Canad. Math. Bull. Vol. **48** (3), 2005 pp. 321–332

On Non-Vanishing of Convolution of Dirichlet Series

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Abstract. We study the non-vanishing on the line $\mathrm{Re}(s) = 1$ of the convolution series associated to two Dirichlet series in a certain class of Dirichlet series. The non-vanishing of various L-functions on the line $\mathrm{Re}(s) = 1$ will be simple corollaries of our general theorems.

Let $f(z) = \sum_{n=1}^{\infty} \hat{a}_f(n)e^{2\pi inz}$ and $g(z) = \sum_{n=1}^{\infty} \hat{a}_g(n)e^{2\pi inz}$ be cusp forms of weight k and level N with trivial character. Let $L_f(s) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$ and $L_g(s) = \sum_{n=1}^{\infty} a_g(n)n^{-s}$ be the L-functions associated to f and g, respectively, where $a_f(n) = \hat{a}_f(n)/n^{\frac{k-1}{2}}$ and $a_g(n) = \hat{a}_g(n)/n^{\frac{k-1}{2}}$. Let

$$L(f \otimes g, s) = \zeta_N(2s) \sum_{n=1}^{\infty} \frac{a_f(n)\overline{a_g(n)}}{n^s}$$

be the Rankin–Selberg convolution of $L_f(s)$ and $L_g(s)$. In [10] Rankin established the analytic continuation of $L(f \otimes g, s)$ (see Theorem 1.5). Rankin's Theorem has numerous number theoretic applications. In [9], Rankin used this theorem to prove the non-vanishing of the modular L-function associated to the discriminant function

$$\Delta(z) = e^{2\pi iz} \prod_{n=1}^{\infty} (1 - e^{2\pi inz})^{24}$$

on the line $\mathrm{Re}(s) = 1$. In fact, Rankin's argument establishes the non-vanishing of L-functions associated to eigenforms for the points on the line $\mathrm{Re}(s) = 1$, except the point $s = 1$. In [8], Ogg proved that the same result is true for $s = 1$. Moreover, he showed the following.

Theorem 0.1 (Ogg) *If f and g are eigenforms with respect to the family of the Hecke operators for $\Gamma_0(N)$ and $\langle f, g\rangle = 0$, then $L(f \otimes g, 1) \neq 0$. Here $\langle f, g\rangle$ denote the Petersson inner product of f and g.*

In this paper we prove similar non-vanishing results (Theorem 2.3, Theorem 3.5 and Theorem 4.2) for the convolution of two Dirichlet series belonging to a certain family of Dirichlet series $\mathcal{S}^*$ (see Definitions 1.1 and 1.2). Our theorems are quite general and clearly demonstrate the close connection between the analytic continuation of a Dirichlet series and its various convolutions to the left of its half plane of

Received by the editors October 3, 2003; revised April 4, 2004.
Research partially supported by NSERC.
AMS subject classification: 11M41.

https://doi.org/10.4153/CMB-2005-030-6 Published online by Cambridge University Press

convergence and its non-vanishing on the line $\mathrm{Re}(s) = 1$. More precisely, for two Dirichlet series F and $G \in \mathcal{S}^*$ with Euler products

$$F(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\Big) \quad \text{and} \quad G(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{b_G(p^k)}{p^{ks}}\Big)$$

valid on $\mathrm{Re}(s) > 1$, we define the *Euler product convolution* of F and G as

$$(F \otimes G)(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\Big).$$

We say $F \in \mathcal{S}^*$ is $\otimes$*-simple* in $\mathrm{Re}(s) \geq \sigma_0$, if $F \otimes F$ has an analytic continuation to $\mathrm{Re}(s) \geq \sigma_0$, except for a possible simple pole at $s = 1$. One of our main results is the following.

Theorem 2.3 *Let $F, G \in \mathcal{S}^*$ be $\otimes$-simple in* $\mathrm{Re}(s) \geq 1$ *and $t \neq 0$. Then*

(i) $(F \otimes F)(1 + it) \neq 0$.

(ii) *If $F \otimes G$ has an analytic continuation to the line* $\mathrm{Re}(s) = 1$ *and $(F \otimes G)(s) = 0$ if and only if $(F \otimes G)(\bar{s}) = 0$ for any s on the line* $\mathrm{Re}(s) = 1$*, then* $(F \otimes G)(1 + it) \neq 0$.

Note that this result does not say anything about the value of $(F \otimes G)(s)$ at $s = 1$. To deal with this case, in Section 3 we prove a non-vanishing theorem, valid on the line $\mathrm{Re}(s) = 1$, for Euler product convolution of two Dirichlet series in $\mathcal{S}^*$ with completely multiplicative coefficients (Theorem 3.5). Finally in Section 4 for Dirichlet series with general coefficients we prove the following.

Theorem 4.2 *Let $\sigma_0 < 1$, and assume the following:*

(i) *F and G (as elements of $\mathcal{S}^*$) are $\otimes$-simple in* $\mathrm{Re}(s) > \sigma_0$*;*

(ii) *$F \otimes G$ has an analytic continuation to the half-plane* $\mathrm{Re}(s) > \sigma_0$*;*

(iii) *At least one of $F \otimes F$, $G \otimes G$, or $F \otimes G$ has zeros in the strip $\sigma_0 < Re(s) < 1$.*

Then $(F \otimes G)(1 + it) \neq 0$ for all real t.

Our general theorems have several applications. The non-vanishing of various classical L-functions will be simple corollaries of our general theorems (see Corollaries 2.4, 2.6 and 4.4). Moreover, as a consequence of our theorems, we will be able to extend Ogg's theorem to the line $\mathrm{Re}(s) = 1$ (Corollary 2.6,(iv)). Another application will result in an extension of Ogg's non-vanishing result to the line $\mathrm{Re}(s) = 1$ and for eigenforms with characters (Corollary 4.4(iv)). Corollary 4.4(ii) gives a generalization of the non-vanishing result of Rankin to eigenforms with characters. Finally non-vanishing of twisted symmetric square L-functions on the line $\mathrm{Re}(s) = 1$ (Corollary 4.4(v)) is a simple consequence of our theorems. Our general theorems could also be applied to the L-functions associated to number fields, however, in applications of this paper we restrict ourselves to Dirichlet and modular L-functions. For results of these types in the context of automorphic forms and representations see [3, 11, 12].

Our approach in this paper is motivated by [7] and [5, §8.4].

Notation In this paper we use the following notations:

$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s$: the Riemann zeta-function,

$\zeta_q(s) = \prod_{p|q}(1 - 1/p)\zeta(s)$: the Riemann zeta-function with the Euler p-factors corresponding to $p \mid N$ removed,

$L_\chi(s) = \sum_{n=1}^{\infty} \chi(n)/n^s$: the Dirichlet L-function associated to a Dirichlet character χ,

$S_k(N)$: the space of cusp forms of weight k and level N with the trivial character,

$S_k(N, \psi)$: the space of cusp forms of weight k and level N with character ψ where $\psi(-1) = (-1)^k$,

$\langle f, g\rangle = \int_{D_0(N)} f(z)\overline{g(z)}y^{k-2}dxdy$: the Petersson inner product of $f, g \in S_k(N, \psi)$. Here, $D_0(N)$ is a fundamental domain for the congruence subgroup $\Gamma_0(N)$,

$L_f(s) = \sum_{n=1}^{\infty} a_f(n)/n^s$: the L-function associated to a cusp form $f \in S_k(N, \psi)$,

$L_{f,\chi}(s) = \sum_{n=1}^{\infty} a_f(n)\chi(n)/n^s$: the twisted L-function associated to a cusp form $f \in S_k(N, \psi)$ and a Dirichlet character χ,

$L(f \otimes g, s) = L_{\psi_1\bar{\psi}_2}(2s) \sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)}/n^s$: the Rankin–Selberg convolution of $L_f(s)$ and $L_g(s)$, where $f \in S_k(N, \psi_1)$ and $g \in S_k(N, \psi_2)$,

$L(\mathrm{sym}^2 f, s) = L(f \otimes f, s)/\zeta_N(s)$: the symmetric square L-function associated to a normalized eigenform f in $S_k(N)$,

$L_\chi(f \otimes g, s) = L_{\psi_1\bar{\psi}_2\chi^2}(2s) \sum_{n=1}^{\infty} a_f(n)\overline{a_g(n)}\chi(n)/n^s$: the twisted Rankin-Selberg convolution of $L_f(s)$ and $L_g(s)$, where $f \in S_k(N, \psi_1)$, $g \in S_k(N, \psi_2)$ and χ is a Dirichlet character,

$L_\chi(\mathrm{sym}^2 f, s) = L_\chi(f \otimes \bar{f}, s)/L_{\psi\chi}(s)$: the twisted symmetric square L-function associated to a normalized eigenform f with character ψ and a Dirichlet character χ.

Note that in the above definitions, we assume that $\mathrm{Re}(s) > 1$ and for a normalized eigenform f we have $a_f(1) = 1$.

1 A Class of Dirichlet Series

We consider the following class of Dirichlet Series.

Definition 1.1 The class $\mathcal{S}^{*}$[1] is the family of Dirichlet series $F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s}$ ($\mathrm{Re}(s) > 1$) satisfying the following properties:

(a) (Euler Product): For $\mathrm{Re}(s) > 1$, we have

$$F(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{b_F(p^k)}{p^{ks}}\Big).$$

[1]We use this notation to emphasize the relation of this class to the Selberg class $\mathcal{S}$. Note that $\mathcal{S} \subset \mathcal{S}^*$. For the definition of the Selberg class $\mathcal{S}$, see [5, Ch. 8].

(b) (Ramanujan's Hypothesis): For any fixed $\epsilon > 0$,

$$a_F(n) = O(n^\epsilon)$$

where the implied constant may depend upon ϵ.

(c) (Analytic Continuation): $F(s)$ has an analytic continuation to the line $\mathrm{Re}(s) = 1$, except for a possible pole at point $s = 1$.

For $F \in \mathcal{S}^*$, we define

$$\bar{F}(s) = \overline{F(\bar{s})} = \sum_{n=1}^{\infty} \frac{\overline{a_F(n)}}{n^s} = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{\overline{b_F(p^k)}}{p^{ks}}\Big).$$

We continue by defining a convolution operation on $\mathcal{S}^*$.

Definition 1.2 For $F, G \in \mathcal{S}^*$, the Euler product convolution of F and G is defined as

$$(F \otimes G)(s) = \prod_p \exp\Big(\sum_{k=1}^{\infty} \frac{k b_F(p^k)\overline{b_G(p^k)}}{p^{ks}}\Big).$$

The following lemma shows that this operation is well-defined on the half plane $\mathrm{Re}(s) > 1$.

Lemma 1.3 *For F, G in $\mathcal{S}^*$, $(F \otimes G)(s)$ is convergent for* $\mathrm{Re}(s) > 1$.

Proof See [5, Exercise 8.4.3]. ■

The next lemma will enable us to express several classical L-functions of number theory as an Euler product convolution of two simpler L-functions. This lemma plays an important role in the applications of our general theorems. The proof is straightforward.

Lemma 1.4

(i) *$\zeta(s)$ is in $\mathcal{S}^*$, and for any F in $\mathcal{S}^*$, we have*

$$(F \otimes \zeta)(s) = F(s).$$

(ii) *For F in $\mathcal{S}^*$, we have*

$$(\zeta \otimes F)(s) = \bar{F}(s).$$

(iii) *If χ is a Dirichlet character (mod q), then $L_\chi(s)$ is in $\mathcal{S}^*$, and*

$$(L_\chi \otimes L_\chi)(s) = \zeta_q(s).$$

(iv) *Let f be a normalized eigenform in $S_k(N, \psi)$. Then $L_f(s)$ is in $\mathcal{S}^*$, and*

$$(L_f \otimes L_\chi)(s) = L_{f,\overline{\chi}}(s).$$

https://doi.org/10.4153/CMB-2005-030-6 Published online by Cambridge University Press

(v) *For any two normalized eigenforms* $f \in S_k(N, \psi_1)$ *and* $g \in S_k(N, \psi_2)$ *and Dirichlet characters* χ_1 *and* χ_2, $L_{f,\chi_1}(s)$ *and* $L_{g,\chi_2}(s)$ *are in* $\mathcal{S}^*$, *and*

$$(L_{f,\chi_1} \otimes L_{g,\chi_2})(s) = L_{\chi_1 \bar{\chi}_2}(f \otimes g, s).$$

In our applications, we also need the following theorem by Rankin [10].

Theorem 1.5 (Rankin) *Let* $f \in S_k(N, \psi_1)$ *and* $g \in S_k(N, \psi_2)$. *Let*

$$\Phi(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f \otimes g, s).$$

Then both $L(f \otimes g, s)$ *and* $\Phi(s)$ *are entire if* $\psi_1 \neq \psi_2$ *or* $\langle f, g \rangle = 0$. *Otherwise, for* $N = 1$ *they are analytic everywhere except that* $L(f \otimes g, s)$ *has a simple pole at* $s = 1$ *and* $\Phi(s)$ *has simple poles at points* $s = 0$ *and* 1, *and for* $N > 1$ *both* $L(f \otimes g, s)$ *and* $\Phi(s)$ *are analytic except a simple pole at* $s = 1$.

2 Mertens' Method

In 1898 Mertens gave a proof for the non-vanishing of $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$. Mertens' proof depends upon the choice of a suitable trigonometric inequality. This line of proof is adaptable for establishing the non-vanishing of various L-functions. For example in [9], Rankin used this method to prove the non-vanishing of $L_f(s)$ on the line $\mathrm{Re}(s) = 1$, $s \neq 1$, where f is an eigenform for $\Gamma_0(N)$. Another example is the proof of the following lemma, due to K. Murty [6], which, similar to Mertens' proof, depends on a certain trigonometric inequality.

Lemma 2.1 *Let* $f(s)$ *be a complex function satisfying the following:*

(i) $f(s)$ *is analytic in* $\mathrm{Re}(s) > 1$ *and non-zero there;*

(ii) $\log f(s)$ *can be written as a Dirichlet series*

$$\sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

with $b_n \geq 0$ *for* $\mathrm{Re}(s) > 1$;

(iii) *on the line* $\mathrm{Re}(s) = 1$, $f(s)$ *is analytic except for a pole of order* $e \geq 0$ *at* $s = 1$.

Then, if $f(s)$ *has a zero on the line* $\mathrm{Re}(s) = 1$, *the order of that zero is bounded by* $e/2$.

Proof See [6, Lemma 3.2]. ■

Here by employing the above lemma we prove a conditional theorem regarding the non-vanishing of $(F \otimes G)(s)$ on the punctured line $\mathrm{Re}(s) = 1$ $(s \neq 1)$. The following definition describes one of the main conditions of our theorem.

Definition 2.2 For $F \in \mathcal{S}^*$ and $\sigma_0 \leq 1$, we say F is $\otimes$-simple in $\mathrm{Re}(s) > \sigma_0$ (resp. $\mathrm{Re}(s) \geq \sigma_0$), if $F \otimes F$ has an analytic continuation to $\mathrm{Re}(s) > \sigma_0$ (resp. $\mathrm{Re}(s) \geq \sigma_0$), except for a possible simple pole at $s = 1$.

The following theorem is the main result of this section.

Theorem 2.3 *Let $F, G \in \mathcal{S}^*$ be $\otimes$-simple in* $\mathrm{Re}(s) \geq 1$ *and $t \neq 0$. Then*

(i) $(F \otimes F)(1 + it) \neq 0$.

(ii) *If $F \otimes G$ has an analytic continuation to the line* $\mathrm{Re}(s) = 1$ *and $(F \otimes G)(s) = 0$ if and only if $(F \otimes G)(\bar{s}) = 0$ for any s on the line* $\mathrm{Re}(s) = 1$*, then* $(F \otimes G)(1 + it) \neq 0$.

Proof (i) Let $f(s) = (F \otimes F)(s)$. We have

$$\log f(s) = \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k)|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s},$$

with $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of Lemma 2.1 with $e = 1$. Therefore, the order of the vanishing of $f(s)$ at point $1 + it$ is $\leq \frac{1}{2}$. This means that $(F \otimes F)(1 + it) \neq 0$.

(ii) Let

$$f(s) = (F \otimes F)(s)\,(F \otimes G)(s)\,(G \otimes F)(s)\,(G \otimes G)(s).$$

Since for $t \neq 0$ all the factors of $f(s)$ have finite values at point $1 + it$, in order to prove that $(F \otimes G)(1 + it) \neq 0$, it suffices to show that $f(1 + it) \neq 0$. Note that

$$\log f(s) = \sum_p \sum_{k=1}^{\infty} \frac{k|b_F(p^k) + b_G(p^k)|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

with $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of Lemma 2.1 with $e \leq 2$, and therefore, the order of the vanishing of $f(s)$ at point $1 + it$ is ≤ 1. Now suppose that $f(1 + it) = 0$. Thus,

$$(F \otimes F)(1 + it)\,(F \otimes G)(1 + it)\,\overline{(F \otimes G)(1 - it)}\,(G \otimes G)(1 + it) = 0.$$

Since by part (i), $(F \otimes F)(1 + it) \neq 0$ and $(G \otimes G)(1 + it) \neq 0$, it follows that $(F \otimes G)(1 + it) = 0$. This is a contradiction, otherwise, the order of the vanishing of $f(s)$ at point $1 + it$ should be 2. ■

Note In Theorem 2.3 in fact we can have $(F \otimes G)(1) = 0$. To see this, Let $F(s) = \sum_{n=1}^{\infty} \frac{(-1)^{\Omega(n)}}{n^s}$ and $G(s) = \zeta(s)$, where $\Omega(n)$ is the total number of prime factors of n. Then we have $(F \otimes G)(s) = \frac{\zeta(2s)}{\zeta(s)}$ and so $(F \otimes G)(1) = 0$.

Corollary 2.4 *Let $f \in S_k(N, \psi)$ be a normalized eigenform for $\Gamma_0(N)$ and let $t \neq 0$. Then*

(i) $\zeta(1 + it) \neq 0$.

(ii) $L(f \otimes f, 1 + it) \neq 0$.

(iii) *For trivial ψ we have* $L(\mathrm{sym}^2 f, 1 + it) \neq 0$. *Here t is any real number including zero.*

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Proof (i) This is a consequence of Theorem 2.3(i) with $F(s) = \zeta(s)$.

(ii) From Lemma 1.4(v) we have $(L_f \otimes L_f)(s) = L(f \otimes f, s)$. By Theorem 1.5 we know that $L_f(s)$ is $\otimes$-simple in the whole plane. So $L_f(s)$ satisfies all the conditions of Theorem 2.3(i) and we have

$$L(f \otimes f, 1 + it) = (L_f \otimes L_f)(1 + it) \neq 0.$$

(iii) Note that $L(\mathrm{sym}^2 f, s) = L(f \otimes f, s)/\zeta_N(s)$. So the result follows from (i) and (iii) for $t \neq 0$. For $t = 0$, $L(\mathrm{sym}^2 f, 1)$ in a non-zero multiple of $\langle f, f \rangle$ (see [10, Theorem 3(iii)]), and therefore it is non-vanishing. ■

Corollary 2.5 *If* $F = \bar{F} \in \mathcal{S}^*$ *is analytic and* $\otimes$*-simple in* $\mathrm{Re}(s) \geq 1$, *then* $F(1 + it) \neq 0$ *for* $t \neq 0$.

Proof This is a simple consequence of (ii) of the previous theorem with $G(s) = \zeta(s)$. ■

The following is a simple corollary of Corollary 2.5 and Theorem 2.3(ii).

Corollary 2.6 *Let* $f \in S_k(N)$ *be an eigenform for* $\Gamma_0(N)$, *let* χ *be a real non-trivial Dirichlet character (mod q), and let* $t \neq 0$. *Then*

(i) $L_\chi(1 + it) \neq 0$.
(ii) $L_f(1 + it) \neq 0$.
(iii) $L_{f,\chi}(1 + it) \neq 0$.
(iv) *Suppose* $g \in S_k(N)$ *is also an eigenform for* $\Gamma_0(N)$. *If* $\langle f, g \rangle = 0$, *then* $L(f \otimes g, 1 + it) \neq 0$.

3 Ingham's Method

One of the main facts regarding Dirichlet series with positive coefficients is the following result of Landau.

Lemma 3.1 (Landau) *A Dirichlet series with non-negative coefficients has a singularity at its abscissa of convergence.*

Proof See [5, Exercise 2.5.14]. ■

In this section, we will show that for two Dirichlet series in $\mathcal{S}^*$ with completely multiplicative coefficients[2], one can apply this lemma of Landau to prove a non-vanishing result, valid on the line $\mathrm{Re}(s) = 1$, for the convolution series. Our result is a generalization of Ingham's proof of the non-vanishing of the Riemann zeta-function on the line $\mathrm{Re}(s) = 1$ [2]. To do this, we start by recalling some results regarding Dirichlet series with completely multiplicative coefficients and completely multiplicative arithmetic functions. The proof of the following lemma is routine.

[2]This means $a_F(mn) = a_F(m)a_F(n)$ for every m and n.

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Lemma 3.2 *For* $F, G \in \mathcal{S}^*$ *with completely multiplicative coefficients,*

$$(F \otimes G)(s) = \sum_{n=1}^{\infty} \frac{a_F(n)\overline{a_G(n)}}{n^s}.$$

Definition 3.3 If $f(n)$ is an arithmetic function, the formal L-series attached to $f(n)$ is defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

If $g(n)$ is also an arithmetic function, the Dirichlet convolution of $f(n)$ and $g(n)$ is defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

The following identity of formal L-series, due to Ramanujan (see [5, p. 185]), will be fundamental in the proof of the main result of this section.

Lemma 3.4 *Let* f_1, f_2, g_1, g_2 *be completely multiplicative arithmetic functions. Then we have*

$$\sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n)(f_2 * g_2)(n)}{n^s} = \frac{L(f_1 f_2, s) L(g_1 g_2, s) L(f_1 g_2, s) L(f_2 g_1, s)}{L(f_1 f_2 g_1 g_2, 2s)}.$$

We are ready to state and prove the main result of this section.

Theorem 3.5 *Let* $F, G \in \mathcal{S}^*$ *be two Dirichlet series with completely multiplicative coefficients. Also assume the following:*

(i) *F and G are* $\otimes$*-simple in* $\mathrm{Re}(s) > \frac{1}{2}$.
(ii) $F \otimes G$ *has an analytic continuation to* $\mathrm{Re}(s) > \frac{1}{2}$.
(iii) $(F \otimes G) \otimes (F \otimes G)$ *is analytic for* $\mathrm{Re}(s) > 1$ *and has a pole at* $s = 1$.
(iv) $(F \otimes F)(s)$, $(G \otimes G)(s)$ *and* $(F \otimes G)(s)$ *have finite limits as* $s \to \frac{1}{2}^+$.[3]

Then $(F \otimes G)(1 + it) \neq 0$ *for all* t.

Proof Let

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}, \quad G(s) = \sum_{n=1}^{\infty} \frac{a_G(n)}{n^s}.$$

Let

$$f_1(n) = a_F(n) n^{-it_0}, \quad f_2(n) = \overline{a_F(n)} n^{it_0}, \quad g_1(n) = a_G(n), \quad g_2(n) = \overline{a_G(n)},$$

and for $\mathrm{Re}(s) > 1$, consider the following Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{|(f_1 * g_1)(n)|^2}{n^s} = \sum_{n=1}^{\infty} \frac{(f_1 * g_1)(n)(f_2 * g_2)(n)}{n^s}.$$

[3]This means $s = \sigma + it \to \frac{1}{2} + it$ for any t as $\sigma \to \frac{1}{2}^+$.

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So, by Lemmas 3.2 and 3.4 and for $\mathrm{Re}(s) > 1$, we have

$$f(s) = \frac{(F \otimes F)(s)\,(F \otimes G)(s + it_0)\,(G \otimes F)(s - it_0)\,(G \otimes G)(s)}{[(F \otimes G) \otimes (F \otimes G)]\,(2s)}.$$

Now suppose that $(F \otimes G)(1 + it_0) = 0$ for a real t_0, then we have in fact the analyticity of $f(s)$ for $\mathrm{Re}(s) > \frac{1}{2}$, and since the coefficients in the series are non-negative, by Lemma 3.1 the Dirichlet series representing $f(s)$ is convergent for $\mathrm{Re}(s) > \frac{1}{2}$. So, for $\eta > 0$, we have

$$f\Big(\frac{1}{2} + \eta\Big) = \sum_{n=1}^{\infty} \frac{|(f_1 * g_1)(n)|^2}{n^{\frac{1}{2}+\eta}} \geq 1.$$

However, since $(F \otimes G) \otimes (F \otimes G)$ has a pole at $s = 1$,

$$[(F \otimes G) \otimes (F \otimes G)]\left(2\Big(\frac{1}{2} + \eta\Big)\right) = [(F \otimes G) \otimes (F \otimes G)]\,(1 + 2\eta) \to \infty$$

as $\eta \to 0^+$. This shows that

$$\lim_{\eta \to 0^+} f\Big(\frac{1}{2} + \eta\Big) = 0,$$

which is a contradiction. So, $(F \otimes G)(1 + it) \neq 0$ for all real t. ■

By choosing $G(s) = \zeta(s)$ in the previous theorem, we have

Corollary 3.6 *Let $F \in \mathcal{S}^*$ be analytic and $\otimes$-simple in $\mathrm{Re}(s) > \frac{1}{2}$. If the coefficients of F are completely multiplicative and $F(s)$ together with $(F \otimes F)(s)$ have finite limits as $s \to \frac{1}{2}^+$, then $F(1 + it) \neq 0$, for all $t \in \mathbb{R}$.*

The following non-vanishing results are simple consequences of the previous corollary.

Corollary 3.7 *Let χ be a non-trivial Dirichlet character and let $f(n)$ be a completely additive[4] arithmetic function and let $t \in \mathbb{R}$. Then*

(i) $L_\chi(1 + it) \neq 0$.

(ii) *If $\sum_{n \leq x} (-1)^{f(n)} \chi(n) = O(x^\delta)$ for $\delta < \frac{1}{2}$, then $L(s) = \sum_{n=1}^{\infty} \frac{(-1)^{f(n)} \chi(n)}{n^s}$ is analytic in $\mathrm{Re}(s) > \delta$ and $L(1 + it) \neq 0$.*

4 Ogg's Method

In this section, we consider the extension of the results of Section 3 to Dirichlet series with general coefficients. Our approach in this section is motivated by a paper by Ogg [8]. The following lemma describes the basic ingredient of this approach.

[4]This means $f(mn) = f(m) + f(n)$ holds for all m and n.

Lemma 4.1 *Let $f(s)$ be a complex function that satisfies the following:*

(i) *$f(s)$ is analytic on the half-plane $\mathrm{Re}(s) > \sigma_0$;*

(ii) *$\log f(s)$ has a representation in terms of a Dirichlet series with non-negative coefficients on the half-plane $\mathrm{Re}(s) > \sigma_1$ ($\sigma_1 > \sigma_0$).*

Then $f(s) \neq 0$ for $\mathrm{Re}(s) > \sigma_0$.

Proof Let σ_2 be the largest real zero of f ($\sigma_0 < \sigma_2 \leq \sigma_1$). Since

$$\log f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

for $\mathrm{Re}(s) > \sigma_1$ ($c(n) \geq 0$), and since $\log f(s)$ is analytic in a neighborhood of the segment $\sigma_2 < \sigma \leq \sigma_1$, then by Lemma 3.1, we have $\log f(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$ for $\mathrm{Re}(s) > \sigma_2$. Thus,

$$\log |f(\sigma)| = \mathrm{Re}(\log f(\sigma)) = \log f(\sigma) = \sum_{n=1}^{\infty} \frac{c(n)}{n^{\sigma}} \geq 0$$

for $\sigma > \sigma_2$. Therefore, $|f(\sigma)| \geq 1$ for $\sigma > \sigma_2$. This contradicts the assumption $f(\sigma_2) = 0$, and therefore f has no real zero $\sigma > \sigma_0$. So $\log f(s)$ is analytic on the interval $(\sigma_0, \sigma_1]$, and Lemma 3.1 in fact shows that $\log f(s)$ exists and is analytic for $\mathrm{Re}(s) > \sigma_0$. This means that $f(s)$ is non-zero for $\mathrm{Re}(s) > \sigma_0$. ■

Here, we prove the main result of this section.

Theorem 4.2 *Let $\sigma_0 < 1$, and assume the following:*

(i) *F and G (as elements of $\mathcal{S}^*$) are $\otimes$-simple in $\mathrm{Re}(s) > \sigma_0$;*

(ii) *$F \otimes G$ has an analytic continuation to the half-plane $\mathrm{Re}(s) > \sigma_0$;*

(iii) *At least one of $F \otimes F$, $G \otimes G$, or $F \otimes G$ has zeros in the strip $\sigma_0 < Re(s) < 1$.*

Then $(F \otimes G)(1 + it) \neq 0$ for all real t.

Proof Suppose that $(F \otimes G)(1 + it_0) = 0$, and let

$$f(s) = (F \otimes F)(s)\,(F \otimes G)(s + it_0)\,(G \otimes F)(s - it_0)\,(G \otimes G)(s).$$

First of all note that $G \otimes F$ is analytic for $\mathrm{Re}(s) > \sigma_0$. Since $(F \otimes G)(1 + it_0) = 0$, then $(G \otimes F)(1 - it_0) = 0$, and since $s = 1$ is a pole of order ≤ 1 for both $F \otimes F$ and $G \otimes G$, we conclude that $f(s)$ is analytic at point $s = 1$, and therefore, analytic for $\mathrm{Re}(s) > \sigma_0$. Now note that for $\mathrm{Re}(s) > 1$,

$$\log f(s) = \sum_{p} \sum_{k=1}^{\infty} \frac{k|b_F(p^k) + b_G(p^k)p^{ikt_0}|^2}{p^{ks}} = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

where $c(n) \geq 0$. So, $f(s)$ satisfies the conditions of the Corollary 4.1 with $\sigma_1 = 1$, and therefore, $f(s) \neq 0$ for $\mathrm{Re}(s) > \sigma_0$. This contradicts our assumption in (iii). ■

Corollary 4.3 *Let $F \in \mathcal{S}^*$ be analytic and $\otimes$-simple in* $\mathrm{Re}(s) \geq \frac{1}{2}$, *then* $F(1+it) \neq 0$.

Proof Let $G(s) = \zeta(s)$. Note that $\zeta(s)$ has zeros in the half-plane $\mathrm{Re}(s) \geq 1/2$ (see [1, p. 97]). Thus by Theorem 4.2, $F(1+it) = (F \otimes G)(1+it) \neq 0$. ■

Corollary 4.4 *Let $f \in S_k(N, \psi_1)$ and $g \in S_k(N, \psi_2)$ be eigenforms for $\Gamma_0(N)$, let χ be a non-trivial Dirichlet character (mod q) and let t be any real number. Then*

(i) $L_\chi(1+it) \neq 0$.
(ii) $L_f(1+it) \neq 0$.
(iii) $L_{f,\chi}(1+it) \neq 0$.
(iv) *If $\psi_1 \neq \psi_2$ or $\langle f, g \rangle = 0$, then $L(f \otimes g, 1+it) \neq 0$.*
(v) *Let $\bar{f}_{\bar{\chi}}(z) = \sum_{n=1}^{\infty} \overline{a_f(n)\chi(n)} e^{2\pi i n z}$. Then if $\psi\chi$ is not a real character of order 2 or $\int_{D_0(Nq^2)} f(z)\overline{\bar{f}_{\bar{\chi}}(z)} y^{k-2}\, dxdy = 0$, we have $L_\chi(f \otimes \bar{f}, 1+it) \neq 0$ and $L_\chi(\mathrm{sym}^2 f, 1+it) \neq 0$. Here $D_0(Nq^2)$ is a fundamental domain for $\Gamma_0(Nq^2)$.*

Proof (i), (ii) and (iii) are simple consequences of Corollary 4.3 and Theorem 4.2.

(iv) By Theorem 1.5 we can show that conditions (i) and (ii) of Theorem 4.2 are satisfied. The result will be obtained if we only show that $L(f \otimes g, s)$ has a zero in the half-plane $\mathrm{Re}(s) < 1$. Again by Theorem 1.5, if $\psi_1 \neq \psi_2$ or $\langle f, g \rangle = 0$, then

$$\Phi(s) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-2s} \Gamma(s)\Gamma(s+k-1)L(f \otimes g, s)$$

is analytic at $s = 0$. Since $\Gamma(s)$ has a pole at $s = 0$, then $L(f \otimes g, 0) = 0$.

(v) First of all note that $\bar{f}_{\bar{\chi}} \in S_k(Nq^2, \psi\chi^2)$ (see [4, p. 127]) and $L_\chi(f \otimes \bar{f}, s) = L(f \otimes \bar{f}_{\bar{\chi}}, s)$. So under the given conditions, by (iv) we have $L_\chi(f \otimes \bar{f}, 1+it) \neq 0$. This together with (i) imply that

$$L_\chi(\mathrm{sym}^2 f, 1+it) = L_\chi(f \otimes \bar{f}, 1+it)/L_{\psi\chi}(1+it) \neq 0. \qquad ■$$

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https://doi.org/10.4153/CMB-2005-030-6 Published online by Cambridge University Press