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THE KERNEL RELATION FOR AN EXTENSION OF COMPLETELY 0-SIMPLE SEMIGROUPS

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Abstract. Let S be an (ideal) extension of a completely 0-simple semigroup S_0 by a completely 0-simple semigroup S_1. Congruences on S can be uniquely represented in terms of congruences on S_0 and S_1. In this representation, for a congruence ρ on S, we express ρ_K, ρ_T, ρ^K and ρ^T, where these denote the least (greatest) congruences with the same kernel (trace) as ρ. Let κ be the least completely 0-simple congruence on S. We provide necessary and sufficient conditions, in terms of the kernel of κ, in order that the relation K be a congruence, and also that $\mathcal{C}(S)/K$ be a modular lattice, where $\mathcal{C}(S)$ denotes the congruence lattice of S.

1. Introduction and summary. The study of congruences on a regular semigroup S is greatly facilitated by the kernel-trace approach which consists in analysing their kernels and their traces. The kernel and the trace of a congruence ρ on S are defined by

$$\ker \rho = \{a \in S \mid a\,\rho\, e \text{ for some } e \in E(S)\}, \quad \operatorname{tr}\rho = \rho|_{E(S)} \tag{1}$$

where $E(S)$ denotes the set of all idempotents of S. These quantities determine the congruence uniquely and induce the following relations K and T on the congruence lattice $\mathcal{C}(S)$:

$$\lambda\, K\, \rho \Leftrightarrow \ker\lambda = \ker\rho, \quad \lambda\, T\, \rho \Leftrightarrow \operatorname{tr}\lambda = \operatorname{tr}\rho. \tag{2}$$

The former is a complete $\wedge$-congruence and the latter a complete congruence on $\mathcal{C}(S)$. These relations can be successfully used for a detailed study of the structure of the lattice $\mathcal{C}(S)$. In particular, the classes of these relations are intervals in $\mathcal{C}(S)$ and we may thus use the notation

$$\rho K = [\rho_K, \rho^K], \quad \rho^T = [\rho_T, \rho^T] \qquad (\rho \in \mathcal{C}(S)). \tag{3}$$

In this way, we arrive at the four operators

$$\rho \to \rho_K, \quad \rho \to \rho^K, \quad \rho \to \rho_T, \quad \rho \to \rho^T \qquad (\rho \in \mathcal{C}(S)) \tag{4}$$

which provide further means for a study of the congruence lattice $\mathcal{C}(S)$. One may also consider necessary and sufficient conditions on a special or an arbitrary semigroup in order that K be a ($\vee$-) congruence. The general reference for this subject is [**3**] and related results may be found in [**4**].

With this general preamble, we concentrate in the paper on some of the topics raised above as they apply to the special situation when S is an ideal extension of a

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completely 0-simple semigroup S_0 by a completely 0-simple semigroup S_1. Reference [**5**] is devoted to the congruence lattice on these semigroups from which we shall draw the form of congruences on S in terms of congruences on S_0 and S_1, respectively, as well as many of their basic properties. Given a congruence ρ on S in such a representation, we shall find explicitly the form of ρ_K, ρ_T, ρ^K and ρ^T as well as necessary and sufficient conditions on S in order that K be a congruence and also that $\mathcal{C}(S)/K$ be a modular lattice. These achievements complete and supplement several of the results with modest partial solutions in [**5**].

As a basis for further consideration, Section 2 contains several elementary results concerning congruences on completely 0-simple semigroups. The construction of congruences as well as many of their properties are stated or, if they are new, also proved in Section 3. The main result in Section 4 provides expressions for ρ_K and ρ_T, whereas in Section 5 for ρ^K and ρ^T for an arbitrary congruence ρ on the cited extension. Necessary and sufficient conditions on S for K to be a congruence are established in Section 6 and for K to be a congruence and $\mathcal{C}(S)/K$ to be modular in Section 7. The theorems in Sections 6 and 7 have simpler versions for strict extensions in Section 8.

2. Congruences on completely 0-simple semigroups. Let

$$S = \mathcal{M}^0(I, G, \Lambda; P)$$

be a Rees matrix semigroup whose elements we denote by (i, g, λ) with multiplication

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

A congruence ρ on any semigroup is proper if ρ is not the universal relation. These congruences on S are described by means of the following device.

Let r be a partition of I, N be a normal subgroup of G and π be a partition of Λ satisfying the following conditions:

(i) if $i\,r\,j$, then for all $\lambda, \mu \in \Lambda$,
  (a) $p_{\lambda i} \neq 0 \Leftrightarrow p_{\lambda j} \neq 0$,
  (b) $p_{\lambda i} \neq 0, p_{\mu i} \neq 0 \Rightarrow p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N$;

(ii) if $\lambda\,\pi\,\mu$, then for all $i, j \in I$,
  (a) $p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu i} \neq 0$,
  (b) $p_{\lambda i} \neq 0, p_{\lambda j} \neq 0 \Rightarrow p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N$.

In such a case (r, N, π) is an *admissible triple* for S, and we define a relation $\rho = \mathcal{C}(r, N, \pi)$ on S by

$$(i, g, \lambda)\,\rho\,(j, h, \mu) \Leftrightarrow i\,r\,j, \quad \lambda\,\pi\,\mu, \quad p_{\nu i}gp_{\lambda k}N = p_{\nu j}hp_{\mu k}N$$

for some [any] $\nu \in \Lambda, k \in I$ such that $p_{\nu i} \neq 0$, $p_{\lambda k} \neq 0$; and $0\,\rho\,0$. Then $\mathcal{C}(r, N, \pi)$ is a proper congruence on S, and conversely, every proper congruence on S can be so written for a unique admissable triple. In addition, for $\rho = \mathcal{C}(r, N, \pi)$ and $\rho' = \mathcal{C}(r', N', \pi')$,

$$\rho \vee \rho' = \mathcal{C}(r \vee r', NN', \pi \vee \pi'), \qquad \rho \wedge \rho' = \mathcal{C}(r \cap r', N \cap N', \pi \cap \pi').$$

For proofs, see [**2**, III.4 and III.5].

We shall presently need the following constructs. For a normal subgroup N of G, we define the relations $N\alpha$ on I and $N\beta$ on Λ, respectively, by

$$\begin{aligned} i\,N\alpha\,j \Leftrightarrow \big(&p_{\lambda i} \neq 0 \Leftrightarrow p_{\lambda j} \neq 0 \;\text{ for all } \lambda \in \Lambda,\\ &p_{\lambda i} \neq 0, p_{\mu i} \neq 0 \;\;\Rightarrow\;\; p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N\big);\\ \lambda\,N\beta\,\mu \Leftrightarrow \big(&p_{\lambda i} \neq 0 \Leftrightarrow p_{\mu i} \neq 0 \;\text{ for all; } i \in I,\\ &p_{\lambda i} \neq 0, p_{\lambda j} \neq 0 \;\;\Rightarrow\;\; p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N\big). \end{aligned}$$

For partitions r of I and π of Λ, let $\overline{r\pi}$ be the normal subgroup of G generated by the set

$$\left\{p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \mid p_{\lambda i}, p_{\mu i}, p_{\mu j}, p_{\lambda j} \neq 0, \text{either } i\,r\,j \text{ or } \lambda\,\pi\,\mu\right\}.$$

We denote by ε and ω the equality and the universal relation on any set if there is no danger of confusion. Recall the definitions contained in (1), (2) and (3).

Lemma 2.1. *Let* $S = \mathcal{M}^0(I, G, \Lambda; P)$ *and* $\rho = \mathcal{C}(r, N, \pi)$, $\rho' = \mathcal{C}(r', N', \pi')$.

(i) $\ker\rho = \{(i, g, \lambda) \in S \mid p_{\lambda i} \neq 0, gp_{\lambda i} \in N\} \cup \{0\}$.

(ii) $\rho\,K\,\rho' \Leftrightarrow N = N'$.

(iii) $\rho_K = \mathcal{C}(\varepsilon, N, \varepsilon)$, $\rho^K = \mathcal{C}(N\alpha, N, N\beta)$.

(iv) *For* $p_{\lambda i} \neq 0$ *and* $p_{\mu j} \neq 0$,

$$(i, p_{\lambda i}^{-1}, \lambda)\,\mathrm{tr}\rho\,(j, p_{\mu j}^{-1}, \mu) \Leftrightarrow\; i\,r\,j, \lambda\,\pi\,\mu.$$

(v) $\rho\,T\,\rho' \Leftrightarrow r = r', \pi = \pi'$.

(vi) $\rho_T = \mathcal{C}(r, \overline{r\pi}, \pi)$, $\rho^T = \mathcal{C}(r, G, \pi)$.

Proof. (i) This is proved in [**4**, Lemma 3.1].

(ii) This follows easily from part (i).

(iii) The first assertion is obvious. For the second, we observe that the definitions of $N\alpha$ and $N\beta$ mimic the conditions in the definition of an admissible triple so that admissibility of (r, N, π) can be written as $r \subseteq N\alpha$ and $\pi \subseteq N\beta$. The maximality of $(N\alpha, N, N\beta)$ relative to N is now obvious which yields the second assertion in view of part (ii).

(iv) Indeed,

$$\begin{aligned} (i, p_{\lambda i}^{-1}, \lambda)\,\mathrm{tr}\rho\,(j, p_{\mu j}^{-1}, \mu) &\Leftrightarrow i\,r\,j,\;\; p_{\lambda i}p_{\lambda i}^{-1}p_{\lambda i}N = p_{\lambda j}p_{\mu j}^{-1}p_{\mu i}N,\;\; \lambda\,\pi\,\mu\\ &\Leftrightarrow i\,r\,j,\;\; p_{\lambda i}p_{\mu i}^{-1}p_{\mu j}p_{\lambda j}^{-1} \in N, \lambda\,\pi\,\mu\\ &\Leftrightarrow i\,r\,j,\;\; \lambda\,\pi\,\mu, \end{aligned}$$

since the middle condition follows from $i\,r\,j$ (also from $\lambda\,\pi\,\mu$).

(v) Using part (iv), we obtain

$$\begin{aligned} \rho\,T\,\rho' &\Leftrightarrow \left(\left(i, p_{\lambda i}^{-1}, \lambda\right)\rho\left(j, p_{\mu j}^{-1}, \mu\right) \Leftrightarrow \left(i, p_{\lambda i}^{-1}, \lambda\right)\rho'\left(j, p_{\mu j}^{-1}, \mu\right) \text{ for all } p_{\lambda i} \neq 0, p_{\mu j} \neq 0\right)\\ &\Leftrightarrow ((i\,r\,j, \lambda\,\pi\,\mu) \Leftrightarrow (i\,r'\,j, \lambda\,\pi'\,\mu)) \Leftrightarrow r = r', \pi = \pi'. \end{aligned}$$

(vi) The definition of $\overline{r\pi}$ mimics the condition in the definition of an admissible triple (r, N, π) which is equivalent to the requirement that $N \supseteq \overline{r\pi}$. This obviously entails the minimality of $(r, \overline{r\pi}, \pi)$ which in view of part (v) implies the first assertion. The second assertion is obvious because of part (v).

Corollary 2.2. *For a completely 0-simple semigroup S, in (4), the first and the fourth mappings are homomorphisms, the second is a $\wedge$-homomorphism and the third is a $\vee$-homomorphism.*

Proof. For $\rho \in \mathcal{C}(S)$, by Lemma 2.1(iii), we obtain that $\rho_K = \omega$ if $\rho = \omega$ and S has zero divisors and $\rho_K = \rho \wedge \mathcal{H}$ otherwise.Using this, the first assertion follows easily from Lemma 2.1(iii)(vi). The second claim follows from Lemma 2.1(iii) since straightforward checking shows that both mappings α and β are $\wedge$-homomorphisms. The third assertion follows similarly form Lemma 2.1(vi); however, the mapping $\rho \to \rho_T$ is a complete $\vee$-homomorphism for arbitrary regular semigroups as proved in [**3**, Theorem 4.13].

The following two examples supplement Corollary 2.2.

Example 2.3. Let $I = \{1, 2\}$, $G = Z_2 \times Z_2$

$$P = \begin{bmatrix} (\overline{0}, \overline{0}) & (\overline{0}, \overline{0}) \\ (\overline{0}, \overline{0}) & (\overline{1}, \overline{1}) \end{bmatrix}$$

and $S = \mathcal{M}^o(I, G, I; P)$. Further let $N = Z_2 \times \{\overline{0}\}$, $N' = \{\overline{0}\} \times Z_2$, $\rho = \mathcal{C}(\varepsilon, N, \varepsilon)$ and $\rho' = \mathcal{C}(\varepsilon, N', \varepsilon)$. Then $NN' = G$ so that $(NN')\alpha = \omega, p_{11}p_{21}^{-1}p_{22}p_{12}^{-1} = (\overline{1}, \overline{1}) \notin N \cup N'$ and thus $N\alpha = N'\alpha = \varepsilon$. Now Lemma 2.1(iii) gives

$$(\rho \wedge \rho')^K = (\mathcal{C}(\varepsilon, N, \varepsilon) \vee \mathcal{C}(\varepsilon, N', \varepsilon))^K = (\mathcal{C}(\varepsilon, NN', \varepsilon))^K = \mathcal{C}(\omega, G, \omega),$$

$$(\rho^K \vee \rho'^K) = (\mathcal{C}(\varepsilon, N, \varepsilon))^K \vee (\mathcal{C}(\varepsilon, N', \varepsilon))^K = \mathcal{C}(\varepsilon, N, \varepsilon) \vee \mathcal{C}(\varepsilon, N', \varepsilon) = \mathcal{C}(\varepsilon, G, \varepsilon).$$

Therefore the mapping $\rho \to \rho^K$ fails to be a $\vee$-homomorphism.

Example 2.4. Let $I = \{1, 2\}, G = Z_2, P = \begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{bmatrix}$ and $S = \mathcal{M}^0(I, G, I; P)$. Further let $\rho = \mathcal{C}(\varepsilon, G, \omega)$ and $\rho' = \mathcal{C}(\omega, G, \varepsilon)$. Now Lemma 2.1(vi) yields $\rho_T = \rho$ and $\rho'_T = \rho'$ so that

$$(\rho \wedge \rho')_T = (\mathcal{C}(\varepsilon, G, \varepsilon))_T = \varepsilon, \quad \rho_T \wedge \rho'_T = \mathcal{C}(\varepsilon, G, \varepsilon).$$

Therefore the mapping $\rho \to \rho_T$ fails to be a $\wedge$-homomorphism.

The next lemma will be useful later. A semigroup T is *reductive* if for any $a, b \in T$, whenever either $ax = bx$ for all $x \in T$ or $xa = xb$ for all $x \in T$, we have $a = b$.

Lemma 2.5. *Let $S = \mathcal{M}^0(I, G, \Lambda; P)$ and $\rho \in \mathcal{C}(S)$. Then S/ρ^K is reductive.*

Proof. If $\ker \rho = S$, then $\rho^K = \omega$ and S/ρ^K is trivial, so reductive. Assume that $\ker \rho \neq S$. Then ρ must be proper and hence $\rho = \mathcal{C}(r, N, \pi)$ for an admissible triple

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(r, N, π). By Lemma 2.1(iii), $\rho^K = \mathcal{C}(N\alpha, N, N\beta)$. We let $(i, g, \lambda), (j, h, \mu) \in S$ and assume that

$$(i, h, \lambda)(k, s, \tau)\, \rho^K\, (j, h, \mu)(k, s, \tau) \tag{5}$$

for all $(k, s, t) \in S$. For $p_{\lambda k} \neq 0$, we get $i\,N\alpha\, j$. For $p_{\lambda k} \neq 0, p_{\theta i} \neq 0, p_{\tau \ell} \neq 0$, we get $p_{\mu k} \neq 0$ from (5) and $p_{\theta j} \neq 0$ since $i\,N\alpha\, j$. Hence $p_{\theta i} g p_{\lambda k} s p_{\tau\ell} N = p_{\theta j} h p_{\mu k} s p_{\tau \ell} N$ so that $p_{\theta i} g p_{\lambda k} N = p_{\theta j} h p_{\mu k} N$.

From (5), we also get $p_{\lambda k} \neq 0 \Leftrightarrow p_{\mu k} \neq 0$. Now assume that $p_{\lambda k} \neq 0, p_{\lambda \ell} \neq 0$. As above, we get $p_{\theta i} g p_{\lambda \ell} N = p_{\theta j} h p_{\mu \ell} N$ where we have also supposed that $p_{\theta i} \neq 0$ so that $p_{\theta j} \neq 0$. It follows that

$$(p_{\theta i} g p_{\lambda k} p_{\mu k}^{-1} h^{-1} p_{\theta j}^{-1})\, (p_{\theta i} g p_{\lambda \ell} p_{\mu \ell}^{-1} h^{-1} p_{\theta j}^{-1})^{-1} \in N$$

whence $p_{\theta i} g p_{\lambda k} p_{\mu k}^{-1} p_{\mu l} p_{\lambda \ell}^{-1} g^{-1} p_{\theta i}^{-1} \in N$ and since N is normal, this implies that $p_{\lambda k} p_{\mu k}^{-1} p_{\mu \ell} p_{\lambda \ell}^{-1} \in N$. Therefore $\lambda\, N\beta\, \mu$, which together with $i\,N\alpha\, j$ and $p_{\theta i} g p_{\lambda k} N = p_{\theta j} h p_{\mu k} N$ whenever $p_{\theta i} \neq 0$ and $p_{\lambda k} \neq 0$, yields $(i, g, \lambda)\, \rho^K\, (j, h, \mu)$.

We have proved that in S/ρ^K, $ax = bx$ for all x implies $a = b$. A dual proof will show that also $xa = xb$ for all x implies that $a = b$. Therefore S/ρ^K is reductive.

3. Extensions. Throughout the remainder of the paper, we fix the following notation: S stands for an (ideal) extension of a completely 0-simple semigroup S_0 by a completely 0-simple semigroup S_1 such that $S_0 S_1 \neq \{0\}$. The reason for the last restriction is that in case $S_0 S_1 = \{0\}$, S is a primitive regular semigroup which was treated in [**4**, (Section 3)] and for which ρ_K, ρ^K, ρ_T and ρ^T can be found readily using Lemma 2.1.

If A is a subset of a semigroup with zero, we write $A^* = A \backslash \{0\}$. For $i = 0, 1$, we denote by ε_i the equality relation on S_i, by ω_i the universal relation on S_i, by ς_i the greatest proper congruence on S_i, by $\mathcal{C}(S_i)$ the lattice of congruences on S_i and by $\mathcal{C}_0(S_i)$ the lattice of proper congruences on S_i. Recall that the natural partial order on a regular semigroup S is defined thus: $a \leq b$ if $a = eb = bf$ for some $e, f \in E(S)$.

From [**5**] (Section 3 and particulary Lemma 3.5), we extract the following description of congruences on S.

Let $\rho_0 \in \mathcal{C}(S_0)$ be such that for every $a \in S_1^*$ and some [all] $b \in S_0^*$ such that $a > b$, we have $ax\, \rho_0\, bx$ and $xa\, \rho_0\, xb$ for all $x \in S_0$. In such a case, we define a relation $[\rho_0]$ on S by

$$a\,[\rho_0]\,b \Leftrightarrow \begin{cases} \overline{a}\,\rho_0\,\overline{b} & \text{if } a, b \in S_1^*, a > \overline{a} > 0, b > \overline{b} > 0, \\ a\,\rho_0\,\overline{b} & \text{if } a \in S_0, b \in S_1^*, b > \overline{b} > 0, \\ \overline{a}\,\rho_0\,b & \text{if } a \in S_1^*, b \in S_0, a > \overline{a} > 0, \\ a\,\rho_0\,b & \text{if } a, b \in S_0, \end{cases}$$

where $\overline{a}$ or $\overline{b}$ can be taken "for all such" or "for some".

Next let $\rho_0 \in \mathcal{C}(S_0)$ and $\rho_1 \in \mathcal{C}_0(S_1)$ be such that

$$a, b \in S_1^*, a\,\rho_1\, b,\;\; x\,\rho_0\, y \Rightarrow ax\,\rho_0\, by,\;\; xa\,\rho_0\, yb.$$

In such a case, define a relation $[\rho_0, \rho_1]$ on S by

$$a\,[\rho_0,\rho_1]\,b \Leftrightarrow \begin{cases} a\,\rho_1\,b & \text{if } a,b \in S_1^* \\ a\,\rho_0\,b & \text{if } a,b \in S_0 \end{cases}.$$

Then the relations $[\rho_0]$ and $[\rho_0,\rho_1]$ are congruences on S. Conversely, every congruence on S can be so represented in a unique way. For an indication of proof, consult [**5**, Theorem 3.2].

If θ is a relation on any semigroup T, we denote by θ^* the congruence on T generated by θ. The following congruence on S will play an important role in many of our considerations. Let

$$\kappa = \{(e,f) \in E(S) \times E(S) \mid e > f > 0\}^*$$

and $\kappa_0 = \kappa|_{S_0}$. Note that $\kappa = [\kappa_0]$ and κ is the least completely 0-simple congruence on S.

We shall also need the following constructs which extend some of those in [**5**, Section 4].

Let $\rho_0 \in \mathcal{C}(S_0)$ be such that $\rho_0 = \rho|_{S_0}$ for some $\rho \in \mathcal{C}(S)$. Define a relation ρ_0' on S_1^* by

$$a\,\rho_0'\,b \Leftrightarrow (x\,\rho_0\,y \Rightarrow ax\,\rho_0\,by,\ xa\,\rho_0\,yb)$$

and let $0\,\rho_0'\,0$.

We show now that ρ_0' is an equivalence relation. First, ρ_0' is reflexive since $\rho_0 = \rho|_{S_0}$ and it is obviously symmetric. Let $a,b,c \in S_1^*$ be such that $a\,\rho_0'\,b$ and $b\,\rho_0'\,c$. Then for any $x\,\rho_0\,y$, we have

$$ax\,\rho_0\,bx\,\rho_0\,by, \qquad xa\,\rho_0\,xb\,\rho_0\,yb$$

so that $a\,\rho_0'\,c$. Therefore ρ_0' is transitive.

Let $\overline{\rho}_0 = (\rho_0')^0$, the greatest congruence on S_1 contained in ρ_0', that is,

$$a\,\overline{\rho}_0\,b \Leftrightarrow xay\,\rho_0'\,xby \text{ for all } x,y \in S_1^1.$$

Note that since $\{(0,0)\}$ is a ρ_0'-*class*, the congruence $\overline{\rho}_0$ is proper.

We shall need a number of properties of congruences introduced in this section.

LEMMA 3.1. *Let* $[\rho_0,\rho_1] \in \mathcal{C}(S)$.
(i) $\ker[\rho_0,\rho_1] = \ker\rho_0 \cup (\ker\rho_1)^*$.
(ii) *If* $a > b > 0$ *and* $a \in \ker\rho_1$, *then* $b \in \ker\rho_0$.

Proof. These statements are proved in [**5**], Lemmas 6.1(ii) and 6.2, respectively.

LEMMA 3.2. *Let* $[\rho_0] \in \mathcal{C}(S)$ *and* $\theta_0 \in \mathcal{C}(S_0)$.
(i) $\ker[\rho_0] = \ker\rho_0 \cup \{a \in S_1^* \mid a > b > 0 \text{ for some } b \in \ker\rho_0\}$.
(ii) *If* $a > b > 0$, *then* $a\,[\rho_0]\,b$.
(iii) *If* $a > b > 0$ *and* $b \in \ker\rho_0$, *then* $a \in \ker[\rho_0]$.
(iv) $[\rho_0]^K = [\rho_0^K]$.
(v) $[\theta_0] \in \mathcal{C}(S) \Leftrightarrow \kappa_0 \subseteq \theta_0$.

Proof. These statements are proved in [**5**], Lemmas 6.1(i), 3.4, 6.1(i), Proposition 6.7 and Lemma 3.6(ii), respectively.

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LEMMA 3.3. *Let* $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$.

(i) $[\lambda_0] \subseteq [\rho_0] \Leftrightarrow \lambda_0 \subseteq \rho_0$.

(ii) $[\lambda_0] \oplus [\rho_0] = [\lambda_0 \oplus \rho_0] \quad (\oplus \in \{\wedge, \vee\})$.

(iii) $[\lambda_0]\, K\, [\rho_0] \Leftrightarrow \lambda_0\, K\, \rho_0$.

Proof. These statements are proved in [**5**], Lemmas 3.3(i) and 4.1 and Corollary 6.3(i), respectively.

LEMMA 3.4. *Let* $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$.

(i) $[\lambda_0] \vee [\rho_0, \rho_1] = [\lambda_0 \vee \rho_0]$.

(ii) $[\lambda_0] \wedge [\rho_0, \rho_1] = \left[\lambda_0 \wedge \rho_0, \overline{\lambda}_0 \wedge \rho_1\right]$.

(iii) $[\lambda_0]\, K\, [\rho_0, \rho_1] \Leftrightarrow \lambda_0\, K\, \rho_0, (a > b > 0, b \in \ker \rho_0 \Rightarrow a \in \ker \rho_1)$.

Proof. These statements are proved in [**5**], Lemma 4.5 and Corollary 6.3(iii), respectively.

LEMMA 3.5. *Let* $[\lambda_0, \lambda_1], [\rho_0, \rho_1] \in \mathcal{C}(S)$.

(i) $[\lambda_0, \lambda_1] \subseteq [\rho_0, \rho_1] \Leftrightarrow \lambda_0 \subseteq \rho_0, \lambda_1 \subseteq \rho_1$.

(ii) $[\lambda_0, \lambda_1] \oplus [\rho_0, \rho_1] = [\lambda_0 \oplus \rho_0, \lambda_1 \oplus \rho_1] \quad (\oplus \in \{\wedge, \vee\})$.

(iii) $[\lambda_0, \lambda_1]\, P\, [\rho_0, \rho_1] \Leftrightarrow \lambda_0\, P\, \rho_0, \lambda_1\, P\, \rho_1 \quad (P \in \{K, T\})$.

Proof. These statements are proved in [**5**], Lemmas 3.3(ii), 4.2 and Corollary 6.3(ii), respectively.

LEMMA 3.6. *Let* $\rho_0 \in \mathcal{C}(S_0)$ *and* $\rho_1 \in \mathcal{C}_0(S_1)$ *be such that* S_0/ρ_0 *is reductive and for* $a, b \in S_1^*$,

$$a\, \rho_1\, b,\ x \in S_0 \Rightarrow ax\, \rho_0\, bx,\ xa\, \rho_0\, xb.$$

Then $[\rho_0, \rho_1] \in \mathcal{C}(S)$.

Proof. Let $a, b \in S_1^*,\ a\, \rho_1\, b,\ x\, \rho_0\, y,\ u \in S_0$. Then $ax\, \rho_0\, bx$ and

$$u(ax)\, \rho_0\, u(bx) = (ub)x\, \rho_0\, (ub)y = u(by)$$

and since S_0/ρ_0 is reductive, we get $ax\, \rho_0\, by$. One proves dually that $xa\, \rho_0\, yb$.

Let $[\rho_0, \rho_1] \in \mathcal{C}(S)$. From the definition of $[\rho_0, \rho_1]$, it follows that if $\lambda_1 \in \mathcal{C}(S_1)$ is such that $\lambda_1 \subseteq \rho_1$, then $[\rho_0, \lambda_1]$ is defined. If S_0/ρ_0 is reductive and $\lambda_0 \in \mathcal{C}(S_0)$ is such that $\rho_0 \subseteq \lambda_0$, then Lemma 3.6 implies that $[\lambda_0, \rho_1]$ is defined. The former procedure decreases the upper congruence and the latter increases the lower congruence.

LEMMA 3.7.

(i) *If* $[\rho_0] \in \mathcal{C}(S)$, *then* $\left[\rho_0, \overline{\rho}_0\right] = [\rho_0] \wedge [\omega_0, \zeta_1]$.

(ii) *Let* $\rho_0 \in \mathcal{C}(S_0)$ *and* $\rho_1 \in \mathcal{C}_0(S_1)$. *Then* $[\rho_0, \rho_1] \in \mathcal{C}(S)$ *if and only if* $\rho_1 \subseteq \overline{\rho}_0$.

(iii) *If* $[\rho_0]$ *is defined, then* $\overline{\rho_0^K} \subseteq \overline{\rho}_0^K$.

(iv) *If* $\overline{\lambda}_0$ *and* $\overline{\rho}_0$ *are defined, then* $\overline{\lambda}_0 \wedge \overline{\rho}_0 \subseteq \overline{\lambda_0 \wedge \rho_0}$.

(v) *If* $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$, *then* $\overline{\lambda}_0 \wedge \overline{\rho}_0 = \overline{\lambda_0 \wedge \rho_0}$.

Proof. (i) This is proved in [**5**, Lemma 4.4].

(ii) *Necessity*. Letting $\rho = [\rho_0, \rho_1]$, we have $\rho_0 = \rho|_{S_0}$ so that $\overline{\rho}_0$ is defined. From the definition of ρ_0' and the fact that $\overline{\rho}_0 \subseteq \rho_0'$, we obtain that $[\rho_0, \overline{\rho}_0]$ is defined. Let $a, b \in S_1^*$ be such that $a\,\rho_1\, b$ and let $x\,\rho_0\, y$. Then $ax\,\rho_0\, by$ and $xa\,\rho_0\, yb$ and thus $a\,\rho_0'\, b$. Hence $\rho_1 \subseteq \rho_0'$ whence $\rho_1 \subseteq \overline{\rho}_0$. Therefore $[\rho_0, \rho_1] \subseteq [\rho_0, \overline{\rho}_0]$.

Sufficiency. This follows directly from the definition of $\overline{\rho}_0$.

(iii) Let $[\rho_0]$ be defined. In view of Lemma 3.2(v), $[\rho_0^K]$ is also defined. Since $\rho_0\, K\, \rho_0^K$, Lemma 3.3(iii) implies that $[\rho_0]\, K[\rho_0^K]$. Hence $[\omega_0, \zeta_1] \wedge [\rho_0]\, K[\omega_0, \zeta_1] \wedge [\rho_0^K]$ which by part (i) yields $[\rho_0, \overline{\rho}_0]\, K[\rho_0^K, \overline{\rho_0^K}]$. Now Lemma 3.5(iii) gives $\overline{\rho}_0\, K\, \overline{\rho_0^K}$ which then implies that $\overline{\rho_0^K} \subseteq \overline{\rho}_0^K$.

(iv) This follows directly from the definition.

(v) If $[\lambda_0], [\rho_0] \in \mathcal{C}(S)$, then by part (i),

$$[\lambda_0, \overline{\lambda}_0] \wedge [\rho_0, \overline{\rho}_0] = [\lambda_0] \wedge [\rho_0] \wedge [\omega_0, \zeta_1] = [\lambda_0 \wedge \rho_0] \wedge [\omega_0, \zeta_1] = [\lambda_0 \wedge \rho_0, \overline{\lambda_0 \wedge \rho_0}]$$

which by Lemma 3.5(ii) yields $\overline{\lambda}_0 \wedge \overline{\rho}_0 = \overline{\lambda_0 \wedge \rho_0}$.

As a consequence of Lemma 3.7(ii), we have that $[\rho_0, \overline{\rho}_0]$ is the greatest congruence on S of the form $[\rho_0, \rho_1]$.

4. A construction of ρ_K and ρ_T. After a lemma of independent interest, we establish a theorem which provides a representation of ρ_K and ρ_T in the form described in the preceding section. We also treat a special case in which ρ_K assumes a simple form.

LEMMA 4.1. *For $[\rho_0] \in \mathcal{C}(S)$, the following conditions are equivalent.*

(i) $[\rho_0]\, K[\lambda_0, \lambda_1]$ *for some* $[\lambda_0, \lambda_1] \in \mathcal{C}(S)$.

(ii) $a > b > 0,\, b \in \ker \rho_0 \Rightarrow a^2 \in S_1^*$.

(iii) $[\rho_0]\, K[\rho_0, \overline{\rho}_0]$.

Proof. We assume first that $\rho_0 \neq \omega_0$.

(i) implies (ii). Let $a > b > 0$ and $b \in \ker \rho_0$. By Lemma 3.2(i), $a \in \ker[\rho_0] = \ker[\lambda_0, \lambda_1]$ and hence $a \in (\ker \lambda_1)^*$. Since λ_1 is proper, a must be contained in a subgroup of S_1 so $a^2 \in S_1^*$.

(ii) implies (iii). By Lemma 3.7(i), we have $[\rho_0] \wedge [\omega_0, \zeta_1] = [\rho_0, \overline{\rho}_0]$ and hence $(\ker \overline{\rho}_0)^* \subseteq \ker[\rho_0] \cap S_1^*$. Conversely, let $a \in \ker[\rho_0] \cap S_1^*$. By Lemma 3.2(i), $a > b > 0$ for some $b \in \ker \rho_0$ and hence, by hypothesis, $a^2 \in S_1^*$. Thus $a\,\mathcal{H}\,e$ for some $e \in E(S_1^*)$. Letting $\lambda = [\rho_0]|_{H_e}$, we obtain a congruence λ on the group H_e such that $a\,\lambda\, a^2$ and thus $a\,\lambda\, e$. Therefore $a\,[\rho_0]\, e$.

Now let $x, y \in S_1^1$, $u\,\rho_0\, v$. Then $(xay)u\,[\rho_0]\,(xey)v$ and $u(xay)\,[\rho_0]\, v(xey)$ so that $(xay)u\,\rho_0\,(xey)v$ and $u(xay)\,\rho_0\, v(xey)$ which shows that $xay\,\rho_0'\, xey$ and thus $a\,\overline{\rho}_0\, e$. Therefore $a \in \ker \overline{\rho}_0$ which proves that $\ker[\rho_0] \cap S_1^* \subseteq (\ker \overline{\rho}_0)^*$ and equality prevails. Consequently $[\rho_0]\, K[\rho_0, \overline{\rho}_0]$ in view of Lemma 3.2(i)(iii).

(iii) implies (i). This is trivial.

We now consider the case $\rho_0 = \omega_0$. Then $[\rho_0] = \omega$. If (i) holds, then $\ker[\lambda_0, \lambda_1] = S$ implies that S_1 has no zero divisors and (ii) holds. If (ii) holds, again S_1 has no zero divisors so $\omega\, K[\omega_0, \overline{\omega}_0]$ where $\overline{\omega}_0 = \zeta_1$ and (iii) holds. Trivially (iii) implies (i).

We are now ready for the first principal result of the paper.

THEOREM 4.2.

(i) *For* $[\rho_0, \rho_1] \in \mathcal{C}(S)$ *and* $P \in \{K, T\}$, *let*

$$\langle \rho_0, \rho_1; P\rangle = \cap\big\{\theta_0 \in \mathcal{C}(S_0) \mid \theta_0\, P\, \rho_0;\, a\,(\rho_1)_P\, b, x\,\theta_0\, y \Rightarrow ax\,\theta_0\, by, xa\,\theta_0\, yb\big\}. \tag{6}$$

Then $[\rho_0, \rho_1]_P = \big[\langle \rho_0, \rho_1\colon P\rangle, (\rho_1)_P\big]$.

(ii) *For* $[\rho_0] \in \mathcal{C}(S)$, *we use the notation*

(N) *if* $a > b > 0, b \in \ker \rho_0$, *then* $a^2 \in S_1^*$.

Then

$$[\rho_0]_K = \begin{cases} \big[\langle \rho_0, \overline{\rho}_0; K\rangle, (\overline{\rho}_0)_K\big] & \text{if } (N) \text{ holds,}\\ \big[(\rho_0)_K \vee \kappa_0\big] & \text{otherwise,}\end{cases}$$

$$[\rho_0]_T = \big[(\rho_0)_T\big].$$

Proof. (i) Let $\rho = [\rho_0, \rho_1]$. By Theorem 3.2 of [**3**], we have

$$\rho_K = \big\{(a, a^2) \in S \times S \mid a\,\rho\, a^2\big\}^*, \tag{7}$$

$$\rho_T = \big\{(e, f) \in E(S) \times E(S) \mid e\,\rho f\big\}^*. \tag{8}$$

First let $a, b \in S$ be such that $a\,\rho_P\, b$. Then there exists a sequence

$$a = x_1u_1y_1,\;\; x_1v_1y_1 = x_2u_2y_2,\;\; \ldots,\;\; x_nv_ny_n = b \tag{9}$$

for some $x_i, y_i \in S^1, u_i\,\rho\, v_i$, for $P = K$, either $u_i^2 = v_i$ or $u_i = v_i^2$ and for $P = T$, $u_i, v_i \in E(S)$, $i = 1, 2, \ldots n$. Notice that either $u_i\,\rho_0\, v_i$ or $u_i\,\rho_1\, v_i$, $i = 1, 2, \ldots, n$. Since $a \in S_1^*$, we have $x_1u_1y_1 \in S_1^*$ so that $x_1u_1y_1\,\rho\, x_1v_1y_1$ yields $x_1v_1y_1 \in S_1^*$. Thus $y_1, v_1 \in S_1^1$ and $u_1\,\rho_1\, v_1$. We may continue this procedure with all the elements of sequence (9). In view of formulae (7) and (8), we conclude that $a\,(\rho_1)_P\, b$. Therefore $\rho_P|_{S_1^*} \subseteq (\rho_1)_P|_{S_1^*}$ and the opposite inclusion is trivial. Consequently $\rho_P = \big[\lambda_0, (\rho_1)_P\big]$ for some $\lambda_0 \in \mathcal{C}(S_0)$.

Let $\mathcal{F}$ be the family of congruences θ on the right hand side of (6). Since $[\rho_0, \rho_1]$ is defined, we have

$$a\,\rho_1\, b,\; x\,\rho_0\, y \Rightarrow ax\,\rho_0\, by,\; dxa\,\rho_0\, yb$$

which evidently implies that

$$a\,(\rho_1)_P\, b,\; x\,\rho_0\, y \Rightarrow ax\,\rho_0\, by,\; xa\,\rho_0\, yb.$$

Hence $\rho_0 \in \mathcal{F}$ so that $\mathcal{F} \neq \phi$. Obviously $\mathcal{F}$ is closed under arbitrary intersections which implies that $\big[\langle \rho_0, \rho_1; P\rangle, (\rho_1)_P\big]$ is defined. Also, for every $\theta \in \mathcal{F}$, we have $\theta\, P\, \rho_0$ and thus $\langle \rho_0, \rho_1; P\rangle\, K\, \rho_0$. Hence $\big[\langle \rho_0, \rho_1; P\rangle, (\rho_1)_P\big]\, P\, [\rho_0, \rho_1]$. The minimality of the former is obvious from the definition of $\langle \rho_0, \rho_1; P\rangle$.

(ii) According to Lemma 4.1, condition (N) is equivalent to $[\rho_0]\, K\big[\rho_0, \overline{\rho}_0\big]$; in such a case, by part (i), we have

$$[\rho_0]_K = \big[\rho_0, \overline{\rho}_0\big]_K = \big[\langle \rho_0, \overline{\rho}_0; K\rangle, (\overline{\rho}_0)_K\big].$$

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Otherwise $[\rho_0]_K = [\lambda_0]$ for some $\lambda_0 \in \mathcal{C}(S_0)$. Hence $[\rho_0]\,K\,[\lambda_0]$; by Lemma 3.3(iii) we have $\rho_0\,K\,\lambda_0$ and thus $\lambda_0 \supseteq (\rho_0)_K$. Since $[\lambda_0]$ is defined, by Lemma 3.2(v), we get $\lambda_0 \supseteq \kappa_0$ and thus $\lambda_0 \supseteq (\rho_0)_K \vee \kappa_0$. Now

$$\ker \rho_0 = \ker \lambda_0 \supseteq \ker\big((\rho_0)_K \vee \kappa_0\big) \supseteq \ker(\rho_0)_K = \ker \rho_0$$

and equality prevails throughout. Therefore $\rho_0\,K\,(\rho_0)_K \vee \kappa_0$. By Lemma 3.2(v), $\big[(\rho_0)_K \vee \kappa_0\big]$ is defined and $[\rho_0]\,K\big[(\rho_0)_K \vee \kappa_0\big]$ by Lemma 3.3(iii). If $[\theta_0]\,K\,[\rho_0]$, then by Lemma 3.3(iii), $\theta_0\,K\,\rho_0$ so $\theta_0 \supseteq (\rho_0)_K$ and since $[\theta_0]$ is defined, we have $\theta_0 \supseteq \kappa_0$ by Lemma 3.2(v). Thus $\theta_0 \supseteq (\rho_0)_K \vee \kappa_0$ whence by Lemma 3.3(i), $[\theta_0] \supseteq \big[(\rho_0)_K \vee \kappa_0\big]$. Consequently $[\rho_0]_K = \big[(\rho_0)_K \vee \kappa_0\big]$.

The last assertion of the theorem is proved in Proposition 6.9 of [**5**].

There is a special case when ρ_K assumes a simple form. The first part of the next result generalizes the first part of Theorem 7.8(i)) of [**5**].

Proposition 4.3. *Assume that $\mathcal{H}$ is a congruence on S. If $[\rho_0, \rho_1] \in \mathcal{C}(S)$, then $[\rho_0, \rho_1]_K = \big[(\rho_0)_K, (\rho_1)_K\big]$. If $[\rho_0] \in \mathcal{C}(S)$ and condition (N) holds for ρ_0, then $[\rho_0]_K = \big[(\rho_0)_K, (\overline{\rho}_0)_K\big]$.*

Proof. Let $[\rho_0, \rho_1] \in \mathcal{C}(S)$. For the first assertion, it is evidently sufficient to prove that $\big[(\rho_0)_K, (\rho_1)_K\big]$ is defined. Hence let $a\,(\rho_1)_K\,b$ and $x\,(\rho_0)_K\,y$. Then $a\,\rho_1\,b$ and $x\,\rho_0\,y$ which by hypothesis implies that $ax\,\rho_0\,by$ and $xa\,\rho_0\,yb$. Suppose that ρ_0 is proper. By Lemma 2.1(iii), we have $(\rho_i)_K = \rho_i \cap \mathcal{H}_i$ for $i = 0, 1$ and thus $a\,\mathcal{H}_1\,b$ and $x\,\mathcal{H}_0\,y$. Hence $a\,\mathcal{H}\,b$ and $x\,\mathcal{H}\,y$ in S and the hypothesis implies that $ax\,\mathcal{H}\,by$ and $xa\,\mathcal{H}\,yb$. Therefore $ax\,\rho_0 \wedge \mathcal{H}_0\,by$ and $xa\,\rho_0 \wedge \mathcal{H}_0\,yb$. In view of Lemma 2.1(iii), we get

$$(\rho_0)_K = \begin{cases} \omega_0 & \text{if } S_0 \text{ has zero divisors and } \rho_0 = \omega_0, \\ \rho_0 \wedge \mathcal{H}_0 & \text{otherwise.} \end{cases}$$

Thus in any case $ax\,(\rho_0)_K\,by$ and $xa\,(\rho_0)_K\,yb$, as required. This establishes the first assertion.

Now assume that $[\rho_0] \in \mathcal{C}(S)$. By Lemma 4.1 and Theorem 4.2, the validity of (N) implies that $[\rho_0]\,K\big[\rho_0, \overline{\rho}_0\big]$. Applying the first statement now gives the second assertion.

5. A construction of ρ^K and ρ^T.

The analysis here is quite parallel to that in the preceding section.

Lemma 5.1. *The following conditions on $\rho = [\rho_0, \rho_1] \in \mathcal{C}(S)$ are equivalent.*

(i) $\rho\,K\,[\lambda_0]$ *for some* $[\lambda_0] \in \mathcal{C}(S)$.

(ii) $\kappa_0 \subseteq \rho_0^K$; $a > b > 0$, $b \in \ker \rho_0 \Rightarrow a \in \ker \rho_1$.

(iii) $\rho^K = \big[\rho_0^K\big]$.

Proof. (i) implies (ii). The hypothesis that $\rho\,K\,[\lambda_0]$ by Lemma 3.4(iii) implies that $\rho_0\,K\,\lambda_0$ and thus $\rho_0^K \supseteq \lambda_0$. Since $[\lambda_0]$ is defined, Lemma 3.2(v) implies that $\lambda_0 \supseteq \kappa_0$. Therefore $\rho_0^K \supseteq \kappa_0$. Let $a > b > 0$ and $b \in \ker \rho_0$. Then $b \in \ker \lambda_0$ and in view of Lemma 3.2(iii), we have $a \in \ker[\lambda_0]$. Hence $a \in \ker[\rho_0, \rho_1]$ so that $a \in \ker \rho_1$ by Lemma 3.1(i).

(ii) implies (iii). Since $\kappa_0 \subseteq \rho_0^K$, Lemma 3.2(v) implies that $[\rho_0^K]$ is defined. Note that $\rho|_{S_0} = \rho_0 \, K \, \rho_0^K = [\rho_0^K]|_{S_0}$ by Lemma 3.2(i). If $a \in (\ker \rho_1)^*$, by [**5**, Lemma 2.2], there exists $b \in S_0$ such that $a > b > 0$, whence $b \in \ker \rho_0$ by Lemma 3.1(ii) and thus $a \in \ker [\rho_0^K]$ by Lemma 3.2(i). Conversely, if $a \in \ker [\rho_0^K] \cap S_1^*$, then by Lemma 3.2(i), there exists $b \in S_0$ such that $a > b > 0$, which by hypothesis implies that $a \in (\ker \rho_1)^*$. Therefore $\rho \, K [\rho_0^K]$. It follows that $\rho^K = [\rho_0^K]^K$ and since by Lemma 3.2(iv), $[\rho_0^K]^K = [\rho_0^K]$, we obtain $\rho^K = [\rho_0^K]$.

(iii) implies (i). This is trivial.

We can now prove the second principal result of the paper.

THEOREM 5.2. *Let $P \in \{K, T\}$.*

(i) *For $[\rho_0] \in \mathcal{C}(S)$, we have $[\rho_0]^P = [\rho_0^P]$.*

(ii) *For $[\rho_0, \rho_1] \in \mathcal{C}(S)$, we use the notation*

(M) $\kappa_0 \subseteq \rho_0^K$; $a > b > 0, b \in \ker \rho_0 \Rightarrow a \in \ker \rho_1$.

Then $[\rho_0, \rho_1]^K = [\rho_0^K]$ if (M) holds and

$$[\rho_0, \rho_1]^P = \left[\rho_0^P, \overline{\rho_0^P} \wedge \rho_1^P\right]$$

for $P = K$ when (M) fails or $P = T$.

Proof. (i) For $P = K$, this is Lemma 3.2(iv) and for $P = T$, this is Proposition 6.9 of [**5**].

(ii) If condition (M) holds, the assertion follows from Lemma 5.1. We consider first the case $P = K$ when (M) fails. Let $\rho = [\rho_0, \rho_1]$ and observe that $\rho^K = [\lambda_0, \lambda_1]$ for some λ_0, λ_1. By Lemma 3.5(iii), we have $\rho_0 \, K \, \lambda_0$ so that $\rho_0 \subseteq \lambda_0 \subseteq \rho_0^K$. Also, by the definition of $[\lambda_0, \lambda_1]$, it holds

$$a \, \lambda_1 \, b, \; x \, \lambda_0 \, y \Rightarrow ax \, \lambda_0 \, by, \; xa \, \rho_0 \, yb$$

which then implies

$$a \, \lambda_1 \, b, \; x \in S_0 \Rightarrow ax \, \rho_0^K \, bx, \; xa \, \rho_0^K \, yb.$$

By Lemma 2.5, S_0/ρ_0^K is a reductive semigroup which by Lemma 3.6 implies that $[\rho_0^K, \lambda_1]$ is defined. Again by Lemma 3.5(iii), we have $[\rho_0^K, \lambda_1] \, K \, \rho$ and by Lemma 3.5(i), $[\lambda_0, \lambda_1] \subseteq [\rho_0^K, \lambda_1]$ which by the maximality of $[\lambda_0, \lambda_1]$ implies that $[\lambda_0, \lambda_1] = [\rho_0^K, \lambda_1]$ whence $\lambda_0 = \rho_0^K$.

We now have $\rho^K = [\rho_0^K, \lambda_1]$ so that, by Lemma 3.5(iii), we get $\rho_1 \, K \, \lambda_1$ and thus $\rho_1 \subseteq \lambda_1 \subseteq \rho_1^K$. Since $[\rho_0^K, \lambda_1]$ is defined, Lemma 3.7(ii) implies that $\lambda_1 \subseteq \overline{\rho_0^K}$ and thus $\lambda_1 \subseteq \overline{\rho_0^K} \wedge \rho_1^K$. It follows that

$$\ker \lambda_1 \subseteq \ker (\overline{\rho_0^K} \wedge \rho_1^K) = \ker \overline{\rho_0^K} \cap \ker \rho_1^K = \ker \overline{\rho_0^K} \cap \ker \lambda_1 \subseteq \ker \lambda_1$$

and thus $\lambda_1 \, K \, \overline{\rho_0^K} \wedge \rho_1^K$. Further

$$a \, \overline{\rho_0^K} \wedge \rho_1^K \, b, \; x \, \rho_0^K \, y \Rightarrow ax \, \rho_0^K \, by, \; xa \, \rho_0^K \, yb,$$

and thus $\left[\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K\right]$ is defined. By Lemma 3.5(iii), we have

$$[\rho_0^K, \lambda_1]\, K \left[\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K\right]$$

and by Lemma 3.5(i) that $[\rho_0^K, \lambda_1] \subseteq [\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K]$ since $\lambda_1 \subseteq \overline{\rho_0^K} \wedge \rho_1^K$. By the maximality of $[\rho_0^K, \lambda_1]$, we conclude that $[\rho_0^K, \lambda_1] = [\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K]$. Therefore $\lambda_1 = \overline{\rho_0^K} \wedge \rho_1^K$, as asserted.

We now consider the case $P = T$. Let $[\rho_0, \rho_1]^T = [\lambda_0, \lambda_1]$. By Lemma 3.5(iii), we have $\rho_0\, T\, \lambda_0$ and $\rho_1\, T\, \lambda_1$. Hence $\lambda_0 \subseteq \rho_0^T$. By ([**3**], Theorem 3.2), we have

$$\rho^T = (\mathcal{L}\tau\mathcal{L}\tau\mathcal{L} \cap \mathcal{R}\tau\mathcal{R}\tau\mathcal{R})^0 \tag{10}$$

where $(\ )^0$ means the greatest congruence contained in $(\ )$ and $\tau = \operatorname{tr}\rho$. Applying this to ρ_0 and letting $\tau_0 = \operatorname{tr}\rho_0$, we get that $\tau_0 \subset \tau$ and thus $\rho_0^T \subset \rho^T$. Hence $\rho_0^T \subseteq \rho^T|_{S_0} = \lambda_0$. Therefore $\lambda_0 = \rho_0^T$.

Now both $[\rho_0^T, \lambda_1]$ and $[\rho_0^T, \overline{\rho_0^T}]$ are defined which by Lemma 3.7(ii) gives that $\lambda_1 \subseteq \overline{\rho_0^T}$. By Lemma 3.5(iii), we have $\rho_1\, T\, \lambda_1$ which implies that $\lambda_1 \subseteq \rho_1^T$ and thus $\lambda_1 \subseteq \overline{\rho_0^T} \wedge \rho_1^T$. Hence

$$\operatorname{tr}\lambda_1 \subseteq \operatorname{tr}\overline{\rho_0^T} \cap \operatorname{tr}\rho_1^T \subseteq \operatorname{tr}\rho_1^T = \operatorname{tr}\rho_1 = \operatorname{tr}\lambda_1$$

and thus $\lambda_1\, T\, \overline{\rho_0^T} \wedge \rho_1^T$. By Lemma 3.5(iii), it follows that

$$[\rho_0^T, \lambda_1]\, T \left[\rho_0^T, \overline{\rho_0^T} \wedge \rho_1^T\right]$$

where the latter is defined since $\overline{\rho_0^T} \wedge \rho_1^T \subseteq \overline{\rho_0^T}$. Also

$$[\rho_0^T, \lambda_1] \subseteq \left[\rho_0^T, \overline{\rho_0^T} \wedge \rho_1^T\right]$$

which by the maximality of the former yields

$$[\rho_0^T, \lambda_1] = \left[\rho_0^T, \overline{\rho_0^T} \wedge \rho_1^T\right].$$

Thus $\lambda_1 = \overline{\rho_0^T} \wedge \rho_1^T$, as asserted.

There is a special case when ρ^T assumes a simple form. The next result generalizes the second part of ([**5**], Theorem 7.8(i)).

Proposition 5.3. *Assume that $\mathcal{H}$ is a congruence on S and let $[\rho_0, \rho_1] \in \mathcal{C}(S)$. Then $[\rho_0, \rho_1]^T = \left[\rho_0^T, \rho_1^T\right]$.*

Proof. In view of Theorem 5.2(ii), it suffices to prove that $\left[\rho_0^T, \rho_1^T\right]$ is defined. Hence let $a\, \rho_1^T\, b$, $x\, \rho_0^T\, y$, $\rho_1 = \mathcal{C}(r_1, N_1, \pi_1)$ (see Section 2), $a = (i, g, \lambda)$, $b = (j, h, \mu)$, $p_{\theta i} \neq 0$ and $p_{\lambda q} \neq 0$. Then $i\, r_1\, j$ and $\lambda\, \pi_1\, \mu$. Since

$$p_{\theta i}(p_{\theta i}^{-1} p_{\lambda q}^{-1}) p_{\lambda q} N_1 = p_{\theta j}(p_{\theta j}^{-1} p_{\mu q}^{-1}) p_{\mu q} N_1$$

letting $a' = (i, p_{\theta i}^{-1} p_{\lambda q}^{-1}, \lambda)$ and $b' = (j, p_{\theta j}^{-1} p_{\mu q}^{-1}, \mu)$, we obtain $a\, \mathcal{H}\, a'$, $b\, \mathcal{H}\, b'$, $a'\, \rho_1\, b'$. Analogously, there exist x', y' such that $x\, \mathcal{H}\, x'$, $y\, \mathcal{H}\, y'$, $x'\, \rho_0\, y'$.

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Since $\mathcal{H}$ is a congruence and $[\rho_0, \rho_1]$ is defined, we get

$$ax\,\mathcal{H}\,a'x'\,\rho_0\,b'y'\,\mathcal{H}\,by$$

which by Lemma 2.1(vi) implies that $ax\,\rho_0^T\,by$. Similarly, we obtain that $xa\,\rho_0^T\,yb$. Consequently $\left[\rho_0^T, \rho_1^T\right]$ is defined.

Corollary 5.4. *Assume that $\mathcal{H}$ is a congruence on S. Then the mapping $\rho \to \rho^T (\rho \in \mathcal{C}(S))$ is a homomorphism.*

Proof. This follows easily from [**3**, Theorem 4.13], Corollary 2.2, Theorem 5.2.(i) and Proposition 5.3.

6. When is K a congruence? We first prove a proposition which specifies precisely when there exist no K-related congruences of different "types". This is followed by necessary and sufficient conditions on S, in terms of $\ker\kappa$, in order that K be a congruence on $\mathcal{C}(S)$.

Proposition 6.1. *The following conditions are equivalent.*

(i) *For no $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$, do we have $[\lambda_0]\,K[\rho_0, \rho_1]$.*
(ii) *For every $[\rho_0] \in \mathcal{C}(S)$, $[\rho_0]_K = \left[(\rho_0)_K\right]$.*
(iii) *For every $[\rho_0, \rho_1] \in \mathcal{C}(S)$, we have $[\rho_0, \rho_1]^K = \left[\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K\right]$.*
(iv) $\kappa_K = \kappa$.
(v) *There exists $x \in \ker\kappa \cap S_1^*$ such that $x^2 \in S_0$.*

Proof. Assume that (i) holds. Then for $[\rho_0] \in \mathcal{C}(S)$, we must have $[\rho_0]_K = [\lambda_0]$ for some λ_0 and hence Theorem 4.2(ii) implies that $[\rho_0]_K = \left[(\rho_0)_K\right]$. Also, for $[\rho_0, \rho_1] \in \mathcal{C}(S)$, we have $[\rho_0, \rho_1]^K = [\lambda_0, \lambda_1]$ for some λ_0, λ_1 and hence by Theorem 5.2(iii), we get

$$[\rho_0, \rho_1]^K = \left[\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K\right].$$

Since $\kappa = [\kappa_0]$, we have by part (ii) that $\kappa_K = \left[(\kappa_0)_K\right]$. Now $\left[(\kappa_0)_K\right]$ being defined by Lemma 3.2(v) implies that $\kappa_0 \subseteq (\kappa_0)_K$ whence $\kappa_0 = (\kappa_0)_K$. But then $\kappa = \kappa_K$. Therefore parts (ii), (iii) and (iv) hold.

Suppose that (i) fails, say $[\lambda_0]\,K[\rho_0, \rho_1]$. Then $[\lambda_0]_K \subseteq [\rho_0, \rho_1]$ and hence $[\lambda_0]_K \neq \left[(\lambda_0)_K\right]$ even if the latter is defined. Also $[\rho_0, \rho_1]^K \supseteq [\lambda_0]$ so that

$$[\rho_0, \rho_1]^K \neq \left[\rho_0^K, \overline{\rho_0^K} \wedge \rho_1^K\right]$$

even if the latter is defined. Further $\kappa \wedge [\lambda_0]\,K\,\kappa \wedge [\rho_0, \rho_1]$ which by Lemma 3.2(v) yields $\kappa\,K[\lambda_0, \lambda_1]$ for some λ_0, λ_1 and hence $\kappa_K = [\theta_0, \theta_1]$ for some θ_0, θ_1. Therefore parts (ii), (iii) and (iv) fail.

Assume that (iv) fails. As we have seen above, we must have $\kappa_K = [\rho_0, \rho_1]$ for some ρ_0, ρ_1. If now $x \in \ker\kappa \cap S_1^*$, then by Lemmas 3.1(i) and 3.2(i), $x \in (\ker\rho_1)^*$ so that $x^2 \in S_1^*$ since ρ_1 is proper. Hence (v) fails.

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Finally suppose that (iv) holds. By Theorem 4.2(ii), condition (N) fails for κ and hence there exist $x > y > 0$ such that $x^2 \in S_0$ and $y \in \ker \kappa_0$. By Lemma 3.2(ii), we have $x \kappa y$ so that $x \in \ker \kappa \cap S_1^*$. Therefore (v) holds.

The following theorem has as a crude precedent ([**5**, Theorem 6.4]) and special cases ([**5**], Theorems 7.6, 7.9 and 8.2).

THEOREM 6.2. *The kernel relation K is a congruence on $\mathcal{C}(S)$ if and only if either S_1 has no zero divisors and $S_1^* \subset \ker \kappa$ or there exists $x \in \ker \kappa \cap S_1^*$ such that $x^2 \in S_0$.*

Proof. Necessity. Suppose first that S_1 has no zero divisors. By Theorem 8.2 of [**5**] we have $A \subseteq \ker \kappa$, where

$$A = \left\{ b \in S_0^* \mid \text{there exists } a \in S_1^* \text{ such that } a > b \right\}.$$

Let $a \in S_1^*$. By [**5**, Lemma 2.2], there exists $b \in S_0$ such that $a > b > 0$. Hence $b \in A$ so $b \in \ker \kappa$ and thus, by Lemma 3.2(i)(iii), $a \in \ker \kappa$. Therefore $S_1^* \subset \ker \kappa$.

Assume next that S_1 has zero divisors. We consider two cases.

Case 1: $[\lambda_0]\, K [\rho_0, \rho_1]$ for some $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$. It follows that

$$[\lambda_0] \vee [\omega_0, \varepsilon_1]\, K [\rho_0, \rho_1] \vee [\omega_0, \varepsilon_1]$$

which by Lemmas 3.4(i) and 3.5(ii) gives $\omega\, K [\omega_0, \rho_1]$ so that $(\ker \rho_1)^* = S_1^*$ and thus $\ker \rho_1 = S_1$. Since ρ_1 is proper, this implies that S_1 has no zero divisors, contrary to the hypothesis. Therefore this case can not occur.

Case 2: $[\lambda_0]\, K [\rho_0, \rho_1]$ for no $[\lambda_0], [\rho_0, \rho_1] \in \mathcal{C}(S)$. In this case, the assertion follows from Proposition 6.1.

Sufficiency. If S_1 has no zero divisors, the argument above shows that $A \subset \ker \kappa$ which by [**5**, Theorem 8.2] yields that K is a congruence. If S_1 has zero divisors, then by Proposition 6.1, we have that $[\lambda_0]\, K [\rho_0, \rho_1]$ never occurs which by Theorem 6.4 of [**5**] implies that K is a congruence.

REMARK 6.3. Recall that ζ_1 denotes the greatest proper congruence on S_1. It is easily seen that $\ker \zeta_1$ is the union of all (maximal) subgroups of S_1. Letting $A = (\ker \zeta_1)^*$ and $B = S_1^* \cap \ker \kappa$, we may paraphrase Theorem 6.2 succinctly as follows.

$$K = K^* \Leftrightarrow \begin{cases} A \subseteq B & \text{if } S_1 \text{ has no zero divisors,} \\ B \nsubseteq A & \text{if } S_1 \text{ has zero divisors.} \end{cases}$$

7. When is K a congruence and $\mathcal{C}(S)/K$ is modular? A sequence of eight lemmas leads to necessary and sufficient conditions on S, in terms of $\ker \zeta_1$ and $\ker \kappa$, in order that K be a congruence and $\mathcal{C}(S)/K$ be modular.

Our first lemma is of general interest.

LEMMA 7.1 *Let R be a regular semigroup for which K is a congruence. Then $\mathcal{C}(R)/K$ is modular if and only if for any $\lambda, \rho, \theta \in \mathcal{C}(R)$,*

$$\ker \lambda \subseteq \ker \rho, \quad \ker (\lambda \wedge \theta) = \ker (\rho \wedge \theta), \quad \ker (\lambda \vee \theta) = \ker (\rho \vee \theta) \qquad (11)$$

implies $\ker \lambda = \ker \rho$.

Proof. First

$$\ker\lambda \subseteq \ker\rho \Leftrightarrow \ker\lambda = \ker(\lambda\wedge\rho) \Leftrightarrow \lambda\, K\lambda\wedge\rho \Leftrightarrow \lambda K \leq \rho K$$

and relations (11) can be written, since K is a congruence, as

$$\lambda K \leq \rho K, \quad \lambda K\wedge\theta K = \rho K\wedge\theta K, \quad \lambda K\vee\theta K = \rho K\vee\theta K$$

and it is well known that this implies $\lambda K = \rho K$ if and only if $\mathcal{C}(R)/K$ is modular.

Our second lemma pertains to congruences on a completely 0-simple semigroup.

LEMMA 7.2. *Let* $C = \mathcal{M}^0(I, G, \Lambda; P)$ *and for* $\lambda, \rho, \theta \in \mathcal{C}(C)$ *assume relations* (11). *Then* $\ker\lambda = \ker\rho$.

Proof. If $\theta = \omega$, then $\ker\lambda = \ker\rho$ directly. If $\rho = \omega$, then $\ker\theta \subseteq \ker\lambda$ and, in view of [**4**], Proposition 2.2 and Lemma 3.2, $\ker\lambda\vee\ker\theta = C$ and thus $\ker\lambda = C = \ker\rho$.

It remains to consider the case when all three congruences λ, ρ and θ are proper. As in Section 2, we have

$$\lambda = \mathcal{C}(r, N, \pi), \quad \rho = \mathcal{C}(r', N', \pi'), \quad \theta = \mathcal{C}(r'', N'', \pi'')$$

for some admissible triples. Since

$$\lambda\wedge\theta = \mathcal{C}(r\cap r'', N\cap N'', \pi\cap\pi''), \quad \lambda\vee\theta = \mathcal{C}(r\vee r'', NN'', \pi\vee\pi'')$$

and similarly for $\rho\wedge\theta$ and $\rho\vee\theta$, in the light of Lemma 2.1(i)(ii), relations (11) imply

$$N \subseteq N', \quad N\cap N'' = N'\cap N'', \quad NN'' = N'N''.$$

Now modularity of the lattice of normal subgroups of G yields that $N = N'$. By Lemma 2.1(ii), we get $\ker\lambda = \ker\rho$.

COROLLARY 7.3. *For any completely 0-simple semigroup* C, K *is a congruence on* $\mathcal{C}(C)$ *and* $\mathcal{C}(C)/K$ *is a modular lattice*.

Proof. The first assertion is the content of [**4**, Lemma 3.2]. The second assertion is a consequence of Lemma 7.2.

We now go back to our extension S of completely 0-simple semigroups S_0 and S_1. In Lemmas 7.4–7.9, we let $\lambda, \rho, \theta \in \mathcal{C}(C)$ with either $\tau = [\tau_0]$ or $\tau = [\tau_0, \tau_1]$ where $\tau \in \{\lambda, \rho, \theta\}$, κ be as before and ζ_1 be the greatest proper congruence on S_1. Note that $(\ker\zeta_1)^*$ is the union of all nonzero (maximal) subgroups of S_1.

LEMMA 7.4.

(i) $\ker[\lambda_0] \subseteq \ker[\rho_0] \Leftrightarrow \ker\lambda_0 \subseteq \ker\rho_0$.

(ii) $\ker[\lambda_0, \lambda_1] \subseteq \ker[\rho_0, \rho_1] \Leftrightarrow \ker\lambda_i \subseteq \ker\rho_i$, $i = 0, 1$.

(iii) $\ker[\lambda_0, \lambda_1] \subseteq \ker[\rho_0] \Leftrightarrow (\ker\lambda_0 \subseteq \ker\rho_0;\ a \in (\ker\lambda_1)^*,\ a > b > 0 \Rightarrow b \in \ker\rho_0)$.

(iv) $\ker[\lambda_0] \subseteq \ker[\rho_0, \rho_1] \Leftrightarrow (\ker\lambda_0 \subseteq \ker\rho_0;\ b \in \ker\lambda_0,\ a > b > 0 \Rightarrow a \in \ker\rho_1)$.

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Proof. This follows easily from Lemmas 3.1(i) and 3.2(i).

LEMMA 7.5. *For* $\tau \in \{\lambda, \rho, \theta\}$, *let* $\tau_0 = \tau|_{S_0}$. *Then relations (11) imply that* $\ker \lambda_0 = \ker \rho_0$.

Proof. In view of Lemmas 3.1(i) and 3.2(i), relations (11) yield $\ker \lambda_0 \subseteq \ker \rho_0$, $\ker(\lambda_0 \wedge \theta_0) = \ker(\rho_0 \wedge \theta_0)$, $\ker(\lambda_0 \vee \theta_0) = \ker(\rho_0 \vee \theta_0)$ which by Lemma 7.2 gives $\ker \lambda_0 = \ker \rho_0$.

LEMMA 7.6. *Assume relations (11) in the following cases:*
(i) $\lambda = [\lambda_0], \rho = [\rho_0], \theta = [\theta_0]$,
(ii) $\lambda = [\lambda_0], \rho = [\rho_0], \theta = [\theta_0, \theta_1]$,
(iii) $\lambda = [\lambda_0, \lambda_1], \rho = [\rho_0, \rho_1], \theta = [\theta_0, \theta_1]$.
Then $\ker \lambda = \ker \rho$.

Proof. (i) This follows easily from Lemmas 3.2(i), 3.3(i)(ii), 7.2 and 7.4(i).
(ii) This follows easily from Lemmas 3.1(i), 3.2(i), 3.4(i)(ii), 7.2 and 7.4(i).
(iii) This follows easily from Lemmas 3.2(i), 3.5(i)(ii), 7.2 and 7.4(ii).

LEMMA 7.7. *Assume that K is a congruence on $\mathcal{C}(S)$ and that relations (11) hold in the following cases:*
(i) $\lambda = [\lambda_0, \lambda_1], \rho = [\rho_0], \theta = [\theta_0]$,
(ii) $\lambda = [\lambda_0, \lambda_1], \rho = [\rho_0], \theta = [\theta_0, \theta_1]$,
(iii) $\lambda = [\lambda_0], \rho = [\rho_0, \rho_1], \theta = [\theta_0]$,
(iv) $\lambda = [\lambda_0], \rho = [\rho_0, \rho_1], \theta = [\theta_0, \theta_1]$.
Then S_1 has no zero divisors and $\ker \lambda = \ker \rho$.

Proof. (i) The second relation in (11) yields

$$\ker([\lambda_0, \lambda_1] \wedge [\theta_0]) = \ker([\rho_0] \wedge [\theta_0])$$

which by Lemmas 3.3(ii) and 3.4(ii) gives

$$\left[\lambda_0 \wedge \theta_0, \lambda_1 \wedge \overline{\theta}_0\right] K [\rho_0 \wedge \theta_0].$$

Now Theorem 6.2 and Proposition 6.1 imply that S_1 has no zero divisors. Further, the second relation in (11), by Lemmas 3.1(i) and 3.2(i), yields

$$\ker(\lambda_1 \wedge \overline{\theta}_0) = \left\{a \in S_1 \mid a > b > 0, b \in \ker(\rho_0 \wedge \theta_0)\right\}. \tag{12}$$

By Lemma 3.2(v), we have $\kappa_0 \subseteq \rho_0 \wedge \theta_0$ and thus by Lemma 7.4(i), $\ker \kappa \subseteq \ker[\rho_0 \wedge \theta_0]$. Since K is a congruence and S_1 has no zero divisors, Theorem 6.2 implies that $S_1^* \subseteq \ker[\rho_0 \wedge \theta_0]$. In view of Lemma 3.7(i), we obtain that the right hand side of (12) equals S_1. Now by (12), we have $\ker(\lambda_1 \wedge \overline{\theta}_0) = S_1$ and thus $\ker \lambda_1 = S_1$. By Lemmas 3.1(i)(ii), 3.7(i) and 7.5, we get

$$\ker \lambda = \ker \lambda_0 \cup (\ker \lambda_1)^* = \ker \rho_0 \cup S_1^* = \ker \rho,$$

as required.

(ii) Similarly as above, we obtain that S_1 has no zero divisors and

$$\ker(\lambda_1 \wedge \theta_1) = \ker(\overline{\rho}_0 \wedge \theta_1), \quad \ker(\lambda_1 \wedge \theta_1) = S_1 = \ker \overline{\rho}_0. \tag{13}$$

From the first and the last equalities in (13), we obtain $\ker \theta_1 \subseteq \ker \lambda_1$. By [**4**] (Proposition 2.2) and the second equality in (13), we get $\ker \lambda_1 \vee \ker \theta_1 = S_1$ which together with $\ker \theta_1 \subseteq \ker \lambda_1$ yields $\ker \lambda_1 = S_1$. Now

$$\ker \lambda = \ker \lambda_0 \cup S_1^* = \ker \rho_0 \cup S_1^* = \ker \rho,$$

as required.

(iii) Similarly as above, we obtain that S_1 has no zero divisors. The second relation in (11) implies that $\ker \rho_1 = S_1$ and thus

$$\ker \lambda = \ker \lambda_0 \cup S_1^* = \ker \rho_0 \cup S_1^* = \ker \rho. \tag{14}$$

(iv) Again, as above it follows that S_1 has no zero divisors. In addition, $\ker \rho_1 = \ker \overline{\lambda}_0 = S_1$ which as in (14) implies that $\ker \lambda = \ker \rho$.

Lemma 7.8. *Let $\lambda = [\lambda_0, \lambda_1]$, $\rho = [\rho_0, \rho_1]$, $\theta = [\theta_0]$, and relations (11) be satisfied.*

(i) *Let S_1 have no zero divisors and K be a congruence on $\mathcal{C}(S)$. Then $\ker \lambda = \ker \rho$.*

(ii) *Let S_1 have zero divisors. Then $\ker \lambda_1 \subseteq \ker \rho_1$ and $\ker(\lambda_1 \wedge \overline{\theta}_0) = \ker(\rho_1 \wedge \overline{\theta}_0)$.*

Proof. (i) By Theorem 6.2, we have $S_1^* \subset \ker \kappa$ and by Lemmas 3.2(v) and 7.4(i), we have $\ker \kappa \subseteq \ker[\theta_0]$ so that $S_1^* \subset \ker[\theta_0]$. In view of Lemma 3.7(i), we get $S_1 \subseteq \ker \overline{\theta}_0$ which by the second relation in (11) yields $\ker \lambda_1 \cap S_1 = \ker \rho_1 \cap S_1$ so that $\ker \lambda_1 = \ker \rho_1$. Now Lemmas 7.5 and 3.5(iii) imply that $\ker \lambda = \ker \rho$.

(ii) By Lemma 7.4(ii), the first relation in (11) yields $\ker \lambda_1 \subseteq \ker \rho_1$. In view of Lemmas 3.4(ii) and 3.5(iii), the second relation in (11) gives $\ker(\lambda_1 \wedge \overline{\theta}_0) = \ker(\rho_1 \wedge \overline{\theta}_0)$.

Lemma 7.9. *Assume that S_1 has zero divisors and that K is a congruence on $\mathcal{C}(S)$. Then $(\ker \zeta_1)^* \subseteq S_1^* \cap \ker \kappa$ if and only if*

$$\begin{aligned} (C) \quad & \lambda = [\lambda_0, \lambda_1], \quad \rho = [\rho_0, \rho_1], \quad \theta = [\theta_0], \quad \ker \lambda_1 \subseteq \ker \rho_1 \\ & \ker(\lambda_1 \wedge \overline{\theta}_0) = \ker(\rho_1 \wedge \overline{\theta}_0) \Rightarrow \ker \lambda_1 = \ker \rho_1. \end{aligned}$$

Proof. Necessity. Assume the antecedent of condition (C). By Lemmas 3.7(i), 3.1(i) and 3.2(i), we obtain

$$\ker \theta_0 \cup (\ker \overline{\theta}_0)^* = (S_0 \cup (\ker \zeta_1)^*) \cap \ker[\theta_0]$$

whence

$$(\ker \overline{\theta}_0)^* = (\ker \zeta_1)^* \cap \ker[\theta_0]. \tag{15}$$

Now

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$$
\begin{aligned}
(\ker \rho_1)^* &= (\ker (\rho_1 \wedge \zeta_1))^* && \text{since } \rho_1 \subseteq \zeta_1 \\
&= (\ker \rho_1)^* \cap (\ker \zeta_1)^* \\
&= (\ker \rho_1)^* \cap (S_1^* \cap \ker \kappa) && \text{by hypothesis} \\
&\subseteq (\ker \rho_1)^* \cap \ker [\theta_0] && \text{by Lemma 3.2(v) and (15)} \\
&= (\ker \rho_1)^* \cap (\ker \overline{\theta}_0)^* = \ker (\rho_1 \wedge \overline{\theta}_0) \\
&= \big(\ker (\lambda_1 \wedge \overline{\theta}_0)\big) && \text{by hypothesis} \\
&\subseteq (\ker \lambda_1)^*
\end{aligned}
$$

so that $\ker \rho_1 \subseteq \ker \lambda_1$ which together with the hypothesis implies that $\ker \lambda_1 = \ker \rho_1$. Therefore (C) holds.

Sufficiency. By contrapositive, suppose that $(\ker \zeta_1)^* \not\subseteq S_1^* \cap \ker \kappa$. Let

$$\lambda_1 = (\zeta_1|_{S_1^*} \cap \kappa|_{S_1^*}) \cup \big\{(0, 0)\big\}$$

so that λ_1 is a proper congruence on S_1. Let $\lambda = [\omega_0, \lambda_1]$, $\rho = [\omega_0, \zeta_1]$ and $\theta = \kappa$. By hypothesis, we get $(\ker \lambda_1)^* \subset (\ker \zeta_1)^*$ and

$$
\begin{aligned}
(\ker (\lambda_1 \wedge \overline{\kappa}_0))^* &= (\ker \lambda_1)^* \cap (\ker \overline{\kappa}_0)^* = (\ker \zeta_1)^* \cap \ker \kappa \cap (\ker \overline{\kappa}_0)^* \\
&= (\ker \zeta_1)^* \cap (\ker \overline{\kappa}_0)^* = (\ker (\zeta_1 \cap \overline{\kappa}_0))
\end{aligned}
$$

which implies that $\ker (\lambda_1 \wedge \overline{\kappa}_0) = \ker (\zeta_1 \wedge \overline{\kappa}_0)$ and (C) fails.

We are finally ready for the theorem of this section.

THEOREM 7.10. *The kernel relation K is a congruence on $\mathcal{C}(S)$ and $\mathcal{C}(S)/K$ is a modular lattice if and only if either S_1 has no zero divisors and $S_1^* \subset \ker \kappa$ or S_1 has zero divisors and $(\ker \zeta_1)^* \subset S_1^* \cap \ker \kappa$.*

Proof. Necessity. If S_1 has no zero divisors, then $S_1^* \subset \ker \kappa$ by Theorem 6.2. If S_1 has zero divisors, then $S_1^* \cap \ker \kappa \not\subseteq (\ker \zeta_1)^*$ by Theorem 6.2, and by Lemmas 7.1, 7.8(ii) and 7.9, we get $(\ker \zeta_1)^* \subseteq S_1^* \cap \ker \kappa$.

Sufficiency. If S_1 has no zero divisors, then Theorem 6.2 implies that K is a congruence. If S_1 has zero divisors, then $S_1^* \cap \ker \kappa \not\subseteq (\ker \zeta_1)^*$ by Theorem 6.2 implies that K is a congruence. In view of Lemma 7.1, we must verify that relations (11) imply that $\ker \lambda = \ker \rho$. The cases considered in Lemmas 7.5–7.9 cover all posibilities for the congruences λ, ρ and θ, as required.

Using the notation introduced in Remark 6.3, we can express the contents of Theorem 7.10 succinctly as follows.

$$K = K^*, \mathcal{C}(S)/K \text{ is modular} \Leftrightarrow \begin{cases} A \subseteq B & \text{if } S_1 \text{ has no zero divisors,} \\ A \subset B & \text{if } S_1 \text{ has zero divisors.} \end{cases}$$

Also note that if S_1 has no zero divisors and K is a congruence, then $\mathcal{C}(S)/K$ is automatically modular.

8. Strict extensions. In both Theorems 6.2 and 7.10, the conditions are expressed in terms of $\ker \kappa$ which does not seem to lend itself to a simple explicit

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form thereby causing certain difficulty in comprehending what these conditions really mean. In order to illustrate these conditions, we shall now specialize S to be a strict extension of S_0 by S_1, that is the multiplication is determined by a partial homomorphism. To this end, we shall need the following construction.

Let

$$S_0 = \mathcal{M}^0(I_0, G_0, \Lambda_0; P), \ S_1 = \mathcal{M}^0(I_1, G_1, \Lambda_1; Q),$$
$$\zeta\colon I_1 \to I_0, \ u\colon I_1 \to G_0, \ \omega\colon G_1 \to G_0, \ \nu\colon \Lambda_1 \to G_0, \ \eta\colon \Lambda_1 \to \Lambda_0$$

be functions such that ω is a homomorphism, $u\colon i \to u_i$, $\nu\colon \lambda \to \nu_\lambda$ and $p_{\lambda i} \neq 0$ implies $p_{\lambda i}\omega = \nu_\lambda q_{\lambda\eta, i\xi} u_i$. Define a function φ by

$$\varphi\colon (i, g, \lambda) \to (i\xi, u_i(g\omega)\nu_\lambda, \lambda\eta) \quad \big((i, g, \lambda) \in S_1^*\big). \tag{17}$$

Then $\varphi\colon S_1^* \to S_0^*$ is a partial homomorphism and conversely, every partial homomorphism $S_1^* \to S_0^*$ can be so constructed. The semigroup S is a strict extension of S_0 by S_1 if there is a partial homomorphism $\varphi\colon S_1^* \to S_0^*$ which determines the multiplication in S in the sense that

$$\begin{array}{ll} ab = (a\varphi)b, ba = b(a\varphi) & \text{if } a \in S_1^*, b \in S_0, \\ ab = (a\varphi)(b\varphi) & \text{if } a, b \in S_1^*, ab = 0 \text{ in } S_1. \end{array}$$

For proofs, see [**1**] (Theorems 3.14 and 4.19). Throughout this section we assume that S is a strict extension as constructed above.

According to [**5**, Proposition 7.1], we have $\kappa_0 = \varepsilon_0$ and, by [**5**, Lemma 7.2],

$$\ker \kappa = E(S_0) \cup \big\{a \in S_1^* \mid a\varphi \in E(S_0)\big\}.$$

Now Theorem 6.2 immediately yields [**5**, Theorem 7.6] which we state as follows.

LEMMA 8.1. *The relation K is a congruence on $\mathcal{C}(S)$ if and only if either S_1 has no zero divisors and $\varphi : S_1^* \to E(S_0)$ or there exists $x \in S_1^*$ such that $x\varphi \in E(S_0)$ and $x^2 \in S_0$.*

The next result takes into account the form of the partial homomorphism φ. A homomorphism with range one element semigroup is said to be *trivial*.

PROPOSITION 8.2. *Let S be the strict extension determined by the partial homomorphism φ* in (17). *Then K is a congruence on $\mathcal{C}(S)$ if and only if either S_1 has no zero divisors and ω is trivial or S contains an element x such that $x > x^2 > 0$.*

Proof. First assume that S_1 has no zero divisors. In view of Lemma 8.1, we must show that $\varphi\colon S_1^* \to E(S_0)$ if and only if ω is trivial.

Assume that $\varphi\colon S_1^* \to E(S_0)$. For any $(i, g, \lambda) \in S_1^*$, in view of (17), we must have $u_i(g\omega)\nu_\lambda = q^{-1}_{\lambda\eta, i\xi}$ whence $g^{-1}\omega = \nu_\lambda q_{\lambda\eta, i\xi} u_i$. Since the right hand side does not depend on g, we conclude that ω is trivial.

Conversely, suppose that ω is trivial. For any $(i, g, \lambda) \in S_1^*$, we have $p_{\lambda i} \neq 0$ since S_i has no zero divisors which implies that $p_{\lambda i}\omega = \nu_\lambda q_{\lambda\eta, i\xi} u_i$. Since ω is trivial, we get $q^{-1}_{\lambda\eta, i\xi} = u_i\nu_\lambda$ which together with $g\omega = 1$ in (17) yields that $(i, g, \lambda)\varphi \in E(S_0)$, as required.

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Suppose now that S_1 has zero divisors. In view of Lemma 8.1, it suffices to prove that for $x \in S$, we have $x \in S_1^*, x\varphi \in E(S_0)$ and $x^2 \in S_0$ if and only if $x > x^2 > 0$. Note that for any $a \in S_1^*$, $b = a\varphi$ is the unique element with the property that $a > b > 0$.

If $x \in S_1^*, x\varphi = e \in E(S_0)$ and $x^2 \in S_0$, then $x^2 = (x\varphi)^2 = e^2 = e = x\varphi$ and hence $x > x\varphi = x^2 > 0$. Conversely, if $x > x^2 > 0$, then $x\varphi = x^2 \in S_0$ and there exists $f \in E(S)$ such $x^2 = fx$ whence

$$x^4 = fx^3 = f(fx)x = fx^2 = fx = x^2$$

and $x\varphi \in E(S_0)$.

Proposition 8.3. *Let S be the strict extension determined by the partial homomorphism φ in (17). Then K is a congruence on $\mathcal{C}(S)$ and $\mathcal{C}(S)/K$ is a modular lattice if and only if ω is trivial and if S_1 has zero divisors, then there exists $x \in S$ such that $x > x^2 > 0$.*

Proof. Necessity. In view of Proposition 8.2, it remains to prove that when S_1 has zero divisors, then ω is trivial. By Theorem 7.10, we have $(\ker \zeta_1)^* \subseteq S_1^* \cap \ker \kappa$. From this, the proof that ω is trivial is essentially the same as in the second paragraph of the proof of Proposition 8.2.

Sufficiency. In view of Theorem 7.10, it suffices to show that in the case that S_1 has zero divisors, ω trivial implies that $(\ker \zeta_1)^* \subseteq S_1^* \cap \ker \kappa$. Now the argument is essentially the same as in the third paragraph of the proof of Proposition 8.2.

Comparing Propositions 8.2 and 8.3 we see that the conditions in them differ only when S_1 has zero divisors in which case the triviality of ω must be added.

REFERENCES

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I. (Amer. Math. Soc., 1961).

2. J. M. Howie, *An introduction to semigroup theory*. (Academic Press, 1977).

3. F. Pastijn and M. Petrich, Congruences on regular semigroups, *Trans. Amer. Math. Soc.* **295** (1986), 607–633.

4. M. Petrich, The kernel relation for certain regular semigroups, *Boll. Un. Mat. Ital.* **7-B** (1983), 87–110.

5. M. Petrich, The congruence lattice for an extension of completely 0-simple semigroups, *Acta Math. Hung.* **64** (1994), 409–435.