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ON PRODUCT VARIETIES OF INVERSE SEMIGROUPS

J. L. BALES

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Abstract

This paper extends results on product varieties of groups to inverse semigroups. We show that if $\mathscr{G}$ is a variety of groups and $\mathscr{V}$ any inverse semigroup variety, then $\mathscr{G}\circ\mathscr{V}$ is a variety. We give a characterization of the identities of $\mathscr{G}\circ\mathscr{V}$ in terms of the identities of $\mathscr{G}$ and of $\mathscr{V}$. We show that if $\mathscr{V}$ does not contain the variety of all groups then it has uncountably many supervarieties. Finally we show that if $\mathscr{H}$ is another variety of groups then

$$(\mathscr{G}\circ\mathscr{H})\circ\mathscr{V}=\mathscr{G}\circ(\mathscr{H}\circ\mathscr{V}).$$

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1. Introduction

An inverse semigroup may be defined as an algebra of type $\langle 2,1\rangle$ with binary operation $(x,y)\to x\cdot y$ (multiplication) and unary operation $x\to x^{-1}$ (taking inverses), satisfying the following set of identities:

1. $x\cdot(y\cdot z)=(x\cdot y)\cdot z$.
2. $x\cdot x^{-1}\cdot x=x$.
3. $x^{-1}\cdot x\cdot x^{-1}=x^{-1}$.
4. $x^{-1}\cdot x\cdot y\cdot y^{-1}=y\cdot y^{-1}\cdot x^{-1}\cdot x$.

It is not difficult to see that the class of all such algebras is in fact the class of all inverse semigroups. This class is a variety and will be denoted by $\mathscr{I}\mathscr{S}$. The lattice of all subvarieties of $\mathscr{I}\mathscr{S}$ is denoted by $\mathscr{L}(\mathscr{I}\mathscr{S})$ and if $\mathscr{U}$ and $\mathscr{V}$ are subvarieties of $\mathscr{I}\mathscr{S}$ with $\mathscr{U}\leqslant\mathscr{V}$ then $\mathscr{L}(\mathscr{U},\mathscr{V})$ denotes the lattice of all varieties $\mathscr{W}$ such that $\mathscr{U}\leqslant\mathscr{W}\leqslant\mathscr{V}$. We will denote the set of identities of a variety $\mathscr{V}$ by $\mathrm{Id}(\mathscr{V})$.

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The class of all semilattices $\mathscr{B}_{1,1}$ forms a subvariety of $\mathscr{I}\mathscr{S}$ and within $\mathscr{I}\mathscr{S}$ has the basis $x = x^2$. Also any variety of groups is a subvariety of $\mathscr{I}\mathscr{S}$. The variety of all groups $\mathscr{G}\!\mathit{p}$ has as basis the identity $xx^{-1} = yy^{-1}$.

Following from the theorem of Petrich (1975) we have that for a variety of groups $\mathscr{G}$, $\mathscr{G} \vee \mathscr{B}_{1,1}$ is the variety of semilattices of groups whose groups are in $\mathscr{G}$. A basis for the variety of all semilattices of groups, $\mathscr{G}\!\mathit{p} \vee \mathscr{B}_{1,1}$ is given, within $\mathscr{I}\mathscr{S}$, by

$$xx^{-1} = x^{-1}x,$$

or by

$$xx^{-1}y = yxx^{-1}.$$

The free inverse semigroup on a countable set of generators in a variety $\mathscr{V}$ is denoted by $F_X(\mathscr{V})$ and we put $F_X = F_X(\mathscr{I}\mathscr{S})$ where $X = \{x_1, x_2, \ldots\}$. Further notation follows Clifford and Preston (1961, 1967) and Grätzer (1968) unless otherwise stated.

In her book Hanna Neumann (1967) defines the product of two varieties of groups $\mathscr{U}$ and $\mathscr{V}$ as the class of all groups that are extensions of a group in $\mathscr{U}$ by a group in $\mathscr{V}$. This class is a variety. Maltsev (1967) extended this to arbitrary classes of algebras.

DEFINITION 1.1. If $\mathscr{U}$ and $\mathscr{V}$ are subclasses of a class $\mathscr{K}$ then the product $\mathscr{U} \circ_{\mathscr{K}} \mathscr{V}$ is defined as consisting of the algebras A from $\mathscr{K}$ such that for some congruence θ on A, $A/\theta \in \mathscr{V}$ and each θ-class which is a subalgebra of A is in $\mathscr{U}$.

If $\mathscr{U}$ and $\mathscr{V}$ are subvarieties of a variety $\mathscr{K}$ then $\mathscr{U} \circ_{\mathscr{K}} \mathscr{V}$ is not necessarily a variety (see Maltsev (1967)). For inverse semigroup varieties we see this from the following example.

EXAMPLE 1.2. The maximal group congruence σ on F_X, $X = \{x_1, x_2, \ldots\}$, has kernel $E(F_X)$, the semilattice of idempotents of F_X, and $F_X/\sigma \cong F_X(\mathscr{G}\!\mathit{p})$. (This follows from the characterizations of F_X by Scheiblich (1973) and Munn (1974).) Thus $F_X \in \mathscr{B}_{1,1} \circ_{\mathscr{I}\mathscr{S}} \mathscr{G}\!\mathit{p}$. But every countable inverse semigroup is a homomorphic image of F_X but certainly not all are extensions of a group by a semilattice.

However, Maltsev proves the following theorem for quasivarieties.

THEOREM 1.3. (Maltsev (1967), Theorem 5.) *For a quasivariety $\mathscr{K}$ of finite type the product of two subquasivarieties is a quasivariety.*

In the product of two inverse semigroup varieties $\mathscr{U} \circ_{\mathscr{I}\mathscr{S}} \mathscr{V}$ we will from now on drop the subscript $\mathscr{I}\mathscr{S}$.

2. On congruences

In this section we develop a theory of fully invariant congruences needed for the proof of the later theorems. First we summarize some results from Clifford and Preston (1967), Chapter 7.

DEFINITION 2.1. A kernel normal system $\mathscr{A} = \{A_i : i \in I\}$ of an inverse semigroup S is a set of disjoint inverse subsemigroups A_i of S such that

(i) for each $e \in E(S)$, $e \in A_i$, for some $A_i \in \mathscr{A}$;

(ii) $a\, A_i a^{-1} \subseteq A_j$, for each $a \in S$, $A_i \in \mathscr{A}$, for some $j \in I$.

(iii) If $a, ab, bb^{-1} \in A_i$ then $b \in A_i$, for every $a, b \in S$ and $A_i \in \mathscr{A}$.

A kernel normal system $\mathscr{A}$ on an inverse semigroup defines a congruence denoted by $\rho_{\mathscr{A}}$. This is given by

$$\rho_{\mathscr{A}} = \{(a,b) \in S \times S \,|\, aa^{-1}, bb^{-1}, ab^{-1} \in A_i, \text{ for some } i \in I\}.$$

Differing from Clifford and Preston, we define the kernel of $\rho_{\mathscr{A}}$ to be the union of the A_i. That is,

$$\begin{aligned} \operatorname{Ker} \rho_{\mathscr{A}} &= \bigcup \{A_i : i \in I\} \\ &= \bigcup \{e\rho_{\mathscr{A}} : e \in E(S)\}. \end{aligned}$$

If ρ is an idempotent separating congruence then $\operatorname{Ker} \rho$ is a semilattice of groups.

THEOREM 2.2. (Clifford and Preston (1967), Theorem 7.54.) *Let E be the set of idempotents of an inverse semigroup S. For each $e \in E$ denote by H_e the maximal subgroup of S with identity element e and take any subgroup N_e of H_e. Put $\mathscr{N} = \{N_e : e \in E\}$ and $N = \bigcup \{N_e : e \in E\}$. Then $\mathscr{N}$ is a kernel normal system of S if and only if* (i) *N is a subsemigroup of S and* (ii) *$a^{-1} Na \subseteq N$ for all $a \in S$.*

Clearly $\rho_{\mathscr{N}}$ is idempotent-separating. The collection $\mathscr{N}$ is called a *group kernel normal system*. To each group kernel normal system there corresponds a unique idempotent separating congruence. The maximum idempotent separating congruence $\mu = \mu_S$ on an inverse semigroup S is given by Howie (1964) as $\mu = \{(x,y) \in S \times S : x^{-1} ex = y^{-1} ey \text{ for all } e \in E\}$.

DEFINITION 2.3. Let S be an inverse semigroup.

(i) A congruence ρ is *fully invariant on S* if and only if for each endomorphism φ of S, and for all $x, y \in S$, if $x\rho y$ then $(x\varphi)\, \rho (y\varphi)$.

(ii) An inverse subsemigroup A of S is *fully invariant in S* if and only if for each endomorphism φ of S, $A\varphi \subseteq A$.

For groups these definitions are 'equivalent' as there is a one-to-one correspondence between normal subgroups and the congruences. This is not so, however, even for semilattices in general, as shown by the following example.

EXAMPLE 2.4. Take the unique three element semilattice $S = \{x, y, 0\}$ with zero 0 and with $xy = 0$, and let ρ be the congruence given by $x\rho = 0\rho = \{0, x\}$ and $y\rho = \{y\}$. Then $\operatorname{Ker}\rho = S$ and so it is fully invariant in S. But ρ is not fully invariant on S; take φ an endomorphism of S given by $x\varphi = y$, $y\varphi = x$ and $0\varphi = 0$. Then we have $0\rho x$, but $0\varphi = 0$, $x\varphi = y$ and 0 is not ρ-equivalent to y.

In the general case we have the following result.

LEMMA 2.5. *If ρ is a fully invariant congruence on an inverse semigroup S then* $\operatorname{Ker}\rho$ *is fully invariant in S.*

PROOF. Let $a \in \operatorname{Ker}\rho = K$ and $\varphi \in \operatorname{end}(S)$. Then $a\rho e$, for some $e \in E(S)$ and so $(a\varphi)\,\rho(e\varphi)$ as ρ is fully invariant. But $e\varphi \in E(S)$ and so $a\varphi \in K$. Thus $K\varphi \subseteq K$ and K is fully invariant in S.

THEOREM 2.6. *Let ρ be an idempotent separating congruence on an inverse semigroup S. Then ρ is fully invariant if and only if* $\operatorname{Ker}\rho$ *is fully invariant.*

PROOF. Necessity follows from the above lemma. To show sufficiency let $K = \operatorname{Ker}\rho$ be fully invariant in S and $\varphi \in \operatorname{end}(S)$. Then $K\varphi \subseteq K$. Take any $x, y \in S$ such that $x\rho y$. Then from Theorem 2.2, $K = \bigcup \{K_e \mid e \in E(S)\}$ where each K_e is a group.

Now we have $xx^{-1}\rho yy^{-1}\rho xy^{-1}$ and so xx^{-1}, yy^{-1}, $xy^{-1} \in K$. So $xx^{-1} = yy^{-1}$ and $yy^{-1}\mathscr{H}xy^{-1}$ as ρ is idempotent generating. Thus

$$x\varphi(x\varphi)^{-1} = (xx^{-1})\,\varphi = (yy^{-1})\,\varphi = (y\varphi)\,(y\varphi)^{-1}$$

and

$$(xy^{-1})\,\varphi((xy^{-1})\,\varphi)^{-1} = (xy^{-1}\,yx^{-1})\,\varphi = (xx^{-1})\,\varphi,$$

$$((xy^{-1})\,\varphi)^{-1}\,(xy^{-1})\,\varphi = (yx^{-1}\,xy^{-1})\,\varphi = (yy^{-1})\,\varphi = (xx^{-1})\,\varphi.$$

Thus $(xy^{-1})\,\varphi \in H_{(xx^{-1})\varphi} \cap K\varphi \subseteq H_{(xx^{-1})\varphi} \cap K$. So

$$(x\varphi)\,(x\varphi)^{-1}, \quad (y\varphi)\,(y\varphi)^{-1}, \quad (x\varphi)\,(y\varphi)^{-1} \in K_{(xx^{-1})\varphi} \quad \text{and} \quad (x\varphi)\,\rho(y\varphi)$$

by the definition (2.1) of ρ.

LEMMA 2.7. *If A is fully invariant in S and B is fully invariant in A then B is fully invariant in S.*

PROOF. For $\varphi \in \operatorname{end}(S)$ we have $A\varphi \subseteq A$ and $\varphi|_A \in \operatorname{end}(A)$ and so $B\varphi|_A \subseteq B$. For any $b \in B \subseteq A$, $b\varphi = b\varphi|_A \in B\varphi|_A \subseteq B$. Thus $B\varphi \subseteq B$.

LEMMA 2.8. *If ρ is an idempotent separating congruence on S and N is any fully invariant full inverse subsemigroup of* Ker ρ *then N is the kernel of a group kernel normal system on S.*

PROOF. As N is a full inverse subsemigroup of Ker ρ and Ker ρ is a semilattice of groups, it suffices to show that $s^{-1}Ns \subseteq N$ for all $s \in S$, by Theorem 2.2. Let $s \in S$ and $K = \operatorname{Ker} \rho$. Define $\varphi: K \to K$ by $k\varphi = s^{-1}ks$. Then φ is a homomorphism since

$$(s^{-1}ks)^{-1} = s^{-1}k^{-1}s = k^{-1}\varphi \quad \text{for all } k \in K,$$

and $(s^{-1}ks)(s^{-1}ls) = s^{-1}kls = (kl)\varphi$ for all $k, l \in K$, as K is a semilattice of groups. Now N is fully invariant and so $s^{-1}Ns = N\varphi \subseteq N$, completing the proof.

We now define verbal inverse subsemigroups, parallel to the case for groups (Neumann (1967), Chapter 1).

DEFINITION 2.9. Let W be a set of inverse semigroup identities on an alphabet X say and let S be an inverse semigroup. The *verbal inverse subsemigroup* $W(S)$ of S is the inverse subsemigroup generated by

$$w(S) = \{(ab^{-1})\alpha \mid a = b \text{ is an identity in } W \text{ and } \alpha \in \operatorname{Hom}(F_X, S)\},$$

where F_X denotes the free inverse semigroup on X.

LEMMA 2.10. *For any inverse semigroups S and S' and any homomorphism $\varphi: S \to S'$ we have $W(S)\varphi \subseteq W(S')$. In particular, $W(S)$ is a fully invariant inverse subsemigroup of S.*

PROOF. Take any $x \in w(S)$. Then $x = (ab^{-1})\alpha$ for some identity $a = b$ from W and for some $\alpha \in \operatorname{Hom}(F_X, S)$. But $\alpha\varphi \in \operatorname{Hom}(F_X, S')$ and so $x\varphi = (ab^{-1})\alpha\varphi \in w(S')$ giving that $w(S)\varphi \subseteq w(S')$. Therefore $W(S)\varphi \subseteq W(S')$ as required.

In particular, if $S = S'$ then $W(S)\varphi \subseteq W(S)$ and so $W(S)$ is a fully invariant inverse subsemigroup of S.

A set of identities is said to be *closed* if it contains all its consequences, that is, if it is the set of identities of some variety.

LEMMA 2.11. *If W is a closed set of identities and S is an inverse semigroup then $W(S) = w(S)$.*

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PROOF. We will show that $w(S)$ is an inverse subsemigroup of S. Let $(ab^{-1})\alpha$, $(cd^{-1})\beta \in w(S)$, where $a = a(x_1, \ldots, x_n)$, $b = b(x_1, \ldots, x_n)$, $c = c(x_1, \ldots, x_m)$, $d = d(x_1, \ldots, x_m)$ and $a = b$ and $c = d$ are laws in W, and $\alpha, \beta \in \mathrm{Hom}(F_X, S)$, where $X = \{x_1, x_2, \ldots\}$. Let $\varphi \in \mathrm{Hom}(F_X, S)$ be the unique extension of

$$f\colon x_i \to \begin{cases} x_i\,\alpha, & i = 1, \ldots, n, \\ x_{i-n}\,\beta, & i = n+1, \ldots, n+m, \\ x_1\,\alpha, & \text{otherwise.} \end{cases}$$

Also let $\bar{c} = c(x_{n+1}, \ldots, x_{n+m})$ and $\bar{d} = d(x_{n+1}, \ldots, x_{n+m})$. Then $\bar{c} = \bar{d}$ is an identity in W.

Also, $ab^{-1} = bb^{-1}$ and $\bar{c}\bar{c}^{-1} = \bar{d}\bar{c}^{-1}$ are in W and so $ab^{-1}\,\bar{c}\bar{c}^{-1} = bb^{-1}\,\bar{c}\bar{c}^{-1}$ is in W. But $bb^{-1}\,\bar{c}\bar{c}^{-1} = \bar{c}\bar{c}^{-1}\,bb^{-1}$ and $\bar{c}\bar{c}^{-1}\,bb^{-1} = \bar{d}\bar{c}^{-1}\,bb^{-1}$ are in W. Thus

$$ab^{-1}\,\bar{c}\bar{c}^{-1} = \bar{d}\bar{c}^{-1}\,bb^{-1} \text{ is in } W. \tag{1}$$

Hence

$$\begin{aligned}(ab^{-1}\alpha)\cdot(cd^{-1}\beta) &= (ab^{-1}\varphi)\cdot(\bar{c}\bar{d}^{-1}\varphi) \\ &= (ab^{-1}\cdot bb^{-1}\cdot \bar{c}\bar{c}^{-1}\cdot \bar{c}\bar{d}^{-1})\,\varphi \\ &= (ab^{-1}\,\bar{c}\bar{c}^{-1}\cdot bb^{-1}\,\bar{c}\bar{d}^{-1})\,\varphi \\ &= [(ab^{-1}\,\bar{c}\bar{c}^{-1})\cdot(\bar{d}\bar{c}^{-1}\,bb^{-1})^{-1}]\,\varphi,\end{aligned}$$

and this is an element of $w(S)$ by (1).

Finally, $(ab^{-1}\alpha)^{-1} = (ba^{-1})\alpha \in w(S)$ as $b = a$ is in W. Thus $w(S)$ is an inverse subsemigroup and so $w(S) = W(S)$.

3. Products of varieties

First we simplify Definition 1.1 for inverse semigroups.

LEMMA 3.1. *For any varieties $\mathscr{U}, \mathscr{V}$ of inverse semigroups, the product $\mathscr{U} \circ \mathscr{V}$ contains precisely those inverse semigroups S such that for some congruence θ on S, $S/\theta \in \mathscr{V}$ and $e\theta \in \mathscr{U}$, for each idempotent e of S.*

PROOF. By Lemma 7.34 of Clifford and Preston (1967), for any congruence θ on any inverse semigroup S, a θ-class is an inverse subsemigroup of S if and only if it contains an idempotent.

THEOREM 3.2. *If $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$ and $\mathscr{G} \leqslant \mathscr{G}\!\not{p}$ then $\mathscr{G} \circ \mathscr{V}$ is a variety.*

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PROOF. By Theorem 1.3, $\mathscr{G} \circ \mathscr{V}$ is a quasivariety and so it suffices to show that it is closed under the taking of homomorphic images. Let $S \in \mathscr{G} \circ \mathscr{V}$ and let θ be a congruence on S such that $S/\theta \in \mathscr{V}$ and $e\theta \in \mathscr{G}$, for all $e \in E(S)$. The kernel of θ, denoted by K, is the semilattice of the groups, $K = \bigcup \{e\theta : e \in E(S)\}$, and θ is idempotent-separating. Let ψ be a homomorphism of S and $T = S\psi$. Then $A = K\psi$ is an inverse subsemigroup of T. Since $\mathscr{G} \vee \mathscr{B}_{1,1}$ is the variety of similattices of groups in $\mathscr{G}$ we have $A \in \mathscr{G} \vee \mathscr{B}_{1,1}$ and so A is a semilattice of groups of $\mathscr{G}$. Let A_f be the subgroup of A with identity $f \in E(T)$, and let $\mathscr{A} = \{A_f : f \in E(T)\}$. Then for any $t \in T$, $a \in A$,

$$t^{-1}at = s^{-1}\psi k\psi s\psi \quad (\text{for some } s \in S,\ k \in K)$$

$$= (s^{-1}ks)\psi \in K\psi = A,$$

as K is the kernel of θ. Thus, by Theorem 2.2, $\mathscr{A}$ is the kernel normal system of a congruence ρ on T. Now define the map $\bar{\psi}: S/\theta \to T/\rho$ by $(s\theta)\bar{\psi} = (s\psi)\rho$ for all $s \in S$. We show that $\bar{\psi}$ is well-defined. If $s\theta = s_1\theta$ for $s, s_1 \in S$ then

$$ss^{-1}\theta = ss_1^{-1}\theta = s_1s_1^{-1}\theta.$$

So ss_1^{-1}, $s_1s_1^{-1}$, $ss^{-1} \in e\theta$, for some $e \in E(S)$. Hence $ss_1^{-1}\psi$, $s_1s_1^{-1}\psi$, $ss^{-1}\psi \in (e\theta)\psi$. But $(e\theta)\psi \subseteq A_f$ where $f = e\psi$; so $(s\psi, s_1\psi) \in \rho$ and $\bar{\psi}$ is well-defined. Further, $\bar{\psi}$ is a homomorphism as ρ is a congruence and ψ a homomorphism. Thus $T/\rho \in \mathscr{V}$ and $T \in \mathscr{G} \circ \mathscr{V}$.

COROLLARY 3.3. *Suppose $\mathscr{G}, \mathscr{H} \leqslant \mathscr{G}\!\!/\!\!\!p$ and $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$ but that $\mathscr{V}$ does not contain all groups. Then $\mathscr{G} \circ \mathscr{V} \leqslant \mathscr{H} \circ \mathscr{V}$ if and only if $\mathscr{G} \leqslant \mathscr{H}$. Hence $\mathscr{G} \circ \mathscr{V} = \mathscr{H} \circ \mathscr{V}$ if and only if $\mathscr{G} = \mathscr{H}$.*

PROOF. If $\mathscr{G} \leqslant \mathscr{H}$ then it is clear that $\mathscr{G} \circ \mathscr{V} \leqslant \mathscr{H} \circ \mathscr{V}$. Conversely, if $\mathscr{G} \circ \mathscr{V} \leqslant \mathscr{H} \circ \mathscr{V}$ we have

$$\mathscr{G} \circ (\mathscr{V} \cap \mathscr{G}\!\!/\!\!\!p) = (\mathscr{G} \circ \mathscr{V}) \cap \mathscr{G}\!\!/\!\!\!p \leqslant (\mathscr{H} \circ \mathscr{V}) \cap \mathscr{G}\!\!/\!\!\!p = \mathscr{H} \circ (\mathscr{V} \cap \mathscr{G}\!\!/\!\!\!p).$$

Also $\mathscr{V} \cap \mathscr{G}\!\!/\!\!\!p \neq \mathscr{G}\!\!/\!\!\!p$ and so by Neumann (1967), 21.21 we have $\mathscr{G} \leqslant \mathscr{H}$.

COROLLARY 3.4. *If $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$ and $\mathscr{V}$ does not contain all groups then $\mathscr{L}(\mathscr{V}, \mathscr{G}\!\!/\!\!\!p \circ \mathscr{V})$ is uncountable.*

PROOF. By the preceding corollary, to each variety of groups $\mathscr{G}$ there corresponds a distinct variety $\mathscr{G} \circ \mathscr{V}$. Thus, as there are uncountably many varieties of groups (Vaughn-Lee (1970)), $\mathscr{L}(\mathscr{V}, \mathscr{G}\!\!/\!\!\!p \circ \mathscr{V})$ is uncountable.

REMARK 3.5. Using the definition of the wreath product of two inverse semigroups given by Houghton (1976), Section 3, it can be shown that $\mathscr{G} \circ \mathscr{V}$ is generated by the wreath product $W = W(F_X(\mathscr{G}), F_Y(\mathscr{V}))$ for $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$.

We will now give a characterization of a basis of the identities of $\mathscr{G} \circ \mathscr{V}$. The identities of a variety of groups can be considered as words in the absolutely free group on a countable set of generators (Neumann (1967), Chapter 1). Any inverse semigroup identity $u(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)$ is equivalent to a pair of identities

$$u(x_1, \ldots, x_n)\, v^{-1}(x_1, \ldots, x_n) = v(x_1, \ldots, x_n)\, v^{-1}(x_1, \ldots, x_n)$$

and

$$u^{-1}(x_1, \ldots, x_n)\, u(x_1, \ldots, x_n) = v^{-1}(x_1, \ldots, x_n)\, v(x_1, \ldots, x_n).$$

Thus there is a basis for $\operatorname{Id}(\mathscr{V})$ consisting of identities of the form

$$u(x_1, \ldots, x_n) = i(x_1, \ldots, x_n)$$

where $i(x_1, \ldots, x_n)$ is an idempotent element in the free inverse semigroup F_X on the set $\{x_i\colon i = 1, 2, \ldots\}$. We will use the notation $\bar{x}$ to denote a string of elements $x_1, \ldots, x_k$ of X.

THEOREM 3.6. *Let $\mathscr{G} \leqslant \mathscr{G}\!\!/\!\!\!p$ and $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$. Then $S \in \mathscr{G} \circ \mathscr{V}$ if and only if S satisfies all the identities of the form*

$$u(v_1(\bar{x}_1), \ldots, v_n(\bar{x}_n)) = i_1(\bar{x}_1) \cdot \ldots \cdot i_n(\bar{x}_n),$$

where $u(x_1, \ldots, x_n) \in \operatorname{Id}(\mathscr{G})$ and $v_j(\bar{x}_j) = i_j(\bar{x}_j)$ is an identity from $\operatorname{Id}(\mathscr{V})$ with $i_j(\bar{x}_j)$ an idempotent of F_X.

PROOF. Let $S \in \mathscr{G} \circ \mathscr{V}$ and take an identity of the form given in the theorem. Then there is a congruence θ on S such that $S/\theta \in \mathscr{V}$ and $e\theta \in \mathscr{G}$ for each $e \in E(S)$. Now taking strings $\bar{s}_j = s_j^1, \ldots, s_j^{m(j)}$ of arbitrary elements of S we have

$$\begin{aligned} u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n))\, \theta &= u(v_1(\bar{s}_1)\, \theta, \ldots, v_n(\bar{s}_n)\, \theta) \\ &= u(v_1(s_1^1\, \theta, \ldots, s_1^{m(1)}\, \theta), \ldots, v_n(s_n^1\, \theta, \ldots, s_n^{m(n)}\, \theta)). \end{aligned}$$

But each $v_j(s_j^1\, \theta, \ldots, s_j^{m(j)}\, \theta) \in S/\theta \in \mathscr{V}$ and $\mathscr{V}$ satisfies $v_j(\bar{x}_j) = i_j(\bar{x}_j)$. Thus $v_j(\bar{s}_j) \in i(\bar{s}_j)\, \theta \subseteq \operatorname{Ker} \theta$ and so $u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n)) \in \operatorname{Ker} \theta$. Now let $e = i_1(\bar{s}_1) \cdot \ldots \cdot i_n(\bar{s}_n)$. Then

$$(u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n)), e) \in \theta$$

and so

$$u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n)) \in e\theta, \quad \text{a group.}$$

Now $\operatorname{Ker} \theta$ is a semilattice of groups and so

$$u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n)) = e \cdot u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n)) = u(e \cdot v_1(\bar{s}_1), \ldots, e \cdot v_n(\bar{s}_n))$$

and $e \cdot v_j(\bar{s}_j) \in e\theta$. But $u(x_1, \ldots, x_n) \in \mathrm{Id}(\mathscr{G})$ and so

$$\begin{aligned} u(v_1(\bar{s}_1), \ldots, v_n(\bar{s}_n)) &= u(e \cdot v_1(\bar{s}_1), \ldots, e \cdot v_n(\bar{s}_n)) \\ &= e \\ &= i_1(\bar{s}_1) \cdot \ldots \cdot i_n(\bar{s}_n). \end{aligned}$$

This is true for all elements of S and so S satisfies the identity.

For the converse let S be an inverse semigroup satisfying all the identities of the given form. Let $\mathrm{Id}(\mathscr{V}) = V$ and put $K = V(S)$ as defined in Section 2.9. Let $a(\bar{x}) = b(\bar{x})$ be in V. We now show $(a(\bar{x})\, b(\bar{x})^{-1})\,\alpha \in \operatorname{Ker}\mu$ by showing that it is μ-related, in S, to $(b(\bar{x})\, b(\bar{x})^{-1})\,\alpha$, an idempotent. Take any idempotent $e \in S$; then $e = (xx^{-1})\,\alpha$ for some $x \in X$.

Now let $u(y_1, y_2) = y_1 y_2 y_2^{-1} y_1^{-1}$ for $y \in X$. Then $u(y_1, y_2) \in \mathrm{Id}(\mathscr{G})$. Further, since V is closed it contains $a(\bar{x})\, b(\bar{x})^{-1} = b(\bar{x})\, b(\bar{x})^{-1}$ and the trivial identity $xx^{-1} = xx^{-1}$. Thus by the assumption we have that

$$u(a(\bar{x})\, b(\bar{x})^{-1}, xx^{-1}) = b(\bar{x})\, b(\bar{x})^{-1}\, xx^{-1}$$

is a law satisfied by S. Therefore, in the inverse semigroup S, we have

$$u(a(\bar{x})\, b(\bar{x})^{-1}, xx^{-1})\,\alpha = (b(\bar{x})\, b(\bar{x})^{-1}\, xx^{-1})\,\alpha,$$

that is,

$$(a(\bar{x})\, b(\bar{x})^{-1}\, xx^{-1} (xx^{-1})^{-1} (a(\bar{x})\, b(\bar{x})^{-1})^{-1})\,\alpha = (b(\bar{x})\, b(\bar{x})^{-1}\, xx^{-1})\,\alpha,$$

whence

$$((a(\bar{x})\, b(\bar{x})^{-1})\,\alpha)\, e((a(\bar{x})\, b(\bar{x})^{-1})\,\alpha)^{-1} = ((b(\bar{x})\, b(\bar{x})^{-1})\,\alpha)\, e((b(\bar{x})\, b(\bar{x})^{-1})\,\alpha)^{-1},$$

giving us that $(a(\bar{x})\, b(\bar{x})^{-1})\,\alpha$ and $(b(\bar{x})\, b(\bar{x})^{-1})\,\alpha$ are μ-related, as required. Hence $(a(\bar{x})\, b(\bar{x})^{-1})\,\alpha \in \operatorname{Ker}\mu$. But as this is an arbitrary one of a set of generators of $V(S)$ we have $V(S) \subseteq \operatorname{Ker}\mu$. Further we know that $V(S)$ is fully invariant and full in S and so by Lemma 2.8, $V(S)$ is the kernel of an idempotent-separating congruence ρ on S.

To complete the proof that $S \in \mathscr{G} \circ \mathscr{V}$ we now show that $S/\rho \in \mathscr{V}$ and that $f\rho \in \mathscr{G}$, for each idempotent $f \in S$.

Let $u(\bar{x}) (= u(x_1, \ldots, x_n))$ be in $\mathrm{Id}(\mathscr{G})$ and let $g_1, \ldots, g_n \in f\rho$ for some $f \in E(S)$. As $V = \mathrm{Id}(\mathscr{V})$ is closed, by Lemma 2.11 each g_i has the form $(a_i(\bar{x}_i)\, b_i(\bar{x}_i)^{-1})\,\alpha_i$ where $a_i(\bar{x}_i) = b_i(\bar{x}_i)$ is in V and $\alpha_i \in \operatorname{Hom}(F_X, S)$, for $i = 1, \ldots, n$. Without loss of generality we can assume that the distinct strings $\bar{x}_i$ contain no element in common.

Now $a_i(\bar{x}_i)\, b_i(\bar{x}_i)^{-1} = b_i(\bar{x}_i)\, b_i(\bar{x}_i)^{-1}$ is in V and so

$$u(a_1(\bar{x}_1)\, b_1(\bar{x}_1)^{-1}, \ldots, a_n(\bar{x}_n)\, b_n(\bar{x}_n)^{-1}) = b_1(\bar{x}_1)\, b_1(\bar{x}_1)^{-1} \cdot \ldots \cdot b_n(\bar{x}_n)\, b_n(\bar{x}_n)^{-1}$$

is satisfied by S. Now we can define $\alpha \in \operatorname{Hom}(F_X, S)$ as an extension of the map t determined as follows: for each y in the ith string $\bar{x}_i$ let $yt = y\alpha_i$. (This will be well-defined if we distinguish between each element of each string, as we can.)

Now

$$\begin{aligned} u(g_1, \ldots, g_n) &= u(a_1(\bar{x}_1)\, b_1(\bar{x}_1)^{-1}\, \alpha_1, \ldots, a_n(\bar{x}_n)\, b_n(x_n^{-1})\, \alpha_n) \\ &= (u(a_1(\bar{x}_1)\, b_1(\bar{x}_1)^{-1}, \ldots, a_n(\bar{x}_n)\, b_n(\bar{x}_n)^{-1}))\, \alpha \\ &= (b_1(\bar{x}_1)\, b_1(\bar{x}_1)^{-1} \cdot \ldots \cdot b_n(\bar{x}_n)\, b_n(\bar{x}_n)^{-1})\, \alpha \end{aligned}$$

as this is satisfied in S. But this last expression is an idempotent in S and ρ is idempotent separating and so, as each of $g_1, \ldots, g_n$ is in $f\rho$, we have

$$u(g_1, \ldots, g_n) = f.$$

Thus $f\rho \in \mathscr{G}$ for all $f \in E(S)$.

Finally, let $v(\bar{x}) = i(\bar{x})$ be any identity from $\operatorname{Id}(\mathscr{V})$, where $\bar{x} = (x_1, \ldots, x_n)$ and $i(\bar{x})$ is idempotent in F_X. Take any elements $s_1\rho, \ldots, s_n\rho \in S/\rho$ and let $\bar{s}\rho = (s_1\rho, \ldots, s_n\rho)$ and $\bar{s} = (s_1, \ldots, s_n)$. Now xx^{-1} and $x^{-1}x$ are identities of $\mathscr{G}$; so

$$v(\bar{s}\rho)\, v(\bar{s}\rho)^{-1} = i(\bar{s}\rho) = v(\bar{s}\rho)^{-1}\, v(\bar{s}\rho).$$

Therefore the identities $v(\bar{x})\, v(\bar{x})^{-1} = i(\bar{x})$ and $v(\bar{x})^{-1}\, v(\bar{x}) = i(\bar{x})$ hold in S/ρ. But $v(\bar{s})\, i(\bar{s}) \in V(S) = \operatorname{Ker}\rho$ and so $v(\bar{s}\rho) = v(\bar{s}\rho)\, i(\bar{s}\rho) = i(\bar{s}\rho)$. Thus S/ρ satisfies $v(\bar{x}) = i(\bar{x})$; so $S/\rho \in \mathscr{V}$ and $S \in \mathscr{G} \circ \mathscr{V}$.

Remark 3.7. It follows from this theorem that $\mathscr{G} \circ \mathscr{V}$ is finitely based if both $\mathscr{G}$ and $\mathscr{V}$ are.

4. Associativity

In this section we will show that for $\mathscr{G}, \mathscr{H} \leqslant \mathscr{G}\!\not{p}$ and $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$,

$$(\mathscr{G} \circ \mathscr{H}) \circ \mathscr{V} = \mathscr{G} \circ (\mathscr{H} \circ \mathscr{V}).$$

Lemma 4.1. *For $\mathscr{G}, \mathscr{H} \leqslant \mathscr{G}\!\not{p}$ and $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$ we have $\mathscr{G} \circ (\mathscr{H} \circ \mathscr{V}) \leqslant (\mathscr{G} \circ \mathscr{H}) \circ \mathscr{V}$.*

Proof. Let $S \in \mathscr{G} \circ (\mathscr{H} \circ \mathscr{V})$. Then there is a congruence ρ on S and a congruence σ on S/ρ such that

$$S/\rho \in \mathscr{H} \circ \mathscr{V},\ e\rho \in \mathscr{G}, \quad \text{for all } e \in E(S),$$

$$(S/\rho)/\sigma \in \mathscr{V},\ (e\rho)\sigma \in \mathscr{H} \quad \text{for all } e \in E(S).$$

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Now $(\rho^\natural \sigma^\natural)(\rho^\natural \sigma^\natural)^{-1}$ is a congruence on S and $S/(\rho^\natural \sigma^\natural)(\rho^\natural \sigma^\natural)^{-1} \cong (S/\rho)/\sigma \in \mathscr{V}$. Now let $K_e = e(\rho^\natural \sigma^\natural)(\rho^\natural \sigma^\natural)^{-1}$. Then $e(\rho \cap (K_e \times K_e)) = e\rho \in \mathscr{G}$ as $e\rho \subseteq K_e$. Further, if $x \in K_e$ then $x\rho\sigma = e\rho\sigma$ and so $x\rho^\natural \in e\rho(\sigma^\natural \sigma^{\natural-1})$. Thus $K_e/(\rho \cap (K_e \times K_e)) = (e\rho)\sigma \in \mathscr{H}$. Thus $S \in (\mathscr{G} \circ \mathscr{H}) \circ \mathscr{V}$.

LEMMA 4.2. *Let $S \in \mathscr{I}\mathscr{S}$ and ρ a congruence on S such that S/ρ satisfies a set of identities W. Then $W(S) \leqslant \operatorname{Ker} \rho$.*

PROOF. Let $a = b$ be any identity from W and take any $\alpha \in \operatorname{Hom}(F_X, S)$. Then $\alpha\rho^\natural \in \operatorname{Hom}(F_X, S/\rho)$ and $a(\alpha\rho^\natural) = b(\alpha\rho^\natural)$. Thus $(ab^{-1})\alpha\rho^\natural = (bb^{-1})\alpha\rho^\natural$ and so $(ab^{-1})\alpha \in \operatorname{Ker}\rho$. Hence $W(S) \subseteq \operatorname{Ker}\rho$.

LEMMA 4.3. *Let $\mathscr{G} \leqslant \mathscr{G}\!\!/\!\!p$, $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$ and $V = \operatorname{Id}(\mathscr{V})$. If $S \in \mathscr{G} \circ \mathscr{V}$ then $V(S)$ is the kernel of an idempotent separating congruence ρ on S for which $S/\rho \in \mathscr{V}$ and $e\rho \in \mathscr{G}$.*

PROOF. Since $S \in \mathscr{G} \circ \mathscr{V}$ there is a congruence θ on S such that $S/\theta \in \mathscr{V}$ and $e\theta \in \mathscr{G}$ for each idempotent $e \in S$. By Lemma 4.2, $V(S) \leqslant \operatorname{Ker}\theta$. Thus by Lemma 2.8, $V(S)$ gives a group kernel normal system with congruence ρ, say. Further ρ is idempotent separating and $e\rho \in \mathscr{G}$ as $V(S) \leqslant \operatorname{Ker}\theta$ implies $e\rho \subseteq e\theta$ for each $e \in E(S)$. To show $S/\rho \in \mathscr{V}$ we let $u(\bar{x}) = v(\bar{x})$ be any identity in V. Then S/θ satisfies this identity so for any $\alpha \in \operatorname{Hom}(F_X, S)$ we have $(u(\bar{x})\alpha)\theta^\natural = (v(\bar{x})\alpha)\theta^\natural$. As θ is idempotent separating we have

$$(u(\bar{x})\,u(\bar{x})^{-1})\,\alpha = (v(\bar{x})\,v(\bar{x})^{-1})\,\alpha$$

and

$$(u(\bar{x})^{-1}\,u(\bar{x}))\,\alpha = (v(\bar{x})^{-1}\,v(\bar{x}))\,\alpha.$$

Also

$$(u(\bar{x})\,u(\bar{x})^{-1}\,\alpha)\,\theta^\natural = (u(\bar{x})\,v(\bar{x})^{-1}\,\alpha)\,\theta^\natural,$$

and as $\theta \subseteq \mathscr{H}_S$ we have

$$(u(\bar{x})\,u(\bar{x})^{-1})\,\alpha\mathscr{H}_S(u(\bar{x})\,v(\bar{x})^{-1})\,\alpha.$$

Thus

$$(u(\bar{x})\,v(\bar{x})^{-1})\,\alpha \in V(S) \cap H_e$$

where $e = (u(\bar{x})\,u(\bar{x})^{-1})\,\alpha$, and so $((u(\bar{x})\,v(\bar{x})^{-1})\,\alpha)\,\rho((u(\bar{x})\,u(\bar{x})^{-1})\,\alpha)$. Thus

$$\begin{aligned} v(\bar{x})\,\alpha &= (v(\bar{x})\,v(\bar{x})^{-1}\,v(\bar{x}))\,\alpha = (v(\bar{x})\,v(\bar{x})^{-1})\,\alpha\cdot(v(\bar{x}))\,\alpha \\ &= ((u(\bar{x})\,u(\bar{x})^{-1})\,\alpha(v(\bar{x}))\,\alpha)\,\rho((u(\bar{x})\,v(\bar{x})^{-1})\,\alpha(v(\bar{x}))\,\alpha) \\ &= (u(\bar{x}))\,\alpha(v(\bar{x})^{-1}\,v(\bar{x}))\,\alpha = (u(\bar{x}))\,\alpha\cdot(u(\bar{x})^{-1}\,u(\bar{x}))\,\alpha \\ &= u(\bar{x})\,\alpha, \end{aligned}$$

and so

$$v(\bar{x})\,\alpha\rho^{\natural} = u(\bar{x})\,\alpha\rho^{\natural}.$$

But any $\beta \in \operatorname{Hom}(F_X, S/\rho)$ is of the form $\alpha\rho^{\natural}$ for some $\alpha \in \operatorname{Hom}(F_X, S)$ and so S/ρ satisfies $u(\bar{x}) = v(\bar{x})$.

We have that $S/\rho \in \mathscr{V}$ as required.

THEOREM 4.4. *Let $\mathscr{G}, \mathscr{H} \leqslant \mathscr{G}\!\!\not{p}$ and $\mathscr{V} \leqslant \mathscr{I}\mathscr{S}$. Then $(\mathscr{G} \circ \mathscr{H}) \circ \mathscr{V} = \mathscr{G} \circ (\mathscr{H} \circ \mathscr{V})$.*

PROOF. By Lemma 4.1 it suffices to show that $\mathscr{G} \circ (\mathscr{H} \circ \mathscr{V}) \geqslant (\mathscr{G} \circ \mathscr{H}) \circ \mathscr{V}$. Let $S \in (\mathscr{G} \circ \mathscr{H}) \circ \mathscr{V}$. Then by Lemma 4.3, $V(S)$ determines a congruence ρ on S with $S/\rho \in \mathscr{V}$ and $e\rho \in \mathscr{G} \circ \mathscr{H}$, for all $e \in E(S)$. Now if $H = \operatorname{Id}(\mathscr{H})$ then by Lemmas 2.7 and 2.10, we have that $H(V(S))$ is fully invariant in S. Now taking the maximal idempotent separating congruence μ on S, we have that $H(V(S))) \leqslant V(S) \leqslant \operatorname{Ker}\mu$. Further, as both V and H are closed sets of laws we have that $H(V(S))$ is full in S. Thus by Lemma 2.8 there is a congruence σ on S with $H(V(S))$ as its kernel.

Now since $e\rho \in \mathscr{G} \circ \mathscr{H}$, for all $e \in E(S)$, we have by Lemma 4.3 that $H(e\rho)$ is the kernel of an idempotent separating congruence on $e\rho$. This is clearly just $\sigma | e\rho$. Now $e\sigma = e(\sigma | e\rho)$ and so $e\sigma \in \mathscr{G}$ for all $e \in E(S)$.

Finally we show that $S/\sigma \in \mathscr{H} \circ \mathscr{V}$. Define a mapping $\tau^{\natural}$ from S/σ onto S/ρ by $(s\sigma)\tau^{\natural} = s\rho$. This is clearly a homomorphism and is well-defined since if $s_1\sigma = s_2\sigma$ then $s_1 s_1^{-1}, s_2 s_2^{-1}, s_1 s_2^{-1} \in G$, a subgroup of $H(V(S))$; but $H(V(S)) \leqslant V(S)$, a semilattice of groups and so $s_1\rho = s_2\rho$. Now

$$\begin{aligned}
\operatorname{Ker}\tau &= \{s\sigma\colon s\sigma\tau^{\natural} = e\sigma\tau^{\natural} \text{ for some } e \in E(S)\}\\
&= \{s\sigma\colon s\rho = e\rho \text{ for some } e \in E(S)\}\\
&= \{s\sigma\colon s \in V(S)\}\\
&= V(S)/(\sigma | V(S)).
\end{aligned}$$

So for each $e \in E(S)$, $(e\sigma)\tau \in V(S)/(\sigma | V(S))$ and this is a semilattice of groups from $\mathscr{H}$ as $\operatorname{Ker}(\sigma | V(S)) = \operatorname{Ker}\sigma = H(V(S))$. Thus $S/\sigma \in \mathscr{H} \circ \mathscr{V}$ and $S \in \mathscr{G} \circ (\mathscr{H} \circ \mathscr{V})$.

This we can express diagrammatically as follows.

$$\begin{tikzcd}
e\sigma \in G & H(V(S)) \arrow[d] \arrow[dr] & & & \\
e\rho \in G \circ \mathscr{H} & V(S) \arrow[r] \arrow[d, "\sigma | V(S)"] & S \arrow[rr] \arrow[dr, "\sigma"] & & S/\rho \in V \\
(e\rho)(\sigma | V(S)) \in H & V(S)/(\sigma | V(S)) \arrow[rr] & & S/\sigma \in H \circ V \arrow[ur, "\tau^{\natural}"'] &
\end{tikzcd}$$

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Department of Mathematics
Monash University
Clayton, Victoria 3168
Australia

Author's current address:
Ridley College
University of Melbourne
Parkville, Victoria 3052
Australia