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INVARIANCE THEOREMS FOR FIRST PASSAGE TIME RANDOM VARIABLES

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1. **Introduction and summary.** Let $X_1, X_2, \ldots$ be i.i.d. r.v. with $EX=\mu>0$, and $E(X-\mu)^2=\sigma^2<\infty$.

Let $S_k=X_1+\cdots+X_k$ and $v_x=\max\{k:S_k\le x\}$, $x\ge 0$ and $v_x=0$ if $X_1>x$. Billingsley [1] proved if $X_1\ge 0$ then

$$T_n(x,\omega) = \frac{v_{nx}(\omega)-(nx/\mu)}{\sigma\mu^{-3/2}\sqrt{n}}$$

converges weakly to the Wiener measure W.

Let $\tau_x(\omega)=\inf\{k\ge 1 \mid S_k>x\}$. In §2 we prove that

$$Z_n(x,\omega) = \frac{\tau_{nx}(\omega)-(nx/\mu)}{\sigma\mu^{-3/2}\sqrt{n}}$$

converges weakly to the Wiener measure when the X's may not necessarily be nonnegative. Also we indicate that this result can be extended to the nonidentical case.

In §3 we prove that certain first passage time random variables of partial sums of i.i.d. r.v. with mean zero (or with positive mean) and finite variance tend to corresponding first passage time r.v. of Brownian motion (or with positive drift).

2. THEOREM 1. *Let $X_1, X_2, \ldots$ be i.i.d. r.v. with $\infty>EX=\mu>0$, $E(X-\mu)^2=\sigma^2<\infty$. Let $S_k=X_1+X_2+\cdots+X_k$. Let*

(1) $$\tau_t = \inf\{k \ge 1 \mid S_k > t\}, t > 0$$

Define

(2) $$Z_n(t,\omega) = \frac{\tau_{nt}-(nt/\mu)}{\sigma\mu^{-3/2}\sqrt{n}}$$

Then $Z_n \xrightarrow{\mathscr{D}} W$, the Wiener measure.

Proof. Without loss of generality we shall assume $\mu>1$.

We first show that

(3) $$\sup_{0\le t\le 1}\left|\frac{\tau_{tn}}{n}-\frac{t}{\mu}\right| \xrightarrow{P} 0 \quad \text{as } n\to\infty.$$

$\tau_{tn}=\inf(k:S_k>tn)$ for $k\ge 1$, a fixed t $(0<t\le 1)$ and n a positive integer tending to ∞. Since the X_k are not necessarily positive, S_k may or may not be greater than

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S_{k-1} but τ_{tn} is a step-function with integer-valued jumps at certain values of tn depending on the observed ω (i.e. on the observed set of values $X_1, X_2, \ldots$). For any given ω, $S_{\tau_{tn}} > tn$ but $S_{\tau_{tn}-1} \leq tn$.

The law of large numbers gives $(\mu-\epsilon)n \leq S_n \leq (\mu+\epsilon)n$ for any ϵ $(0<\epsilon<\mu)$ and for sufficiently large n. Therefore

$$tn < S_{\tau_{tn}} \leq (\mu+\epsilon)\tau_{tn}$$

and

$$tn \geq S_{\tau_{tn}-1} \geq (\mu-\epsilon)(\tau_{tn}-1),$$

that is

$$\frac{t}{\mu+\epsilon} < \frac{\tau_{tn}}{n} \leq \frac{t}{\mu-\epsilon}+\frac{1}{n} \quad \text{for } t > 0 \text{ and } n \to \infty. \tag{4}$$

From (4) it follows that $\tau_{tn}/n \to t/\mu$ a.e. as $n \to \infty$. Since τ_{tn} is everywhere left-continuous $|\tau_{tn}/n - t/\mu| \xrightarrow{P} 0$ as $n \to \infty$ for any fixed $t>0$. If $t=0$, τ_0 will be a positive integer m $(>1$ if some negative X_i precedes the first positive value), but since $E(X_i)>1$, the probability of large m is vanishingly small, and in any case $E(m)<\infty$. Then

$$\sup_{0\leq t\leq 1}\left|\frac{\tau_{tn}}{n}-\frac{t}{\mu}\right| \xrightarrow{P} 0.$$

Define

$$\begin{aligned} U_n(t) &= \tau_{tn}/n \quad \text{if } \tau_{tn} \leq tn, \\ &= t/\mu \quad\quad \text{otherwise.} \end{aligned}$$

Let $u(t)=t/\mu$, then

$$\sup_{0\leq t\leq 1}|U_n(t)-u(t)| \leq \sup_{0\leq t\leq 1}\left|\frac{\tau_{tn}}{n}-\frac{t}{\mu}\right| \xrightarrow{P} 0 \quad \text{as } n\to\infty,$$

so U_n converges in probability in the sense of Skorohod topology to $u(t)$ of $C[0, 1]$, since $C[0, 1]$ is a subspace of $D[0, 1]$, with relative topology.

Let

$$X_n(t) = \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{[nt]}(X_i-\mu).$$

Therefore by Donsker's theorem [1],

$$X_n \xrightarrow{\mathscr{D}} W, \quad \text{so} \quad X_n \circ U_n \xrightarrow{\mathscr{D}} W \circ u.$$

Define

$$Y_n(t) = \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{\tau_{nt}}(X_i-\mu).$$

Then by the definition of τ_{tn},

$$Y_n(t) - X_{\tau_{tn}}/\sigma\sqrt{n} \leq \frac{nt-\mu\tau_{nt}}{\sigma\sqrt{n}} < Y_n(t).$$

With our definition of U_n,

$$Y_n = X_n \circ U_n \quad \text{if } \tau_n/n \le 1.$$

Since $\max_{i\le n}(|X_i|/\sqrt{n}) \xrightarrow{P} 0$, it follows that

$$\sup_{t\le 1} |X_{\tau_{nt}}|/\sigma\sqrt{n} \xrightarrow{P} 0.$$

Let $Z_n^*(t) = (nt-\mu\tau_{nt})/\sigma\sqrt{n}$, then

$$Z_n^* \xrightarrow{\mathscr{D}} W \circ u.$$

Therefore

$$\mu^{1/2}Z_n^* \xrightarrow{\mathscr{D}} W \quad \text{(by scaling property of } W).$$

Therefore $Z_n \xrightarrow{\mathscr{D}} W$. Hence the theorem. Q.E.D.

Let $M(x) = \max(k \mid S_k \le x)$, then

$$M(x)+1 = \tau_x.$$

COROLLARY (Heyde [2]). *Let $X_1, X_2, \ldots$ be i.i.d. r.v. with $EX=\mu>0$*, var (X) $=\sigma^2<\infty$. *Then*

$$\lim_{x\to\infty} \Pr\left\{\frac{M(x)-x\mu^{-1}}{(x\sigma^2\mu^{-3})^{1/2}} < a\right\} = \frac{1}{(2\Pi)^{1/2}}\int_{-\infty}^{a} \exp(-1/2u^2)\,\mathrm{du}.$$

REMARK. Let $X_1, X_2, \ldots$ be independent r.v. with $EX_i=\mu>0$ and $E(X_i-\mu)^2$ $=\sigma^2<\infty$ for all i and suppose that $\{X_n\}$ obey Lindberg's condition; then $Z_n \xrightarrow{\mathscr{D}} W$.

By the classical Kolmogorov's strong law for independent random variables, $S_n/n \to \mu$ a.e.

By Prohorov's functional central limit theorem [3],

$$X_n(t) \xrightarrow{\mathscr{D}} W.$$

So, as before, $\tau_{tn}/n \to t/\mu$ a.e. as $n\to\infty$.

Lindberg's condition implies

$$P\left(\max_{1\le i\le n}\left|\frac{X_i}{\sqrt{n\sigma}}\right| \ge \epsilon\right) = P\left(\bigcup_{i=1}^{n}\left\{\frac{|X_i|}{\sqrt{n\sigma}} \ge \epsilon\right\}\right)$$

$$\le \sum_{i=1}^{n} P\left(\frac{|X_i|}{\sqrt{n\sigma}} \ge \epsilon\right) \le \frac{1}{\epsilon^2 n\sigma^2}\sum_{i=1}^{n}\int_{|x|\ge\epsilon\sqrt{n\sigma}} x^2\,dF_i(x).$$

Therefore

$$\max_{i\le n}\frac{|X_i|}{\sqrt{n}} \xrightarrow{P} 0.$$

3. Let $\eta=\eta_a=\inf(t\ge 0 \mid W(t)>a)$, $a>0$ where $W(t)$ is the standard Brownian motion.

Let $\tau_a = \inf\{k > 1 \mid S_k > a\}$ where $S_k = X_1 + \cdots + X_k$ and $\{X_k\}$ are independent random variables with $EX_k = 0$, and $\{X_k\}$ satisfies Lindberg's condition. For simplicity let us assume $EX_k^2 = 1$.

THEOREM 2. *$\tau_{a\sqrt{n}}/n$ converges in distribution to η.*

Proof. By Prohorov's theorem [3], $S_{[nt]}/\sqrt{n} \xrightarrow{\mathscr{D}} W$, the Wiener measure, and also $\sup_{0 \le t \le T} (S_{[nt]}/\sqrt{n}) = \max_{i \le [nT]} (S_i/\sqrt{n})$ converges in distribution to $\sup_{0 \le t \le T} W(t)$.

It is well known that

$$P(\eta > T) = P\left(\sup_{0 \le t \le T} W(t) < a\right) = \frac{2}{\sqrt{2\Pi T}} \int_0^a e^{-x^2/2T}\, dx$$

$$= \frac{2}{\sqrt{2\Pi}} \int_0^{a/\sqrt{t}} e^{-x^2/2}\, dx$$

Now

$$P\left(\max_{i \le [nT]} \frac{S_i}{\sqrt{n}} < a\right) = P\left(\max_{i \le [nT]} S_i < a\sqrt{n}\right)$$

$$= P(\tau_{a\sqrt{n}} > [nT])$$

Therefore $P[(\tau_{a\sqrt{n}}/n) > T] \to P(\eta > T)$ as $n \to \infty$ if T is a continuity point of the distribution of η.

Now let $X_1, X_2, \ldots$ be i.i.d. r.v. with $EX = \delta > 0$ and $EX_i^2 = 1$.

Let h be a fixed continuous function on $[0, T]$.

Define

$$F_h[f] = \inf\,[t \ge 0 \mid f(t) \ge h(t)] \quad \text{if this exists,}$$

$$= T \qquad \text{otherwise.}$$

Let $f(t-) = \lim_{s \uparrow t} f(s)$ for each $t \in (0, T]$ and $f(0-)$ be $f(0)$. Then define

$$F_h^-[f] = \inf\,[t \ge 0 \mid f(t-) \ge h(t)] \quad \text{if this exists,}$$

$$= T \qquad \text{otherwise.}$$

LEMMA. *The functional $F_h[.]$ is continuous in J_1-topology of Skorohod* [3] *at every $f \in D[0, T]$ for which* (i) *$F_h[f] = T$ or $f(t_n) > h(t_n)$ for a sequence of points $t_n \downarrow F_n[f]$ and* (ii) $F_h[f] = F_h^-[f]$.

Proof. Let $f_n \to f$ in J_1-topology and let λ_n be a sequence of homomorphisms of $[0, T]$ onto itself such that $\lambda(0) = 0$, $\lambda(T) = T$.

Let $\rho_n = F_h(f_n)$, and assume that some subsequence (ρ_{nk}) of (ρ_n) tends to ρ_0. Then $f_n(\rho_n) \ge h(\rho_n)$ for each ρ_n, and hence $\underline{\lim}_{k \to \infty} f_{n_k}(\rho_{n_k}) \ge h(\rho_0)$.

But $\lim_{n \to \infty} |f_n(\rho_n) - f(\lambda_n(\rho_n))| = 0$. Therefore $\underline{\lim}_{k \to \infty} f(\lambda_{n_k}(\rho_{n_k})) \ge h(\rho_0)$. Since $\lim_{k \to \infty} \lambda_{n_k}(\rho_{n_k}) = \rho_0$, it follows that either $f(\rho_0) \ge h(\rho_0)$ or $f(\rho_0^-) \ge h(\rho_0)$, so that

$\rho_0 \geq F_h[f]$. Suppose $\rho_0 > F_h[f]$. Then there exists $F_h(f) < \rho_0^* < \rho_0$ such that $f(\rho_0^*) > h(\rho_0^*)$. If $0 < \epsilon < (\rho_0 - \rho_0^*)/2$, then

$$\begin{aligned}\overline{\lim_{k\to\infty}} |f_{n_k}(\rho_0^*+\epsilon)-f(\rho_0^*+\epsilon)| &= \overline{\lim_{k\to\infty}} |f_{n_k}(\rho_0^*+\epsilon)-f(\lambda_{n_k}(\rho_0^*+\epsilon))| \\ &\quad + \overline{\lim_{k\to\infty}} |f(\lambda_{n_k}(\rho_0^*+\epsilon))-f(\rho_0^*+\epsilon)| \\ &= \overline{\lim_{k\to\infty}} |f(\lambda_{n_k}(\rho_0^*+\epsilon))-f(\rho_0^*+\epsilon)| \\ &\leq |f(\rho_0^*+\epsilon)-)-f(\rho_0^*+\epsilon)|.\end{aligned}$$

By the right continuity of f at ρ_0^*, we can choose ϵ so small that $f_{n_k}(\rho_0^*+\epsilon) > h(\rho_0^*+\epsilon)$ for k sufficiently large. This means $\rho_0 \leq \rho_0^*+\epsilon$ which is a contradiction (since $\epsilon > 0$ is arbitrary). Therefore $\rho_0 = F_h(f)$ and hence F_h is continuous at f. Q.E.D.

Suppose $\delta > 0$ and $W_\delta = \{W_\delta(t); t \geq 0, W_\delta(0) = 0\}$, be a Wiener process with drift δ per unit time.

Let

$$\begin{aligned}\eta_\delta(a) &= \inf\{t \geq 0 \mid W_\delta(t) \geq a\}, \quad a > 0, \quad \delta > 0 \\ &= \inf\{t \geq 0 \mid W(t) \geq a - \delta t\}.\end{aligned}$$

Let $X_1, X_2, \ldots$ be i.i.d. r.v. with $EX = \delta > 0$, $E(X-\delta)^2 < \infty$.

Let $\tau_x = \inf\{k \geq 1 \mid S_k > x\}$.

THEOREM 3. *$\tau_{a\sqrt{n}+k\delta}/n$ converges in distribution to $\eta_\delta(a)$, whose probability density is given by*

$$P_{\eta_\delta}(T) = (2\Pi T^3)^{-1/2} \exp(-(a-\delta T)^2/2T).$$

Proof. Consider $W(t)$ as a random element of $D[0, T]$ with its extended measure as its distribution.

Let $h(t) = a - \delta t$. Then W and h satisfy the conditions of our lemma.

Now again, by Donsker's theorem [1],

$$f_n(t, \omega) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} (X_i - \delta) \xrightarrow{\mathscr{D}} W.$$

Then by Skorohod's theorem [1] if F is any real-valued functional on $D[0, T]$ which is J_1-continuous, the distribution of $F[f_n(., \omega)]$ converges to the distribution of $F[W(., t)]$.

It is easy to see

$$F[f_n(., \omega)] \approx \frac{1}{n} \inf\{k \geq 1 \mid S_k > a\sqrt{n}+k\delta\} = \frac{\tau_{a\sqrt{n}+k\delta}}{n}.$$

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References

1. P. Billingsley, Convergence of probability measures, Wiley, New York, 1968.

2. C. C. Heyde, *Asymptotic renewal results for a natural generalisation of classical renewal theory*, J. Roy. Statist. Soc. Series B, **29** (1967), 141–150.

3. K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York, 1967.

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