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Absolute Nörlund summability of Fourier series of functions of bounded variation

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The authors prove two theorems. The first theorem generalizes theorems due to T. Singh and O.P. Varshney, concerning absolute Nörlund summability of Fourier series of functions of bounded variation. The second theorem generalizes theorems of L.S. Bosanquet and H.P. Dikshit.

1. *Introduction and theorems*

1.1. Let $\sum a_n$ be an infinite series and (s_n) be the sequence of its partial sums. Let (p_n) be a sequence of positive numbers and let $P_n = p_0 + p_1 + \ldots + p_n$ for $n \geqq 0$ and $p_{-1} = P_{-1} = 0$. We suppose that $P_n \to \infty$ as $n \to \infty$. The sequence (t_n) defined by

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \quad (n = 1, 2, \ldots) \tag{1}$$

is called the Nörlund means of the series $\sum a_n$. If the sequence (t_n) is of bounded variation, that is, $\sum |t_n - t_{n-1}| < \infty$, then the series $\sum a_n$ is said to be absolutely Nörlund summable or $|N, p_n|$ summable.

Let f be an integrable function, periodic with period 2π, and its

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Fourier series be

$$(2) \qquad f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x) .$$

If the series (2) is $|N, p_n|$ summable, then we say that the Fourier series of f is $|N, p_n|$ summable at the point x and we write $f \in |N, p_n|$.

We use the notations

$$\varphi(t) = \varphi_x(t) = f(x+t) + f(x-t) - 2f(x) ,$$

$$\varphi_1(t) = \frac{1}{t} \int_0^t \varphi(u) du \quad (t > 0) .$$

1.2. T. Singh [1] has proved the following theorem (*cf.* T. Pati [2], H.P. Dikshit [3] and O.P. Varshney [4]):

THEOREM A. *If the function* φ *is of bounded variation on the interval* $(0, \pi)$ *and if the sequence* (p_n) *is non-increasing and convex, and satisfies the condition*

$$(3) \qquad \sum_{k=1}^{n} \frac{P_k}{k} \leq AP_n \quad \text{for all} \quad n \geq 1 ,$$

then $f \in |N, p_n|$.

On the other hand, O.P. Varshney [5] has proved the following:

THEOREM B. *If the function* $\varphi(t)\log(K/t)$ $(K > \pi)$ *is of bounded variation over the interval* $(0, \pi)$, *then* $f \in |N, p_n|$ *where*

$$(4) \qquad p_n = \frac{1}{n+1} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n+1}\right) \quad \text{for all} \quad n \geq 0 .$$

T. Pati [6] (*cf.* R. Mohanty and B.K. Ray [7]) has proved that the sequence (4) in Theorem B cannot be replaced by

$$p_n = \frac{1}{n+1} \quad \text{for all} \quad n \geq 0 .$$

The sequence (4) does not satisfy the condition (3) and then Theorem B is not contained in Theorem A.

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We shall first prove the following

THEOREM 1. *Let* $a \geqq 0$ *and* K *be a constant* $> \pi$. *If the function* $\varphi(t)(\log K/t)^a$ *is of bounded variation over the interval* $(0, \pi)$ *and if the sequence* (p_n) *is non-increasing and satisfies the condition*

$$(5) \qquad \sum_{k=n}^{\infty} \frac{1}{kP_k} \leqq A \frac{(\log n)^a}{P_n} \quad \text{for all} \quad n > 1 ,$$

then $f \in |N, p_n|$.

The case $a = 0$ in Theorem 1 is a generalization of Theorem A, since the condition of convexity of (p_n) is dropped and the condition (3) implies (5) (see [1]). In the case $a = 1$, we get the following:

COROLLARY. *If the function* $\varphi(t)\log(K/t)$ *is of bounded variation on the interval* $(0, \pi)$, *then* $f \in |N, p_n|$, *where*

$$p_n = \frac{1}{n+1} \left(\log(n+1)\right)^b \quad \text{for all} \quad n \geqq 0 \quad \text{and some} \quad b > 0 .$$

The case $b = 1$ in the corollary is Theorem B and this corollary does not hold for $b = 0$ by Pati's theorem.

1.3. We shall next consider the case that φ_1 is of bounded variation. L.S. Bosanquet [11] has proved the

THEOREM C. *If the function* φ_1 *is of bounded variation over the interval* $(0, \pi)$, *then the Fourier series of* f *is* $|C, a|$ *summable at the point* x *for any* $a > 1$.

This was generalized by H.P. Dikshit [12] in the following form.

THEOREM D. *If the function* φ_1 *is of bounded variation over the interval* $(0, \pi)$ *and if the sequence* (p_n) *is a non-decreasing and concave sequence satisfying the conditions*

(i) *the sequence* $\left((n+1)p_n/P_n\right)$ *is of bounded variation and*

(ii) $\displaystyle\sum_{k=n+1}^{\infty} \frac{1}{P_k} \leqq A \frac{n}{P_n}$ *for all* $n \geqq 1$,

then $f \in |N, p_n|$.

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We shall prove the following generalization.

THEOREM 2. *Suppose that the sequence* (p_n) *is non-decreasing and concave and satisfies the condition*

(6) $$\sum_{k=1}^{\infty} \frac{1}{P_k} < \infty .$$

Then $f \in |N, p_n|$ *for any* f *satisfying the condition that* φ_1 *is of bounded variation on the interval* $(0, \pi)$, *if and only if*

(7) $$\sum_{k=n+1}^{\infty} \left| \frac{P_{k-n}}{P_k} - \frac{P_{k-n-1}}{P_{k-1}} \right| \leqq A \quad \text{for all} \quad n \geqq 1 .$$

The condition (7) is satisfied when the sequence (P_{n-s}/P_n) is non-decreasing for each $s \geqq 1$ or the sequence (p_n/P_n) is non-increasing. Further the non-decreasing and concave sequence (p_n) satisfies the following relations which are used in the proof of Theorem 2:

(8) $$np_n \leqq AP_n \quad \text{for all} \quad n \geqq 1 ,$$

(9) $$\sum_{n=j+1}^{\infty} \left(p_{n-j}-p_{n-j-1}\right)/P_{n-1} \leqq A/j \quad \text{for all} \quad j \geqq 1 ,$$

and

(10) $$p_{n-j+1} - 2p_{n-j} + p_{n-j-1} \geqq 0 \quad \text{for all} \quad j \geqq 1 .$$

These are proved easily, so that we omit the proof.

Theorem 2 is a generalization of Theorems C and D.

By Theorem 2, we know that $f \in \left|N, \left(\log(n+1)\right)^a\right|$ $(a > 1)$ when φ_1 is of bounded variation, since the sequence $p_n = \left(\log(n+1)\right)^a$ $(a > 1)$ satisfies the conditions. But we don't know the case $a = 1$.

1.4. For the proof of Theorem 1 we use the following lemma due to E. Hille and J.D. Tamarkin [8] (see [9]).

LEMMA. *If the sequence* (p_n) *is positive non-increasing, then*

$$\left|\sum_{k=0}^{N} p_k \sin(n-k)t\right| \leqq AP_{[1/t]} \quad or \quad \leqq Ap_0/t$$

for any N *, any* n *and any* $t > 0$.

2. Proof of the theorems

2.1. Proof of Theorem 1: the case $\alpha = 0$. By (2) and by integration by parts, we have

$$A_j(x) = \frac{2}{\pi}\int_0^\pi \varphi(t)\cos jt\,dt = -\frac{2}{j\pi}\int_0^\pi \sin jt\,d\varphi(t)$$

and we can suppose $A_0(x) = 0$ without loss of generality. By the formula (1),

$$-\Delta t_{n-1} = t_n - t_{n-1} = \frac{1}{P_n P_{n-1}} \sum_{j=1}^{n} \left(P_n p_{n-j} - P_{n-j} p_n\right) A_j(x) ,$$

and then

$$\frac{\pi}{2}\sum_{n=1}^{\infty} |\Delta t_n| \leqq \int_0^\pi |d\varphi(t)| \left(\sum_{n=1}^{\infty} \left|\sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \frac{\sin jt}{j}\right|\right) .$$

It is enough to prove that the sum on the right side is uniformly bounded in t . We write, putting $s = [1/t]$,

$$\sum_{n=1}^{\infty} \left|\sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \frac{\sin jt}{j}\right| = \sum_{n=1}^{s} + \sum_{n=s+1}^{\infty} = U + V .$$

Since $P_{n-j}/P_n \uparrow 1$ as $n \uparrow \infty$ for each j ,

$$U \leqq t \sum_{n=1}^{s} \sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} = t \sum_{j=1}^{s} \sum_{n=j}^{s} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}}\right)$$

$$= t \sum_{j=1}^{s} \left(\frac{P_{s-j}}{P_s} - \frac{P_{-1}}{P_{j-1}}\right) \leqq A .$$

Now

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$$V = \sum_{n=s+1}^{\infty} \left| \sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \frac{\sin jt}{j} \right| \leqq \sum_{n=s+1}^{\infty} \left| \sum_{j=1}^{s} \right| + \sum_{n=s+1}^{\infty} \left| \sum_{j=s+1}^{n} \right| ,$$
$$= W + X$$

where

$$W \leqq t \sum_{n=s+1}^{\infty} \sum_{j=1}^{s} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}} \right) = t \sum_{j=1}^{s} \sum_{n=s+1}^{\infty} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}} \right)$$
$$= t \sum_{j=1}^{s} \left(1 - \frac{P_{s-j}}{P_s} \right) \leqq A$$

and

$$X = \sum_{n=s+1}^{2s+1} \left| \sum_{j=s+1}^{n} \right| + \sum_{n=2s+2}^{\infty} \left| \sum_{j=s+1}^{[n/2]} \right| + \sum_{n=2s+2}^{\infty} \left| \sum_{j=[n/2]+1}^{\infty} \right|$$
$$= X' + Y + Z .$$

X' is bounded by the estimation similar to W. Writing $[n/2] = m$,

$$Y = \sum_{n=2s+2}^{\infty} \left| \sum_{j=s+1}^{m} \frac{P_n p_{n-j} - P_{n-j} p_n}{jP_n P_{n-1}} \frac{\cos(j-1/2)t - \cos(j+1/2)t}{2\sin t/2} \right|$$
$$\leqq \sum_{n=2s+2}^{\infty} \left\{ \left| \frac{P_n p_{n-m} - P_{n-m} p_n}{mP_n P_{n-1}} \frac{\cos(m+1/2)t}{2\sin t/2} \right| \right.$$
$$+ \left| \sum_{j=s+1}^{m-1} \Delta \left(\frac{P_n p_{n-j} - P_{n-j} p_n}{jP_n P_{n-1}} \right) \frac{\cos(j+1/2)t}{2\sin t/2} \right|$$
$$\left. + \left| \frac{P_n p_{n-s-1} - P_{n-s-1} p_n}{(s+1) P_n P_{n-1}} \frac{\cos(s+1/2)t}{2\sin t/2} \right| \right\}$$
$$= Y_1 + Y_2 + Y_3 ,$$

where

$$Y_1 \leqq \frac{A}{t} \sum_{n=2s+2}^{\infty} \frac{1}{n} \left(\frac{P_{n-m}}{P_n} - \frac{P_{n-m-1}}{P_{n-1}} \right) \leqq \frac{A}{t} \sum_{n=2s+2}^{\infty} \frac{p_{m-1}}{nP_n} + \frac{A}{t} \sum_{n=2s+2}^{\infty} \frac{1}{n^2} \leqq A ,$$

$$Y_2 \leqq \frac{A}{t} \sum_{n=2s+2}^{\infty} \sum_{j=s+1}^{n-1} \left\{ \frac{1}{j} \left(\frac{p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}}{P_n} \right) + \frac{1}{j^2} \left(\frac{P_{n-j-1}}{P_n} - \frac{P_{n-j-2}}{P_{n-1}} \right) \right\}$$
$$\leqq \frac{A}{t} \sum_{j=s+1}^{\infty} \sum_{n=2j+1}^{\infty} \leqq \frac{A}{t} \sum_{j=s+1}^{\infty} \left(\frac{p_j}{jP_{2j}} + \frac{1}{j^2} \right) \leqq A ,$$

and similarly Y_3 is also bounded. Finally, using the lemma and condition (5) with $a = 0$,

$$Z = \sum_{n=2s+2}^{\infty} \left| \sum_{j=m+1}^{n} \left(\frac{1}{jP_{n-1}} p_{n-j} \sin jt - \frac{p_n}{P_n P_{n-1}} \frac{P_{n-j}}{j} \sin jt \right) \right|$$

$$\leq AP_s \sum_{n=2s+2}^{\infty} \frac{1}{nP_{n-1}} + \frac{A}{t} \sum_{n=2s+2}^{\infty} \frac{p_n P_{n-m-1}}{nP_n P_{n-1}} \leq A .$$

Thus we have proved Theorem 1 in the case $a = 0$.

2.2. Proof of Theorem 1: the case $a > 0$. We put

$$h(t) = \varphi(t)(\log K/t)^a \quad \text{for} \quad 0 < t \leq \pi .$$

Then, by (2) and by integration by parts,

$$A_j(x) = \frac{2}{\pi} \int_0^{\pi} \varphi(t) \cos jt\, dt$$

$$= \frac{2}{\pi} h(\pi) \int_0^{\pi} \frac{\cos ju}{(\log K/u)^a} du - \frac{2}{\pi} \int_0^{\pi} dh(t) \int_0^t \frac{\cos ju}{(\log K/u)^a} du .$$

Putting $s = [1/t]$, we have

$$\frac{\pi}{2} \sum_{n=1}^{\infty} |\Delta t_n| = \sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} A_j(x) \right|$$

$$\leq A|h(\pi)| \sum_{n=1}^{\infty} \left| \sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \int_0^{\pi} \frac{\cos ju}{(\log K/u)^a} du \right|$$

$$+ A \int_0^{\pi} |dh(t)| \left(\sum_{n=1}^{s} + \sum_{n=s+1}^{\infty} \right) \left| \sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \int_0^t \frac{\cos ju}{(\log K/u)^a} du \right|$$

$$= R + S + T .$$

Since the sequence (p_n) is non-increasing and

$$\left| \int_0^t \frac{\cos ju}{(\log K/u)^a} du \right| \leq \frac{t}{(\log K/t)^a} ,$$

we get

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$$S \leqq \int_0^\pi |dh(t)| \frac{t}{(\log K/t)^\alpha} \sum_{n=1}^{s} \sum_{j=1}^{n} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}}\right)$$

$$= \int_0^\pi \frac{t|dh(t)|}{(\log K/t)^\alpha} \sum_{j=1}^{s} \left(\frac{P_{s-j}}{P_s} - \frac{P_{-1}}{P_{j-1}}\right) \leqq \int_0^\pi \frac{|dh(t)|}{(\log K/t)^\alpha} \leqq A .$$

Now, by integration by parts,

$$\int_0^t \frac{\cos ju}{(\log K/u)^\alpha} du = \frac{\sin jt}{j(\log K/t)^\alpha} + \frac{\alpha}{j} \int_0^t \frac{\sin ju}{u(\log K/u)^{\alpha+1}} du ,$$

and then

$$T \leqq A \int_0^\pi \frac{|dh(t)|}{(\log K/t)^\alpha} \left(\sum_{n=s+1}^{\infty} \left|\sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \frac{\sin jt}{j}\right|\right)$$

$$+ A \int_0^\pi |dh(t)| \left(\sum_{n=s+1}^{\infty} \left|\int_0^t \left(\sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \frac{\sin ju}{j}\right) \frac{du}{u(\log K/u)^{\alpha+1}}\right|\right)$$

$$= U + V .$$

U is bounded by the estimation of V in the last section, using the condition (5) with $\alpha > 0$ and

$$V \leqq A \int_0^\pi |dh(t)| \left(\sum_{n=s+1}^{\infty} \sum_{j=1}^{n} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}}\right) \frac{1}{j(\log Kj)^{\alpha+1}}\right)$$

$$\leqq A \int_0^\pi |dh(t)| \left(\sum_{n=s+1}^{\infty} \sum_{j=1}^{s} + \sum_{n=s+1}^{\infty} \sum_{j=s+1}^{n}\right)$$

$$\leqq A \int_0^\pi |dh(t)| \left(\sum_{j=1}^{s} \frac{1}{j(\log Kj)^{\alpha+1}} \left(1 - \frac{P_{s-j}}{P_s}\right) + \sum_{j=s+1}^{\infty} \frac{}{j(\log j)^{\alpha+1}}\right) \leqq A .$$

Therefore T is also bounded. It remains to prove R is bounded. Using integration by parts,

$$R = A|h(\pi)| \sum_{n=1}^{\infty} \left|\sum_{j=1}^{n} \frac{P_n p_{n-j} - P_{n-j} p_n}{P_n P_{n-1}} \int_0^\pi \frac{\sin jt}{j(\log K/t)^{\alpha+1}} dt\right|$$

$$\leqq A \sum_{n=1}^{\infty} \sum_{j=2}^{n} \frac{1}{j(\log j)^{\alpha+1}} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}}\right)$$

$$= A \sum_{j=2}^{\infty} \frac{1}{j(\log j)^{\alpha+1}} \sum_{n=j}^{\infty} \left(\frac{P_{n-j}}{P_n} - \frac{P_{n-j-1}}{P_{n-1}}\right) = A \sum_{j=2}^{\infty} \frac{1}{j(\log j)^{\alpha+1}} < A ,$$

by the following inequality

$$\left|\int_0^{\pi} \frac{\sin jt}{t(\log K/t)^{\alpha+1}}\,dt\right| \leq \left|\int_0^{\pi/j} \frac{\sin jt}{t(\log K/t)^{\alpha+1}}\,dt\right| \leq \frac{A}{(\log j)^{\alpha+1}} \quad \text{for all} \quad j \geq 2 .$$

This inequality is easily seen from the fact that the function $t(\log K/t)^{\alpha+1}$ is non-decreasing on the interval $(0, \pi)$, by taking $K = \pi e^{\alpha+1}$, which does not lose any generality.

Thus Theorem 1 is completely proved.

2.3. Proof of Theorem 2. We suppose that $\varphi_1(\pi) = 0$ without any loss of generality. By (2)

$$A_j(x) = \frac{2j}{\pi}\int_0^{\pi} t\sin jt\varphi_1(t)dt = -\frac{2j}{\pi}\int_0^{\pi} d\varphi_1(t)\int_0^t u\sin ju\,du$$

and then

$$\begin{aligned}\frac{\pi}{2}\sum_{n=1}^{\infty}|\Delta t_n| &\leq \sum_{n=1}^{\infty}\left|\sum_{j=1}^{n}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\int_0^{\pi}d\varphi_1(t)\int_0^t ju\sin ju\,du\right| \\ &\leq \int_0^{\pi}|d\varphi_1(t)|\left\{\sum_{n=1}^{\infty}\left|\sum_{j=1}^{n}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\int_0^t ju\sin ju\,du\right|\right\} \\ &= \int_0^{\pi}R(t)|d\varphi_1(t)| .\end{aligned}$$

We shall show that, if (p_n) is non-decreasing and concave and satisfies the condition (6), then R is bounded if and only if (7) holds. By (6), we can suppose that $p_1 = p_0$ and the sequence $(p_n, n \geq 1)$ is concave. We write

$$\begin{aligned}R &= \sum_{n=1}^{\infty}\left|\sum_{j=1}^{n}\frac{P_np_{n-j}-P_{n-j}p_n}{P_nP_{n-1}}\int_0^t ju\sin ju\,du\right| = \sum_{n=1}^{s} + \sum_{n=s+1}^{\infty} \\ &= S + T ,\end{aligned}$$

where $s = [1/t]$. We shall put

$$x_j = \sum_{k=1}^{j} k\int_0^t u\sin ku\,du ,$$

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then

(11) $$|x_j| \leqq \sum_{k=1}^{j} k^2 \int_0^t u^2 du \leqq Aj^3t^3$$

and x_j is bounded uniformly in t, since

(12) $$x_j = \frac{t\sin(j+1/2)t}{2\sin t/2} + \int_0^t \frac{\sin(j+1/2)u}{2\sin u/2}\, du .$$

Now using Abel's transformation,

$$\begin{aligned} S &= \sum_{n=1}^{s} \left| \sum_{j=1}^{n} \left(\frac{p_{n-j}}{P_{n-1}} - \frac{P_{n-j}p_n}{P_nP_{n-1}}\right)(x_j - x_{j-1}) \right| \\ &\leqq A \sum_{n=1}^{s} \frac{1}{P_n} + At^3 \sum_{n=1}^{s} \left| \sum_{j=1}^{n-1} j^3 \left\{\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} + p_{n-j}\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\right\} \right| \\ &\leqq A + At^3 \sum_{j=1}^{s-1} j^3 \left(\sum_{n=j+1}^{s} \frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} + \frac{p_1}{P_j}\right) \leqq A \end{aligned}$$

by (11), (6) and (9). On the other hand

$$\begin{aligned} T &= \sum_{n=s+1}^{\infty} \left| \frac{p_0 x_n}{P_n} + \sum_{j=1}^{n-1} \left(\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}p_n}{P_nP_{n-1}}\right) x_j \right| \\ &\leqq A \sum_{n=s+1}^{\infty} \frac{1}{P_n} + \sum_{n=s+1}^{\infty} \left| \sum_{j=1}^{s} \right| + \sum_{n=s+1}^{\infty} \left| \sum_{j=s+1}^{n-1} \right| = A + U + V , \end{aligned}$$

where

$$U \leqq At^3 \sum_{j=1}^{s} j^3 \sum_{n=s+1}^{\infty} \left\{\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} + p_{n-j}\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\right\} \leqq A$$

by (11) and (9), and by (12) we can write

$$\begin{aligned} V &\leqq A \sum_{n=s+1}^{\infty} \left| \sum_{j=s+1}^{n-1} \left(\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}p_n}{P_nP_{n-1}}\right) \sin(j+1/2)t \right| \\ &\quad + \sum_{n=s+1}^{\infty} \left| \int_0^t \frac{du}{2\sin u/2} \left\{ \sum_{j=s+1}^{n-1} \left(\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}p_n}{P_nP_{n-1}}\right) \sin(j+1/2)u \right\} \right| \\ &= AW + X . \end{aligned}$$

Now again using Abel's transformation, we get

$$(2\sin t/2)W = \sum_{n=s+1}^{\infty} \left| \sum_{j=s+1}^{n-1} \left(\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}p_n}{P_n P_{n-1}} \right) (\cos jt - \cos(j+1)t) \right|$$

$$= \sum_{n=s+1}^{\infty} \left(\frac{p_0}{P_{n-1}} - \frac{p_1}{P_n} \right)$$

$$+ \sum_{n=s+1}^{\infty} \sum_{j=s+2}^{n-1} \left(\frac{p_{n-j-1}-2p_{n-j}+p_{n-j+1}}{P_{n-1}} + \frac{(p_{n-j+1}-p_{n-j})p_n}{P_n P_{n-1}} \right)$$

$$+ \sum_{n=s+1}^{\infty} \left(\frac{p_{n-s-1}-p_{n-s-2}}{P_{n-1}} + \frac{p_{n-s-1}p_n}{P_n P_{n-1}} \right)$$

$$= Y_1 + Y_2 + Y_3$$

by (10), where

$$Y_1 \leqq A \sum_{n=s+1}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \leqq \frac{A}{P_s} \leqq At$$

by (8),

$$Y_2 \leqq A \sum_{n=s+1}^{\infty} \sum_{j=s+2}^{n-1} \frac{p_{n-j-1}-2p_{n-j}+p_{n-j+1}}{P_{n-1}} \leqq A \sum_{n=s+1}^{\infty} \frac{p_{n-s-1}-p_{n-s-2}}{P_{n-1}}$$

$$\leqq At$$

by (9) and similarly $Y_3 \leqq At$. Therefore W is bounded, so that it remains to estimate X . We can easily see that

$$\int_0^t \frac{\sin(j+1/2)u}{2\sin u/2}\,du = \frac{\pi}{2} - \int_t^{\pi} \frac{\sin(j+1/2)u}{2\sin u/2}\,du = \frac{\pi}{2} - y_j$$

where $|y_j| \leqq A/jt$, and hence

$$X = \sum_{n=s+1}^{\infty} \left| \int_0^t \frac{du}{2\sin u/2} \left(\sum_{j=s+1}^{n-1} \left(\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}p_n}{P_{n-1}P_n} \right) \sin(j+1/2)u \right) \right|$$

$$\leqq \frac{\pi}{2} \sum_{n=s+1}^{\infty} \left| \sum_{j=s+1}^{n-1} \left(\frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} - \frac{p_{n-j}p_n}{P_{n-1}P_n} \right) \right|$$

$$+ \frac{A}{t} \sum_{n=s+1}^{\infty} \left(\sum_{j=s+1}^{n-1} \frac{p_{n-j}-p_{n-j-1}}{jP_{n-1}} + \sum_{j=s+1}^{n-1} \frac{p_{n-j}p_n}{P_{n-1}P_n} \right) = \frac{\pi}{2} X_1 + X_2$$

where

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$$X_2 \leqq \frac{A}{t} \sum_{j=s+1}^{\infty} \frac{1}{j}\left(\sum_{n=j+1}^{\infty} \frac{p_{n-j}-p_{n-j-1}}{P_{n-1}} + \sum_{n=j+1}^{\infty} p_{n-j}\left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right)\right)$$

$$\leqq \frac{A}{t} \sum_{j=s+1}^{\infty} \frac{1}{j^2} + \frac{A}{t} \sum_{j=s+1}^{\infty} \frac{1}{jP_j} \leqq A$$

by Abel's transformation, (6) and (9) and

$$X_1 = \sum_{n=s+1}^{\infty} \left| \frac{P_{n-s-1}-p_0}{P_{n-1}} - (P_{n-s-1}-p_0)\frac{p_n}{P_{n-1}P_n} \right|$$

$$= \sum_{n=s+1}^{\infty} \left| \frac{P_{n-s-1}}{P_n} - \frac{P_{n-s-2}}{P_{n-1}} \right| + O\left(\sum_{n=s+1}^{\infty} \frac{1}{P_n} \right) .$$

Thus we have proved that

$$R = \frac{\pi}{2} \sum_{n=s+1}^{\infty} \left| \frac{P_{n-s-1}}{P_n} - \frac{P_{n-s-2}}{P_{n-1}} \right| + O(1) ,$$

by (6), therefore the condition (7) is necessary and sufficient for boundedness of R . Thus Theorem 2 is proved.

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