ldots \cap Q_m$ of A as an intersection of quasi-primary ideals Q_i of index r_i and radical P_i, the component Q_i can be replaced by the ideal $A + P_i^{r_i}$; indeed, $A + P_i^{r_i}$ is the minimal quasi-primary ideal of index r_i which can be substituted for Q_i in the decomposition. Of course, any exponent of P_i greater than r_i can also be used. Since $Q_1 \cap \ldots \cap Q_m$ is in shortest form, the P_i are exactly the minimal prime ideals of A and the decomposition

$$(A + P_1^{r_1}) \cap \ldots \cap (A + P_m^{r_m})$$

is also in shortest form. Since the minimal prime ideals of A are determined only by A, we have proved the following results of Fuchs **(1**, Theorems 5 and 6**)**.

THEOREM 2. *Every ideal A of R can be represented as a finite intersection of quasi-primary ideals. Given two shortest representations of A, there is a one-to-one correspondence between the components such that corresponding components have the same radical.*

We now establish Krull's intersection theorem **(3)** without the use of the Noetherian primary decomposition. (The referee has called our attention to the fact that Herstein **(2)** has recently also obtained such a proof by different methods.)

THEOREM 3. *Let M be any ideal of R, and $A = \bigcap_{n=1}^{\infty} M^n$. Then $A = (0)$ if and only if no element of $1 - M$ is a zero-divisor.*

Proof. The proof in **(5**, p. 216**)** utilizes the primary decomposition only to obtain the preliminary result $AM \supseteq A$. Now, applying Lemma 1, we have that for n sufficiently large

$$AM = (AM + M^n) \cap (AM:M^n) \supseteq A \cap A = A,$$

as required.

Next we turn our attention to the primary decomposition. Recall that for P any prime ideal and A any ideal of R the P-component of A is

$$A_P = \{x|xt \in A \text{ for some } t \notin P\},$$

and that since R is Noetherian, there exists an s not in P such that $A_P = A:(s)$. We collect several simple results concerning P-components in the following lemma.

LEMMA 2. (i) *If A is any ideal, P a prime ideal, and B an ideal contained in A_P, then $A:B \not\subseteq P$.*

(ii) *If P is a minimal prime ideal of A, then A_P is the (unique) minimal P-primary ideal containing A, and is a minimal primary ideal containing A.*

(iii) *If Q is a minimal primary ideal containing A, then Q is equal to A_P for some minimal prime ideal P of A.*

(iv) *If A and B are any ideals and P is a prime ideal, then $P \supseteq A:B$ if and only if $P \supseteq A_P:B_P$.*

(v) *P is a minimal prime ideal of A if and only if P is a prime ideal containing A and $A:P^n \not\subseteq P$ for some positive integer n.*

Proof. (i) Since $sA_P \subseteq A$ for some s not in P, it is immediate that $A:A_P \not\subseteq P$, whence if $B \subseteq A_P$ so that $A:B \supseteq A:A_P$, then $A:B \not\subseteq P$.

(ii) Now let P be a minimal prime ideal of A, and $P_2, \ldots, P_k$, the remaining minimal prime ideals of A. Clearly $A_P \subseteq P$. There exists an n such that $(PP_2 \ldots P_k)^n \subseteq A$. If $k = 1$, then $P^n \subseteq A \subseteq A_P$, while if $k > 1$, then there exists some y in $(P_2 \ldots P_k)^n$ such that y is not in P, whence $P^n y \subseteq A$ and so $P^n \subseteq A_P$. Thus we have $P^n \subseteq A_P \subseteq P$.

If $xy \in A_P$ and $y \notin P$, then there exists $s \notin P$ such that $xys \in A$. Then $ys \notin P$ implies that $x \in A_P$, and A_P is P-primary.

Now if Q is any P-primary ideal containing A, then $x \in A_P$ implies that there exists $s \notin P$ such that $xs \in A \subseteq Q$, whence $x \in Q$, so that A_P is the minimal P-primary ideal containing A. Clearly A_P is also a minimal primary ideal containing A.

(iii) It is well known (e.g. **4**, p. 51) that Krull's intersection theorem implies (again without essential use of the primary decomposition) that for any prime ideal P of R the intersection of all symbolic prime powers, $P^{(n)}$, of P or, equivalently, of the ideals $(P^n)_P$ is equal to the P-component, $(0)_P$, of the zero ideal. Noting that if $A \subseteq P$, then under the canonical homomorphism, f, of R modulo A the ideal $\bigcap_{n=1}^{\infty} (A + P^n)_P$ corresponds to the ideal

$$\bigcap_{n=1}^{\infty} f(P)^{(n)} = (0)_{f(P)},$$

we see that

$$\bigcap_{n=1}^{\infty} (A + P^n)_P = A_P.$$

If, now, Q is a minimal primary ideal containing A, then, *a fortiori*, Q is a minimal P-primary ideal containing A for some prime ideal P. Thus, Q is equal to $(A + P^n)_P$ for all sufficiently large n and so is equal to A_P. But A_P P-primary implies that P is a minimal prime ideal of A.

(iv) Suppose that P is prime and $P \supseteq A:B$. Now $sA_P \subseteq A$ for some s not in P. If $x \in A_P:B_P$, then $xB_P \subseteq A_P$ and $sxB_P \subseteq A$. Thus, $sx \in A:B_P \subseteq A:B \subseteq P$, whence $x \in P$ and $A_P:B_P \subseteq P$.

Conversely, suppose P is prime and $P \not\supseteq A:B$. Then $sB \subseteq A$ for some s not in P. Also, $tB_P \subseteq B$ for some t not in P. Hence $stB_P \subseteq A$ with st not in P, so that $B_P \subseteq A_P$, $A_P:B_P = R \not\subseteq P$.

(v) Suppose that P is a minimal prime ideal of A. Then A_P is P-primary by (ii), $P^n \subseteq A_P$ for some n, and by (i), $A:P^n \not\subseteq P$.

Conversely, suppose that P is prime, $A \subseteq P$, and $A:P^n \not\subseteq P$ for some positive integer n. The result is trivial if $P = A$; so suppose A is properly contained in P and let C be any ideal which contains A and is properly contained in P. There exists a p in P such that $p \notin C$ and an s not in P such that $sP^n \subseteq A$. Hence, $sp^n \in A \subseteq C$ and C is not prime. Thus, P must be a minimal prime ideal of A.

This completes the proof of Lemma 2.

To obtain the primary decomposition of A, let S_1 be the intersection of all ideals A_P for P a minimal prime ideal of A, or, equivalently, the intersection of all minimal primary ideals containing A, let $S_{k+1} = S_k \cap B_k$, where B_k is the intersection of all A_P for P a minimal prime ideal of $A:S_k$, and consider the chain of ideals thus obtained.

Since the S_k form a descending chain, the ideals $A:S_k$ form an ascending chain, and hence for some n, $A:S_n = A:S_{n+1} = \ldots$. If $A:S_n \neq R$, then $A:S_n$ has a minimal prime ideal P, $S_{n+1} \subseteq A_P$ and $A:S_{n+1} \not\subseteq P$. Thus $A:S_n = R$. Thus the chain of ideals $S_1 \supset S_2 \supset \ldots \supset S_n = A$ associates with A a finite set of prime ideal divisors $P_1, \ldots, P_q$ of A such that

$$A = A_{P_1} \cap \ldots \cap A_{P_q}.$$

We have that S_1 is the intersection of primary ideals whose radicals are the minimal prime ideals of A. Assume, inductively, that for some k, $1 \leqslant k < n$, each $A_{P'}$ in $S_k = \cap A_{P'}$ has a primary decomposition for which the prime radicals are contained in P'.

Now let P be a minimal prime ideal of $A:S_k$. Then for some s not in P we have $(S_k)_P = S_k:(s)$ and $s(S_k)_P \subseteq S_k$. Since $A:S_k$ contains a product of its minimal prime ideals, for all integers r greater than some integer m and some q not in P we have $P^r qS_k \subseteq A$. Then $P^r qs(S_k)_P \subseteq A$, and since $qs \notin P$, thus $P^r(S_k)_P \subseteq A_P$ and $(S_k)_P \subseteq A_P:P^r$.

In the assumed primary decomposition of each $A_{P'}$ in $S_k = \cap A_{P'}$ the prime ideal P does not occur as one of the prime radicals, since, by Lemma 2, $S_k \subseteq A_{P'}$ implies $A:S_k \not\subseteq P'$, whereas $A:S_k \subseteq P$. Now $(S_k)_P$ also has a primary decomposition for which the prime radicals are all contained in P (**4**, p. 17) and hence properly contained in P. Thus, if $yP^r \subseteq A_P \subseteq (S_k)_P$, then we must have $y \in (S_k)_P$, and so $A_P:P^r \subseteq (S_k)_P$. We conclude that $A_P:P^r = (S_k)_P$.

Now take r sufficiently large that $A_P:P^r = A_P:P^{r+1} = \ldots, A_P:P^r = (S_k)_P$, and $A_P = (A_P:P^r) \cap (A_P + P^r)$. Then $A_P = (S_k)_P \cap (A_P + P^r)$, and taking P-components we obtain $A_P = (S_k)_P \cap (A_P + P^r)_P$. But $(A_P + P^r)_P = (A + P^r)_P$; hence $A_P = (S_k)_P \cap (A + P^r)_P$. Since P is the unique minimal prime ideal of $A + P^r$, $(A + P^r)_P$ is P-primary and A_P has a primary decomposition, all components of which have radicals contained in P.

Thus, by induction, A has a primary decomposition. We note that if P is a minimal prime ideal of A, then for some m, $P^m \subseteq A_P$ and

$$A_P = (A_P)_P = (A_P + P^m)_P = (A + P^m)_P.$$

Moreover, since for P a minimal prime ideal of $A:S_k$, $A_P = (S_k)_P \cap (A + P^r)_P$, it is clear that

$$S_{k+1} = S_k \cap B_k = S_k \cap [\cap (A + P_i^{r_i})_{P_i}],$$

where the P_i are the minimal prime ideals of $A:S_k$, and hence do not occur as radicals of any of the primary ideals in the decomposition of S_k. Thus, each P occurring as a minimal prime ideal of A or of some $A:S_k$ is the radical of exactly one P-primary ideal (namely $(A + P^r)_P$, the unique minimal P-primary ideal containing $(A + P^r)$) in the primary decomposition of A thus obtained.

We now show that this decomposition of A as a finite intersection of primary ideals $Q_i = (A + P_i^{r_i})_{P_i}$ is automatically a normal decomposition of A. Since the P_i are all distinct, it suffices to show that no Q_i is redundant. Suppose, on the contrary, that some $Q_i = Q = (A + P^r)_P$ is redundant, where P is a minimal prime ideal of $A:S_k$, so that

$$A = Q_1 \cap \ldots \cap Q_m \cap \ldots \cap Q_n \cap Q = Q_1 \cap \ldots \cap Q_n,$$

where $Q_1, \ldots, Q_m$ are those Q_j whose radicals are contained in the radical P of Q. Then

$$A_P = Q_1 \cap \ldots \cap Q_m \cap Q = Q_1 \cap \ldots \cap Q_m$$

and Q is also redundant in the resulting decomposition of A_P. But this decomposition is precisely $A_P = (S_k)_P \cap Q$, which implies that $A_P = (S_k)_P$ and $A_P:(S_k)_P = R$. But by Lemma 2 (iv), $A:S_k \subseteq P$ implies $A_P:(S_k)_P \subseteq P \neq R$. Hence the original primary decomposition of A is irredundant, and therefore also normal.

We have proved the following theorem:

THEOREM 4. *Let A be any ideal of R. Then by the above process A is represented as an (automatically normal) intersection of a finite number of primary ideals of the form $(A + P^r)_P$.*

https://doi.org/10.4153/CJM-1965-017-x Published online by Cambridge University Press

The usual uniqueness theorems concerning primary decompositions now follow by standard arguments.

For decompositions of the above form, an additional uniqueness property can be obtained by use of the following theorem.

THEOREM 5. *Let*

$$A = Q_1 \cap \ldots \cap Q_k = Q_1' \cap \ldots \cap Q_k',$$

where Q_i *and* Q_i' *are primary with radical* P_i, $i = 1, \ldots, k$, *and* $P_i \neq P_j$ *if* $i \neq j$. *Then*

$$A = Q_1 \cap \ldots \cap Q_{j-1} \cap Q_j' \cap Q_{j+1} \cap \ldots \cap Q_k.$$

Proof. Assume the indexing chosen so that $P_1, \ldots, P_{j-1}$ are those prime ideals of A not contained in $P = P_j$. Then $A = Q_1 \cap \ldots \cap Q_{j-1} \cap A_P$. But for n sufficiently large, $A_P = Q_j' \cap (A_P:P^n)$ and $A_P:P^n = Q_{j+1} \cap \ldots \cap Q_k$. Thus,

$$A = Q_1 \cap \ldots \cap Q_{j-1} \cap Q_j' \cap Q_{j+1} \cap \ldots \cap Q_k.$$

THEOREM 6. *Any ideal* A *of* R *has a normal decomposition*

$$A = \bigcap_{i=1}^{n} (A + P_i^{m_i})_{P_i}$$

which is unique in that each exponent m_i *is the minimum exponent that yields a primary component of* A *of the given form.*

Proof. By Theorem 4, A has a normal decomposition

$$A = \bigcap_{i=1}^{n} (A + P_i^{r_i})_{P_i}.$$

By Theorem 5 the exponents associated with the prime ideals P_i can be minimized independently and, moreover, each minimum m_i thus obtained depends only upon A and P_i.

REFERENCES

1. L. Fuchs, *On quasi-primary ideals*, Acta Math. Szeged, *3* (1947), 174–183.
2. I. N. Herstein, *Due risultati classici sugli anelli*, Univ. e Polit. Torino Rend. Sem. Mat., *21* (1961–62), 99–102.
3. W. Krull, *Idealtheorie*, Ergebn. Math., IV, *3* (1948).
4. D. Northcott, *Ideal theory*, (Cambridge, 1953).
5. O. Zariski and P. Samuel, *Commutative algebra*, vol. I (New York, 1958).

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