# STABILITY ON THE BASIS OF ORTHOGONAL TRAJECTORIES

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1. Introduction. We consider a system of differential equations of second order given by

$$(1) \qquad \begin{cases} x' = P(x, y) \\ y' = Q(x, y) \end{cases}$$

($' = d/dt$) where $P$ and $Q$ have continuous first partial derivatives with respect to $x$ and $y$ in some open and simply connected set $R$ containing $O = (0, 0)$ which we assume to be the only singular point in $R$. In fact, let $R$ be the whole plane; for if not then the following discussion can be modified.

The differential equation for the orbits of our system is

$$(1)' \qquad dy/dx = Q(x, y)/P(x, y) .$$

If it is possible to solve (1)' then by examining the direction field determined by (1) we may determine the stability properties of the null solution to (1). But to solve (1)' is in general impossible unless it is exact. Even if (1)' is not exact we know that there exists an integrating factor; however, to find this factor is to solve a partial differential equation which is often more of a problem than the original one.

This paper was motivated by the fact that it is sometimes easy to find an integrating factor for the orthogonal system even though one is not readily found for the given equation. This was first noticed in the Liénard equation $x'' + f(x) x' + g(x) = 0$

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under the substitution $x' = y - \int_0^x f(t)\,dt$ from which we obtain $y' = - g(x)$. An integrating factor is then easily found for the orthogonal trajectories. For the van der Pol equation this integrating factor is $\frac{1}{x^2}$. The question then arises as to what stability properties of (1) can be inferred from

$$(2) \qquad \begin{aligned} x' &= -Q(x,y) \\ y' &= P(x,y), \end{aligned}$$

or from

$$(2)' \qquad dy/dx = - P(x,y)/Q(x,y),$$

which give the orthogonal trajectories to (1)'.

For linear systems with constant coefficients in canonical form the question is quickly answered, since the null solution to (1) is a node if and only if (2) is a center and (1) is a saddle if and only if (2) is a saddle. A focus gives rise to even a simpler and more general treatment. (For this terminology see Lefschetz [1; p. 184].) If (1) is nonlinear the treatment is not nearly so straightforward, but the results are many and varied.

We state the following definitions for reference.

A spherical neighborhood of a point $P$ with radius $T$ will be denoted by $S(P, T)$.

The null solution to (1) is Liapunov stable if for every $e > 0$ there exists $d > 0$ such that if $X = (x, y)$ is in $S(O, e)$ then the solution $f(X, t)$ through $X$ remains in $S(O, d)$ for all $t > 0$. If also $f(X, t) \to O$ as $t \to \infty$ then $O$ is asymptotically stable.

A set $K$ is positively invariant if every solution which enters it at $t = 0$ remains in it for all $t > 0$.

We shall denote the distance between two sets $H$ and $K$ by $d(H, K)$.

A curve $C$ is a transversal if its tangent never coincides with the direction field of a given differential equation on $C$.

2. Orthogonal analysis. Following the types of configurations of orbits for linear systems we may obtain a number of theorems of the following nature about nonlinear systems. These results are far from exhaustive, but the methods here can be used on exceptional cases.

THEOREM 1. Let $G$ be a nonzero periodic solution to (2) with inward normal $(X, Y)$. Then the inner product $M = [(X, Y), (P, Q)]$ does not vanish on $G$.

Proof. Let $(x, y)$ be a point on $G$ and $M(x, y) > 0$. Now $M$ is continuous on $G$ so there exists an open interval $I$ on $G$ containing $(x, y)$ on which $M > 0$. Extend $I$ as large as possible. Then $I$ is either $G$ or there exists a point $(a, b)$ on $G$ such that $M(a, b) = 0$ and $M > 0$ on $G$ between $(a, b)$ and $(x, y)$. But $M = 0$ if and only if the normal to $G$ is orthogonal to $(P(a, b), Q(a, b))$. This can not be since $G$ itself is orthogonal to $(P(a, b), Q(a, b))$. Hence such a point $(a, b)$ with $M(a, b) = 0$ does not exist and so $M > 0$ for all $(x, y)$ on $G$. The proof is symmetric for $M < 0$.

We shall retain these definitions of $M$ and $(X, Y)$ throughout the remainder of the discussion.

THEOREM 2. Let $G$ be a nonzero periodic solution of (2) and let $(X, Y)$ be an inward normal to $G$ at each point $(x, y)$. Then $G$ together with its interior is a positively invariant set for (1) if and only if $M = [(X, Y), (P, Q)] > 0$ for all points on $G$.

Proof. By Theorem 1 $M$ is positive or negative at all points of $G$. If $M > 0$ then the normal to $G$ points in the same direction as the vector $(P, Q)$. Hence every solution of (1) which intersects $G$ does so from the outside of $G$ to the inside and so $G$ together with its interior form a positively invariant set.

Suppose $G$ and its interior form a positively invariant set. Then every trajectory of (1) which intersects $G$ does so from the outside to the inside of $G$. But then $M > 0$ since the normal to $G$ and $(P, Q)$ point in the same direction.

Assume there exists G as above with $M > 0$. We may apply the Poincaré-Bendixson (P-B) theorem and assert that either all solutions to (1) inside G tend to O or there exists a periodic solution to (1) inside G.

THEOREM 3. There do not exist periodic solutions to (1) arbitrarily close to a periodic solution G to (2).

Proof. For each (x, y) on G there exists a small neighborhood N(x, y) such that the vector field [P, Q] in N(x, y) differs from (X(x, y), Y(x, y)) by less than $\pi/4$ [3; p. 39]. The collection of all such N(x, y) is an open cover of G so there exists a finite subcover $\{N_i(x,y)\}_{i=1}^{i=k}$. Let $R = \bigcup_{i=1}^{i=k} N_i(x,y)$ and the complement of R, $R^c = W$. W has one maximal compact connected component W' interior to G. Let the distance between G and W' be $k > 0$. Since $M > 0$ between G and all points inside G of distance k from G, there can exist no periodic solution of (1) in this region.

COROLLARY. If every solution to (2) is periodic then there exists no periodic solution to (1).

Example. Given

$$x' = x^3 + xy^2$$

$$y' = y^3 + yx^2$$

we obtain

$$dy/dx = (y^3 + yx^2)/(x^3 + xy^2)$$

which is not exact and an integrating factor is not readily found. We therefore consider the orthogonal system

$$dy/dx = -(x^3 + xy^2)/(y^3 + yx^2)$$

whose solution is

$$V(x,y) = (x^2 + y^2)^2 = c .$$

Hence all solutions to the orthogonal system are periodic and so no solution to the given system is periodic. An inward normal to $V = c$ is $(-x^3 - xy^2, -y^3 - yx^2)$ whose inner product with $(x', y')$ is negative. By the Poincaré-Bendixson theorem we may conclude that O is unstable.

The following theorem follows immediately using the Poincaré-Bendixson theorem.

THEOREM 4. Let G and G' be two distinct periodic solutions to (2). If M on G differs in sign from M on G', then there exists a periodic solution to (1) between G and G'.

If there exists a nonzero periodic solution to (2) which is inside all other nonzero periodic solutions to (2) we shall call it the last periodic solution.

THEOREM 5. If (2) has no last periodic solution and $M > 0$ on each then the O solution of (1) is Liapunov stable.

Proof. Since there is no last periodic solution to (2) we can find a sequence $G_i$ of periodic solutions such that $d(G_i, O) \to 0$ as $i \to \infty$. So for any $e > 0$, there exists i such that $G_i$ is in $S(O, e)$. Let $d(G_i, O) = d$. Then the solution $f(X, t)$ through X remains in $S(O, e)$ so long as X is in $S(O, d)$.

Example. Given

$$x' = -x - y(x^2+y^2) \sin [\pi/(x^2+y^2)^{1/2}]$$

$$y' = -y + x(x^2+y^2) \sin [\pi/(x^2+y^2)^{1/2}]$$

we obtain

$$dy/dx = \frac{[-y + xr^2 \sin(\pi/r)]}{[-x - yr^2 \sin(\pi/r)]}$$

where $r^2 = x^2 + y^2$. This equation is not exact, but the orthogonal system can be written as

$$dr/d\theta = r^3 \sin(\pi/r)$$

in polar coordinates which has limit cycles at $r = 1/n$ for every integer $n$. An inward normal to the limit cycles is given by $(-x, -y)$ whose inner product with $(x', y')$ is positive on the limit cycles. Hence O is Liapunov stable by Theorem 5.

THEOREM 6. If O is a saddle point for (2), then O is a saddle point for (1).

Proof. Since the O solution of (2) is a saddle point, by definition there are four solutions whose orbits intersect at O. Two of these solutions enter O as $t$ increases and the other two enter O as $-t$ increases. If one of the solutions enters O then the two adjacent ones leave O. All other solutions are asymptotic to these four and none of the others enter or leave O. Label these asymptotes $R$, $R'$, $R''$, and $R'''$ in a counterclockwise direction.

LEMMA 1. There exists an orthogonal trajectory (i.e., trajectory of (1)) which intersects O between $R$ and $R'$. Such trajectories also exist between $R'$ and $R''$, $R''$ and $R'''$, and between $R'''$ and $R$. These trajectories ($K$, $K'$, $K''$, and $K'''$) are the asymptotes of the saddle point of (1).

Proof. Consider a point $X$ on $R$ with $X \neq O$ but $X$ as close to O as we please. Let the solution to (1) through $X$ be $f(X,t)$. Now as $t$ increases $f(X,t)$ leaves $R$ and enters either the region between $R$ and $R'$ or the region between $R$ and $R'''$. For the argument which follows we may assume it is the region between $R$ and $R'''$. (The argument for the other is symmetric.) $f(X,t)$ crosses orbits of (2) arbitrarily close to $R$, but $f(X,t)$ does not cross $R'''$; for if $f(X,t)$ crosses $R'''$, then there are orbits of (2) arbitrarily close to $R$ and $R'''$ which $f(X,t)$ crosses twice. This can not happen by

LEMMA 2. $f(X,t)$ does not cross any orbit asymptotic to $R$ and $R'''$ twice.

Proof. Let such an orbit be L and let $Y_1$ and $Y_2$ be the first two times which $f(X,t)$ crosses L after leaving R. Then there exist $t_1$ and $t_2$ such that $f(X, t_1) = Y_1$ and $f(X, t_2) = Y_2$. A simple closed curve C is formed by that part of $f(X,t)$ for $t_1 \leq t \leq t_2$ together with the compact part of L between $Y_1$ and $Y_2$. The region bounded by C is dense with trajectories of (1) and each such trajectory must cross L twice since there are no singular points inside C and hence no periodic solutions of (1). Thus there exists a trajectory of (1) which is tangent to L contradicting the fact that solutions to (2) are orthogonal to solutions of (1). This proves Lemma 2.

Returning now to the proof of Lemma 1, we see that $f(X,t)$ must remain between R and R''' since a second application of Lemma 2 shows that $f(X,t)$ can not approach O and it can not again cross R. Thus, by the Poincaré-Bendixson theorem, $f(X,t)$ eventually leaves any bounded neighborhood of the origin provided that R and R''' leave any such neighborhood.

The same arguments show that $f(X,-t)$, for $t > 0$, remains between R and R' and eventually leaves any bounded neighborhood of the origin.

Now consider any point Y on R''' with $Y \neq O$ but Y as close to O as we please. Let the solution through Y be $f(Y,t)$. Suppose that as t increases $f(Y,t)$ enters the region between R and R'''. Then, by the same arguments as used before in this lemma $f(Y,t)$ remains between R and R''' and eventually leaves any neighborhood of O. Note in addition that $f(Y,t)$ does not cross $f(X,t)$.

Consider two sequences of points $\{X_n\}$ and $\{Y_n\}$ on R and R''' respectively with $X_i \neq O$ and $Y_i \neq O$ for any i. Let $\lim_{n\to\infty} |X_n| = \lim_{n\to\infty} |Y_n| = 0$. Since $f(X_n, t)$ and $f(Y_n, t)$ remain between R and R''', and no $f(X_n, t)$ intersects any $f(Y_n, t)$, there must exist a separatrix K between the $f(X_n, t)$



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and $f(Y_n, t)$, and K is a solution to (1). Now K must enter O as t decreases and leave any neighborhood of O as t increases, provided that R and R''' leave any neighborhood of O.

The same arguments give the three other separatrices asserted in Lemma 1 completing the proof of this lemma and, in fact, the theorem.

If O is a simple singularity [2; p. 87] then the theorem is immediate since the Poincaré indices of O with respect to both (1) and (2) are equal; but the index of (2) is -1 since it is a saddle point and so O with respect to (1) is a saddle point. However, the weight of this theorem goes far beyond that of linearization theorems since the latter are local statements whereas once we have the $R^i$ and $K^i$ we know the extent to which the saddle point goes. That is, the existence of the $K^i$ is assured and we can continue these curves orthogonal to the orbits of (2) which are asymptotic to the $R^i$. We know that no solution of (1) can cross the $K^i$ so we can employ the P-B theorem to assert that all solutions depart until the $K^i$ hit a limit cycle. We shall elaborate on this in an example.

Given

$$x' = y + x^4$$

$$y' = x + y^6$$

we obtain

$$dy/dx = (x + y^6)/(y + x^4)$$

whose orthogonal trajectories are given by $dy/dx = -(y+x^4)/(x+y^6)$. The solution to the last equation is $V = xy + \frac{x^5}{5} + \frac{y^7}{7} = c$ which gives rise to a saddle point as does the given system. The asymptotes to the orthogonal system become unbounded, so the same is true for the given system; otherwise, by the P-B theorem, the asymptotes to the given system tend to a periodic solution. Let G be a periodic solution. Then its Poincaré index is + 1 with respect to both the given system and the orthogonal system. But on $G$ the Poincaré index of the orthogonal system is still $-1$ since it is a saddle point. Hence $G$ does not exist.

THEOREM 7. Let $Pdx + Qdy = 0$ be exact and let the solution to $Pdx + Qdy = 0$ be written as $V(x, y) = c$. If $V$ is a positive definite or negative definite function in some $S(O, A)$ then $V$ is a Liapunov function for (1).

Proof. By definition $\partial V/\partial x = P$, $\partial V/\partial y = Q$, so $\dot{V} = P^2 + Q^2 > 0$ except at $O$.

COROLLARY. If $V$ is positive definite, then $O$ is unstable. If $V$ is negative definite, then $O$ is asymptotically stable.

Notice that Theorem 7 and its corollary represent a counterpart to the work of G.K. Pojarickii ([4] or [5; pp. 24-25]).

Suppose $Pdx + Qdy = 0$ is not exact, but $z(x, y)$ is an integrating factor. Let the solution to $z\,Pdx + z\,Qdy = 0$ be $V(x, y) = c$. Then $\dot{V} = z(P^2 + Q^2)$ has the same sign as does $z(x, y)$ at any point $(x, y)$. If $z$ is always positive or always negative except at $(0, 0)$ then Theorem 7 and its corollary apply. If $V$ is not definite then $V = 0$ has at least one (non-trivial) real solution curve through the origin. Even a stronger statement could be made using the Weierstrass preparation theorem provided only that $V$ is analytic. Let these solution curves divide some disc, of radius $A$ centered at $O$, into $n$ distinct simply connected regions labelled consecutively as $G_1, \ldots, G_n$. The following theorem is a direct consequence of Cetaev's theorem [6].

THEOREM 8. If $z(x, y)$ has the same sign as $V$ at each point of some $G_i$, then $O$ is unstable.

We shall say that the graph of a function $V(x, y) = c$ is a spiral locally if there exists $A > 0$ such that for any $A' > 0$, but $A' < A$, some straight line $L$ intersects $V(x, y) = c$ twice in $S(O, A')$ at points $Y_1$ and $Y_2$ in such a manner that the part of $L$ between $Y_1$ and $Y_2$ together with the part of

$V(x,y) = c$ between $Y_1$ and $Y_2$ form a simple closed curve containing O in its interior.

THEOREM 9. If $Pdx + Qdy = 0$ can be integrated as $V(x,y) = c$ which is a spiral locally; if for some c there exists a transversal T with respect to (2) which has a unique normal at each point from O to $\partial S(O,A)$ so that the normal to T pointing towards the interior of the spiral, say $(X,Y)$, satisfies $M = [(P,Q),(X,Y)] > 0$ on T except at O; and if the inward normal, say $(X',Y')$, to the spiral satisfies $M' = [(P,Q),(X',Y')] > 0$ on the spiral except at O, then O is asymptotically stable. If $M, M' < 0$ then O is unstable.

Proof. Let $e > 0$ be given with $e < A$. Consider that part of $V = c$ inside $S(O,e)$. Let T intersect $V = c$ at the first two consecutive points inside $S(O,e)$, say B and B', which make a simple closed curve K containing O formed by BB' together with the spiral inward from B to B'. Since T is a transversal the existence of B and B' is assured. Let X be inside K. The solution $f(X,t)$ does not cross K since M and M' are positive. Let $d(K,O) = d$. If X is in $S(O,d)$, then $f(X,t)$ remains in $S(O,e)$ for all $t > 0$. Since M and M' are positive there can be no periodic solution. Hence $f(X,t) \to O$ as $t \to \infty$ by the P-B theorem.

Assume that M and M' are negative. Parametrize the spiral S by $x = x(s)$, $y = y(s)$ so that $(x,y) \to O$ as $s \to \infty$. We may then pick a point X on S and T arbitrarily close to O so that the first intersection of S outward (s decreasing) with T after X is D, and the part of S between X and D together with XD on T form a simple closed curve K'. Since $M, M' < 0$, $f(X,t)$ can not cross K' inward. By the P-B theorem either $f(X,t)$ is cyclic, approaches a cyclic characteristic or leaves $S(O,A)$. M and $M' < 0$ prevent the existence of periodic solutions.

Example. Given

$$x' = -x + y(x^2 + y^2)^{1/2}$$

$$y' = -y - x(x^2 + y^2)^{1/2},$$

we consider the orthogonal trajectories given by

$$(-x + y(x^2+y^2)^{1/2})\,dx + (-y - x(x^2+y^2)^{1/2})\,dy = 0 .$$

This equation is not exact, but $(x^2 + y^2)^{-3/2}$ is an integrating factor. The solution is $V = (x^2 + y^2)^{-1/2} - \arctan(y/x) = c$ whose polar form is $r = 1/(\theta + c)$ which is locally a spiral. Notice that the $x$ axis is a transversal with normal $(0,1)$ pointing into the spiral and that $[(0,1),(x',y')] > 0$. The inward normal to the spiral is $(x',y')$ so $[(x',y'),(x',y')] = x'^2 + y'^2 > 0$ except at $O$. Hence $O$ is asymptotically stable by Theorem 9.

3. The Liénard equation. We shall conclude with a non-trivial example. Consider

$$x'' + f(x)x' + G(x) = 0 .$$

Let $F(x) = \int_0^x f(t)dt$ and $\frac{d}{dx} G(x) = g(x)$. Under the substitution $x' = y - F(x)$ we obtain the equivalent system

$$x' = y - F(x) ,$$

$$y' = -G(x) .$$

The equation for the orbits is given by $\frac{dy}{dx} = \frac{-G(x)}{y - F(x)}$, so the equation for the orthogonal trajectories is $\frac{dy}{dx} = \frac{y - F(x)}{G(x)}$ or $[F(x) - y]dx + G(x)dy = 0$. Let $z(x,y)$ be an integrating factor so that $z[F(x) - y]dx + z\,G(x)dy = 0$ is exact. Then $\frac{\partial z}{\partial x} G(x) + \frac{\partial z}{\partial y}[y - F(x)] = -z[1 + g(x)]$. Using the method of Lagrange we obtain $\frac{dx}{G(x)} = \frac{dy}{y-F(x)} = \frac{dz}{-z[1+g(x)]}$. Then $\frac{-[1+g(x)]}{G(x)} dx = \frac{dz}{z}$ has a solution $z = \exp\{-\int_a^x \frac{1+g(t)}{G(t)} dt\}$.

Integrating for the orthogonal trajectories we obtain



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$$V(x, y) = \int_{x_0}^{x} z[F(x) - y_0]dx + \int_{y_0}^{y} zG(x)dy .$$

We leave the example here since an ad hoc assumption on F and G would destroy the generality. For specific F and G we may apply theorems already proved. In addition, the form of the solution may inspire further generalizations of our results.


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