A GENERALIZATION OF THE HARDY SPACES

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1. Introduction. The Hardy spaces for right half-planes, $\mathfrak{H}_p(\sigma)$, σ real, $1 \leqslant p \leqslant \infty$, are defined to consist of all those functions $f(s)$, holomorphic for $\text{Re } s > \sigma$, for which $\mu_p(f, x)$ exists and is bounded for $x > \sigma$, where

$$\mu_p(f, x) = \left\{\frac{1}{2\pi} \int_{-\infty}^{\infty} |f(x + iy)|^p dy\right\}^{1/p}, \qquad 1 \leqslant p < \infty,$$

and

$$\mu_\infty(f, x) = \sup_y |f(x + iy)|.$$

These spaces have been studied extensively (see, for example, **3**, Chapter 8, and **2**, §19.1).

In particular, it is known that if $e^{-\sigma t}F(t) \in L_p(0, \infty)$, $1 \leqslant p \leqslant 2$, and if f is the Laplace transform of F, then $f \in \mathfrak{H}_q(\sigma)$, where now and henceforth $p^{-1} + q^{-1} = 1$, and if $f \in \mathfrak{H}_p(\sigma)$, $1 \leqslant p \leqslant 2$, then there is a function F, with $e^{-\sigma t}F(t) \in L_q(0, \infty)$, such that f is the Laplace transform of F. These results are essentially due to Doetsch (**1**). (Doetsch proved them for $\sigma = 0$ and $1 < p \leqslant 2$; the extension to non-zero σ, and to $p = 1$ is easy and we shall take the results as known.) In an earlier article (**5**), we generalized the Hardy spaces somewhat and found corresponding generalizations of Doetsch's results.

Here we propose further to generalize these spaces and to obtain the corresponding generalizations of Doetsch's theorems. The generalized Hardy spaces are defined in §2 and some preliminary lemmas are proved, while the generalizations of Doetsch's theorems occupy §3.

As might be expected, the generalization of $\mathfrak{H}_2(\sigma)$ possesses certain properties not shared by the other spaces, and consequently we devote §4 to the elaboration of some of these.

2. Generalized spaces. Let α be a non-constant non-decreasing function defined on $[0, \infty)$, with $\alpha(0) = 0$, and with the property that the integral

$$\int_0^\infty e^{-tx} d\alpha(x) \equiv \phi(t)$$

converges for all $t > 0$. Let $c = c(\alpha)$ be the first point in $[0, \infty)$, where α varies; that is c is the point such that if $x > c$, $\alpha(x) > 0$, and if $\alpha(x) > 0$, $x \geqslant c$.

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With each such α, and each p, $1 < p \leqslant \infty$, we associate a space $\mathfrak{H}_p^\alpha$ defined as follows: $f \in \mathfrak{H}_p^\alpha$ if and only if (i) $f \in \mathfrak{H}_p(\sigma)$ for each $\sigma > c/q$, and (ii) $\|f\|_{\alpha,p} < \infty$, where

$$\|f\|_{\alpha,p} = \left\{ \int_{c+}^{\infty} (\mu_p(f, x/q))^q d\alpha(x) + \sup_{x>c} \alpha(c+) \cdot (\mu_p(f, x/q))^q \right\}^{1/q}.$$

The presence of the second term in the definition of the norm may appear a trifle unnatural, but it can be given a form, which may appear natural, as follows. Let m be the Lebesgue-Stieltjes measure generated on $[0, \infty)$ by α, and if $m(\{c\}) \neq 0$—that is if $\alpha(c+) \neq 0$—define $\mu_p(f, c/q)$ to be the supremum for $x > c$ of $\mu_p(f, x/q)$ (this is natural for from (i) of our definition, and **(2**, Theorem 19.1.4**)**, $\mu_p(f, x/q)$ is a decreasing function of x). Then

$$\|f\|_{\alpha,p} = \left\{ \int_0^{\infty} (\mu_p(f, x/q))^q dm(x) \right\}^{1/q}.$$

We also define $\mathfrak{H}_1^\alpha$ to consist of those functions in $\mathfrak{H}_1(\sigma)$ for all $\sigma > c$ with the property that $\|f\|_{\alpha,1} < \infty$, where

$$\|f\|_{\alpha,1} = \sup_{x>c} \alpha(x)\mu_1(f, x).$$

The spaces $\mathfrak{H}_p(\sigma)$ are special cases of the $\mathfrak{H}_p^\alpha$, coming from choosing $\alpha(x) = H(x - \sigma)$, where H is the Heaviside function

$$H(x) = \begin{cases} 0, & x \leqslant 0, \\ 1, & x > 0. \end{cases}$$

Also if $\alpha(x) = (x - \omega)^{q\lambda}/q\lambda$, $\lambda > 0$, we obtain the spaces $\mathfrak{H}_p^\lambda(\omega)$ that we discussed in **(5)**. Hence our generalization includes these previous cases, but of course many more. For example $\alpha(x) = [x]$ leads to quite new spaces.

It should be noted that if $\alpha(c+) \neq 0$, then $\mathfrak{H}_p^\alpha$ is a subspace of $\mathfrak{H}_p(c/q)$. For, if $x > c$,

$$\mu_p(f, x/q) \leqslant \|f\|_{\alpha,p}/(\alpha(c+))^{1/q}, \qquad 1 < p \leqslant \infty,$$

and

$$\mu_1(f, x) \leqslant \|f\|_{\alpha,1}/\alpha(c+).$$

If $\alpha(c+) = 0$, as in the second example above, then $\mathfrak{H}_p^\alpha$ need not be a subspace of $\mathfrak{H}_p(c/q)$.

The following lemmas are needed in the next section.

LEMMA 1. *If*

$$\int_0^{\infty} \phi(t)|F(t)|^p dt < \infty,$$

then $e^{-\sigma t}F(t) \in L_p(0, \infty)$ *for each* $\sigma > c/p$, *and for* $\sigma = c/p$ *if* $\alpha(c+) \neq 0$.

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Proof. Clearly, if $x \geqslant 0$,

$$(*) \qquad \phi(t) = \int_0^\infty e^{-tu}d\alpha(u) \geqslant \int_0^x e^{-tu}d\alpha(u) \geqslant e^{-tx}\int_0^x d\alpha(u) = e^{-tx}\alpha(x).$$

Hence, if $\alpha(p\sigma) > 0$,

$$\int_0^\infty e^{-p\sigma t}|F(t)|^p dt \leqslant (\alpha(p\sigma))^{-1}\int_0^\infty \phi(t)|F(t)|^p dt < \infty,$$

and $e^{-\sigma t}F(t) \in L_p(0, \infty)$. But $\alpha(p\sigma) > 0$ if $\sigma > c/p$. Also, letting $x \to c+$ in the inequality $\phi(t) \geqslant e^{-tx}\alpha(x)$, we obtain $\phi(t) \geqslant e^{-ct}\alpha(c+)$, so that if $\alpha(c+) > 0$,

$$\int_0^\infty e^{-ct}|F(t)|^p dt \leqslant (\alpha(c+))^{-1}\int_0^\infty \phi(t)|F(t)|^p dt < \infty,$$

and $e^{-\sigma t}F(t) \in L_p(0, \infty)$ for $\sigma = c/p$.

LEMMA 2. *If* $\sigma > c$,

$$\int_0^\infty (\phi(t))^{-1}e^{-\sigma t}t^n dt < \infty.$$

Proof. From (*), $\phi(t) \geqslant e^{-\frac{1}{2}(\sigma+c)t}\alpha(\frac{1}{2}(\sigma + c))$, and $\alpha(\frac{1}{2}(\sigma + c)) > 0$.

3. Generalized theorems. Theorems 1 and 2 correspond respectively to Theorems 2 and 3 of **(1)**.

THEOREM 1. *If* $1 \leqslant p \leqslant 2$, *and*

$$\int_0^\infty \phi(t)|F(t)|^p dt < \infty,$$

then F has a Laplace transform f, and $f \in \mathfrak{H}_q^\alpha$, *and*

$$||f||_{\alpha,q} \leqslant \left\{\int_0^\infty \phi(t)|F(t)|^p dt\right\}^{1/p}.$$

Proof. By Lemma 1, $e^{-\sigma t}F(t) \in L_p(0, \infty)$ if $\sigma > c/p$, and hence F has a Laplace transform f in $\mathfrak{H}_p(\sigma)$ for each $\sigma > c/p$. Further, since for $x > c/p$,

$$f(x - iy) = \int_0^\infty e^{iyt}(e^{-xt}F(t))dt,$$

and $e^{-xt}F(t) \in L_p(0, \infty)$, it follows that for each such x, $f(x - iy)$ is the Fourier transform of a function in $L_p(0, \infty)$, and hence by **(7**, Theorem 74**)**, if $1 < p \leqslant 2$,

$$\mu_q(f, x) = \left\{\frac{1}{2\pi}\int_{-\infty}^\infty |f(x + iy)|^q dy\right\}^{1/q}$$
$$= \left\{\frac{1}{2\pi}\int_{-\infty}^\infty |f(x - iy)|^q dy\right\}^{1/q} \leqslant \left\{\int_0^\infty e^{-pxt}|F(t)|^p dt\right\}^{1/p}.$$

This inequality also holds for $p = 1$, for

$$|f(x+iy)| \leqslant \int_0^\infty e^{-xt}|F(t)|dt,$$

so that

$$\mu_\infty(f, x) \leqslant \int_0^\infty e^{-xt}|F(t)|dt.$$

Thus

$$\int_{c+}^\infty (\mu_q(f, x/p))^p d\alpha(x) \leqslant \int_{c+}^\infty \int_0^\infty e^{-xt}|F(t)|^p dt d\alpha(x)$$
$$= \int_0^\infty |F(t)|^p \int_{c+}^\infty e^{-tx} d\alpha(x) dt.$$

If $\alpha(c+) = 0$, this last inequality is just the statement

$$(||f||_{\alpha,q})^p \leqslant \int_0^\infty \phi(t)|F(t)|^p dt,$$

for

$$\phi(t) = \int_{c+}^\infty e^{-tx} d\alpha(x) + \alpha(c+)e^{-ct}.$$

If $\alpha(c+) \neq 0$, then

$$\int_0^\infty e^{-ct}|F(t)|^p dt < \infty,$$

so that

$$\mu_q(f, x/p) \leqslant \left\{ \int_0^\infty e^{-ct}|F(t)|^p dt \right\}^{1/p}.$$

Hence

$$(||f||_{\alpha,q})^p = \int_{c+}^\infty (\mu_q(f, x/p))^p d\alpha(x) + \sup_{x>c} (\alpha(c+)\cdot(\mu_q(f, x/p))^p)$$
$$\leqslant \int_0^\infty |F(t)|^p \int_{c+}^\infty e^{-tx} d\alpha(x) dt + \alpha(c+) \int_0^\infty e^{-ct}|F(t)|^p dt$$
$$= \int_0^\infty \phi(t)|F(t)|^p dt.$$

THEOREM 2. *If $f \in \mathfrak{H}_p^\alpha$, $1 < p \leqslant 2$, then there is a function F, with*

$$\int_0^\infty \phi(t)|F(t)|^q dt < \infty,$$

such that

$$f(s) = \int_0^\infty e^{-st}F(t)dt, \qquad \text{Re } s > c/q, \qquad \textit{and} \left\{ \int_0^\infty \phi(t)|F(t)|^q dt \right\}^{1/q} \leqslant ||f||_{\alpha,p}.$$

Proof. Since $f \in \mathfrak{H}_p(\sigma)$ for each $\sigma > c/q$, by (1, Theorem 3), for each such σ there is a function F_σ, with $e^{-\sigma t}F_\sigma(t) \in L_q(0, \infty)$, such that

$$f(s) = \int_0^\infty e^{-st}F_\sigma(t), \qquad \operatorname{Re} s > \sigma.$$

But by the uniqueness theorem for Laplace transforms (8, Chapter 2, Corollary 9.3*b*), if σ_1 and σ_2 are larger than c/q, $F_{\sigma_1}(t) = F_{\sigma_2}(t)$ almost everywhere. Hence if F is one of the F_σ's, $e^{-\sigma t}F(t) \in L_q(0, \infty)$ if $\sigma > c/q$, and

$$f(s) = \int_0^\infty e^{-st}F(t)dt, \qquad \operatorname{Re} s > c/q.$$

Further, from (1, Theorem 3), for each $x > c/q$,

$$\lim_{r\to\infty} (q) \int_{-r}^{r} e^{ity}f(x+iy)dy = \begin{cases} e^{-xt}F(t), & t > 0, \\ 0, & t < 0, \end{cases}$$

almost everywhere, where $\lim(q)$ denotes the limit in mean of order q. This means that for each $x > c/q$, the Fourier transform of $f(x+iy)$, which is in $L_p(-\infty, \infty)$ relative to y, is equal a.e. to $e^{-xt}F(t)$ if $t > 0$ and to zero if $t < 0$. Hence by (7, Theorem 74), if $x > c/q$,

$$\left\{\int_0^\infty e^{-qxt}|F(t)|^q dt\right\}^{1/q} \leqslant \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+iy)|^p dy\right\}^{1/p} = \mu_p(f, x).$$

Thus

$$\int_0^\infty |F(t)|^q \int_{c+}^\infty e^{-tx}d\alpha(x) = \int_{c+}^\infty \int_0^\infty e^{-xt}|F(t)|^q dt d\alpha(x)$$
$$\leqslant \int_{c+}^\infty (\mu_p(f, x/q))^q d\alpha(x).$$

If $\alpha(c+) = 0$, this says that

$$\int_0^\infty \phi(t)|F(t)|^q dt \leqslant (||f||_{\alpha,p})^q.$$

If $\alpha(c+) \neq 0$, then

$$\sup_{x>c} \mu_p(f, x/q) < \infty$$

and hence from Fatou's Lemma

$$\int_0^\infty e^{-ct}|F(t)|^p dt \leqslant \varliminf_{x\to c+} \int_0^\infty e^{-xt}|F(t)|^p dt \leqslant \varliminf_{x\to c+} (\mu_p(f, x/q))^q$$
$$= \sup_{x>c} (\mu_p(f, x/q))^q,$$

so that

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$$\int_0^\infty \phi(t)|F(t)|^q dt = \int_0^\infty |F(t)|^q \int_{c+}^\infty e^{-tx} d\alpha(x) dt + \alpha(c+) \int_0^\infty e^{-ct}|F(t)|^q dt$$
$$\leqslant \int_{c+}^\infty (\mu_p(f, x/q))^q d\alpha(x) + \sup_{x>c} (\alpha(c+)\cdot(\mu_p(f, x/q))^q)$$
$$= (||f||_{\alpha,p})^q.$$

The reader has undoubtedly noticed that Theorem 2 does not cover the case $p = 1$. Here the results are slightly different. From (**8**, Chapter 2, Theorem 2.2*a*) it follows that $\alpha(x) = O(e^{tx})$ for any $t > 0$. Then clearly

$$\psi(t) = \sup_{x>c} e^{-tx}\alpha(x)$$

is finite for all positive t.

THEOREM 3. *If $f \in \mathfrak{H}_1^\alpha$, then there is a function F with $\psi F \in L_\infty(0, \infty)$, such that*

$$f(s) = \int_0^\infty e^{-st}F(t)dt, \qquad \text{Re}\, s > c, \qquad \textit{and}\ \text{ess.}\sup_{t>0} \psi(t)|F(t)| \leqslant ||f||_{\alpha,1}.$$

Proof. Since $f \in \mathfrak{H}_1(\sigma)$ for all $\sigma > c$, then, as in the previous theorem, F exists with $e^{-\sigma t}F(t) \in L_\infty(0, \infty)$, $\sigma > c$, so that

$$f(s) = \int_0^\infty e^{-st}F(t)dt, \qquad \text{Re}\, s > c,$$

and if $x > c$, $t > 0$,

$$e^{-xt}F(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ity}f(x + iy)dy,$$

almost everywhere. Hence if $x > c$, then for almost all $t > 0$

$$e^{-xt}|F(t)| \leqslant \frac{1}{2\pi}\int_{-\infty}^\infty |f(x + iy)|dy = \mu_1(f, x)$$
$$\leqslant \frac{1}{\alpha(x)} ||f||_{\alpha,1}.$$

Hence for almost all $t > 0$,

$$\alpha(x)e^{-xt}|F(t)| \leqslant ||f||_{\alpha,1},$$

so that taking suprema with respect to x, for almost all $t > 0$,

$$\psi(t)|F(t)| \leqslant ||f||_{\alpha,1},$$

that is $\psi F \in L_\infty$, and

$$\text{ess.}\sup_{t>0} \psi(t)F(t) \leqslant ||f||_{\alpha,1}.$$

It is worth noting that from (*), $\psi(t) \leqslant \phi(t)$ for all t.

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4. The case $p = 2$. Theorems 1 and 2 deal with the same $\mathfrak{H}_p^\alpha$ space only for $p = 2$. Together they then yield the following theorem.

THEOREM 4. *$f \in \mathfrak{H}_2^\alpha$ if and only if there is a function F, with*

$$\int_0^\infty \phi(t)|F(t)|^2 dt < \infty,$$

such that

$$f(s) = \int_0^\infty e^{-st}F(t)dt, \qquad \operatorname{Re} s > \tfrac{1}{2}c.$$

However, in the case of $\mathfrak{H}_2(0)$, another condition, originally due to Shohat **(6)**, is known for f to lie in $\mathfrak{H}_2(0)$ and thus to be the Laplace transform of a function in $L_2(0, \infty)$. This is that

$$\sum_{n=0}^{\infty} |q_n|^2 < \infty,$$

where

$$q_n = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{r!} f^{(r)}(\tfrac{1}{2}).$$

Shohat's proof makes use of the Laguerre polynomials

$$L_n(t) = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{r!}(-t)^r.$$

Here we propose to find a similar condition that f be in $\mathfrak{H}_2^\alpha$.

From Lemma 2 it follows that

$$\int_0^\infty (\phi(t))^{-1} e^{-(c+1)t} t^{2n} dt < \infty$$

so that the members of the sequence $\{(\phi(t))^{-\frac{1}{2}} e^{-\frac{1}{2}(c+1)t} t^n\}$ are in $L_2(0, \infty)$. Let $\{\Phi_n(t)\}$ be the sequence obtained from this sequence by the Gram-Schmidt process. Clearly $\{\Phi_n(t)\}$ is a complete orthonormal sequence in $L_2(0, \infty)$ and

$$\Phi_n(t) = (\phi(t))^{-\frac{1}{2}} e^{-\frac{1}{2}(c+1)t} P_n(t),$$

where $P_n(t)$ is a real polynomial of degree exactly n, say

$$P_n(t) = \sum_{r=0}^{n} a_{nr}(-t)^r, \qquad a_{nn} \neq 0.$$

Let

$$\Psi_n(s) = \int_0^\infty (\phi(t))^{-1} e^{-st} P_n(t) dt,$$

the integral existing for $\operatorname{Re} s > c$, by Lemma 2.

THEOREM 5. *Given a function $f(s)$ holomorphic for* $\operatorname{Re} s > \frac{1}{2}c$, *then a necessary and sufficient condition that a function F exist, with*

$$\int_0^\infty \phi(t)|F(t)|^2 dt < \infty,$$

so that

$$f(s) = \int_0^\infty e^{-st} F(t)dt, \qquad \operatorname{Re} s > \tfrac{1}{2}c,$$

is that

$$\sum_{n=0}^{\infty} |q_n|^2 < \infty,$$

where

$$q_n = \sum_{r=0}^{n} a_{nr} f^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}).$$

When this condition holds,

$$F(t) = (\phi(t))^{-\frac{1}{2}} \lim_{n\to\infty} (2)(\phi(t))^{-\frac{1}{2}} e^{-\frac{1}{2}(c+1)t} \sum_{m=0}^{n} q_m P_m(t)$$

almost everywhere,

$$\int_0^\infty \phi(t)|F(t)|^2 dt = \sum_{n=0}^{\infty} |q_n|^2,$$

and

$$f(s) = \sum_{n=0}^{\infty} q_n \Psi_n(s + \tfrac{1}{2}c + \tfrac{1}{2}).$$

Proof of necessity. Suppose

$$f(s) = \int_0^\infty e^{-st} F(t)dt, \qquad \operatorname{Re} s > \tfrac{1}{2}c,$$

where

$$\int_0^\infty \phi(t)|F(t)|^2 dt < \infty.$$

Let $G(t) = (\phi(t))^{\frac{1}{2}} F(t)$. Then $G \in L_2(0, \infty)$, and

$$(G, \Phi_n) = \int_0^\infty G(t)\Phi_n(t)dt = \int_0^\infty e^{-\frac{1}{2}(c+1)t} P_n(t) F(t)dt$$

$$= \sum_{r=0}^{n} a_{nr} \int_0^\infty e^{-\frac{1}{2}(c+1)t}(-t)^r F(t)dt = \sum_{r=0}^{n} a_{nr} f^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}) = q_n.$$

Hence from the Parseval equality

$$\sum_{n=0}^{\infty} |q_n|^2 = \sum_{n=0}^{\infty} |(G, \Phi_n)|^2 = \int_0^\infty |G(t)|^2 dt = \int_0^\infty \phi(t)|F(t)|^2 dt.$$

Also,

$$(\phi(t))^{\frac{1}{2}} F(t) = G(t) = \lim_{n\to\infty} (2) \sum_{m=0}^{n} (G, \Phi_m)\Phi_m(t),$$

almost everywhere. In other words

$$F(t) = (\phi(t))^{-\frac{1}{2}} \lim_{n\to\infty} (2)(\phi(t))^{-\frac{1}{2}} e^{-\frac{1}{2}(c+1)t} \sum_{m=0}^{n} q_m P_m(t).$$

Further, if Re $s > \frac{1}{2}c$,

$$f(s) = \int_0^\infty e^{-st} F(t)dt = \int_0^\infty (\phi(t))^{-\frac{1}{2}} e^{-st} G(t)dt.$$

But from Lemma 2, if Re $s > \frac{1}{2}c$, $(\phi(t))^{-\frac{1}{2}} e^{-st} \in L_2(0, \infty)$. Hence

$$f(s) = \int_0^\infty (\phi(t))^{-\frac{1}{2}} e^{-st} \Big(\lim_{n\to\infty} (2)(\phi(t))^{-\frac{1}{2}} e^{-\frac{1}{2}(c+1)t} \sum_{m=0}^{n} q_m P_m(t)\Big) dt$$

$$= \lim_{n\to\infty} \sum_{m=0}^{n} q_m \int_0^\infty (\phi(t))^{-1} e^{-(s+\frac{1}{2}c+\frac{1}{2})t} P_m(t)dt$$

$$= \sum_{n=0}^{\infty} q_n \Psi_n(s + \tfrac{1}{2}c + \tfrac{1}{2}).$$

Proof of sufficiency. Suppose

$$\sum_{n=0}^{\infty} |q_n|^2 < \infty.$$

Then by the Riesz-Fischer theorem, G exists in $L_2(0, \infty)$ such that

$$(G, \Phi_n) = q_n.$$

Let $F(t) = (\phi(t))^{-\frac{1}{2}} G(t)$, and

$$f_1(s) = \int_0^\infty e^{-st} F(t)dt, \qquad \text{Re } s > \tfrac{1}{2}c.$$

We shall show that $f_1 = f$.

Note that

$$\sum_{r=0}^{n} a_{nr} f_1^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}) = \int_0^\infty e^{-\frac{1}{2}(c+1)t} \sum_{r=0}^{n} a_{nr}(-t)^r F(t)dt$$

$$= \int_0^\infty (\phi(t))^{-\frac{1}{2}} e^{-\frac{1}{2}(c+1)t} P_n(t)G(t)dt = (G, \Phi_n) = q_n,$$

so that if $f_2 = f - f_1$,

$$\sum_{r=0}^{n} a_{nr} f_2^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}) = 0.$$

But since $P_n(-t)$ is a polynomial of degree exactly n, there are numbers b_{nm} such that

$$t^n = \sum_{m=0}^{n} b_{nm} P_m(-t).$$

This yields

$$t^n = \sum_{m=0}^{n} b_{nm} \sum_{r=0}^{m} a_{mr} t^r = \sum_{r=0}^{n} t^r \sum_{m=r}^{n} b_{nm} a_{mr},$$

and hence

$$\sum_{m=r}^{n} b_{nm}\, a_{mr} = 0, \qquad r < n,$$

and

$$b_{nn}\, a_{nn} = 1.$$

Consequently,

$$0 = \sum_{m=0}^{n} b_{nm} \sum_{r=0}^{m} a_m\, f_2^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}) = \sum_{r=0}^{n} f_2^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}) \sum_{m=r}^{n} b_{nm}\, a_m$$
$$= f_2^{(n)}(c + \tfrac{1}{2}),$$

and hence $f_2 \equiv 0$, and $f_1 = f$.

COROLLARY. *A necessary and sufficient condition that a function f, holomorphic for* Re $s > \frac{1}{2}c$, *lie in $\mathfrak{H}_2^{\alpha}$ is that*

$$\sum_{n=0}^{\infty} |q_n|^2 < \infty,$$

where

$$q_n = \sum_{r=0}^{n} a_{nr} f^{(r)}(\tfrac{1}{2}c + \tfrac{1}{2}).$$

5. Inversion for $p = 2$. If $f \in \mathfrak{H}_2(0)$, then the function F, of which f is the Laplace transform, is given by

$$F(t) = \lim_{\gamma\to\infty}(2)\, \frac{1}{\pi} \int_0^{\infty} f(s) E(st, \gamma) ds,$$

where, for $x > 0$,

$$E(x, \gamma) = \int_0^{\gamma} \operatorname{Re}(x^{-\frac{1}{2}+iy}\Gamma(\tfrac{1}{2} + iy))dy.$$

This formula is due to Paley and Wiener (**4**, p. 39). Here we shall generalize this to a formula appropriate for $\mathfrak{H}_2^{\alpha}$, under a special condition. This condition is that there should be a non-decreasing function β on $(0, \infty)$, with $\beta(0) = 0$, such that

$$(\phi(t))^{\frac{1}{2}} = \int_0^{\infty} e^{-tx} d\beta(x), \qquad t > 0.$$

It follows from (**8**, Chapter 2, Theorem 11.5) that at every point of continuity of α,

$$\alpha(t) = \int_0^{t} \beta(t - u) d\beta(u),$$

so that β is a square root of α relative to Stieltjes convolution.

Note that $c' = c(\beta) \geqslant \frac{1}{2}c(\alpha)$. For if $c' < \frac{1}{2}c(\alpha)$, and $2c' < x < c(\alpha)$, then

$$0 = \alpha(x) = \int_0^{x} \beta(x - u) d\beta(u) = \int_{c'-}^{(x-c')+} \beta(x - u) d\beta(u) > 0.$$

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THEOREM 6. *If $f \in H_2^\alpha$, then the function F whose Laplace transform is f is given by*

$$F(t) = (\phi(t))^{-\frac{1}{2}} \lim_{\gamma\to\infty}(2) \frac{1}{\pi} \int_0^\infty f(s)E_\alpha(s, t, \gamma)ds,$$

where

$$E_\alpha(s, t, \gamma) = \int_0^s E((s-u)t, \gamma)d\beta(u).$$

Proof. Let $G(t) = (\phi(t))^{\frac{1}{2}}F(t)$. Then $G \in L_2(0, \infty)$, and

$$f(s) = \int_0^\infty e^{-st}(\phi(t))^{-\frac{1}{2}}G(t)dt.$$

Hence, if $s > 0$,

$$\int_0^\infty f(u+s)d\beta(u) = \int_0^\infty d\beta(u) \int_0^\infty e^{-(u+s)t}(\phi(t))^{-\frac{1}{2}}G(t)dt$$
$$= \int_0^\infty (\phi(t))^{-\frac{1}{2}}e^{-st}G(t)dt \int_0^\infty e^{-ut}d\beta(u) = \int_0^\infty e^{-st}G(t)dt,$$

provided we justify the interchange of the order of the integrations. For this, by (**8**, Chapter 1, Theorem 15*c*), it suffices to show that

$$\int_0^\infty d\beta(u) \int_0^\infty e^{-(u+s)t}(\phi(t))^{-\frac{1}{2}}|G(t)|dt < \infty.$$

But this integral is equal to

$$\int_0^\infty (\phi(t))^{-\frac{1}{2}}e^{-st}|G(t)|dt \int_0^\infty e^{-ut}d\beta(u) = \int_0^\infty e^{-st}|G(t)|dt < \infty$$

since $G \in L_2(0, \infty)$.

Hence, by the result of Paley and Wiener quoted above,

$$(\phi(t))^{\frac{1}{2}}F(t) = G(t) = \lim_{\gamma\to\infty}(2) \frac{1}{\pi} \int_0^\infty E(\sigma t, \gamma) \int_0^\infty f(u+\sigma)d\beta(u)d\sigma$$
$$= \lim_{\gamma\to\infty}(2) \frac{1}{\pi} \int_0^\infty \int_0^\infty E(\sigma t, \gamma)f(u+\sigma)d\sigma d\beta(u)$$
$$= \lim_{\gamma\to\infty}(2) \frac{1}{\pi} \int_0^\infty \int_u^\infty E((s-u)t, \gamma)f(s)dsd\beta(u)$$
$$= \lim_{\gamma\to\infty}(2) \frac{1}{\pi} \int_0^\infty f(s) \int_0^s E((s-u)t, \gamma)d\beta(u)ds,$$
$$= \lim_{\gamma\to\infty}(2) \frac{1}{\pi} \int_0^\infty f(s)E_\alpha(s, t, \gamma)ds,$$

again provided the interchanges of the order of the integrations are justified.

For this, we first note that by the Paley and Wiener result above, if g is the Laplace transform of a function in $L_2(0, \infty)$, then

$$\int_0^\infty |E(st, \gamma)g(s)|ds < \infty.$$

Also, as previously,

$$\int_0 |f(u+\sigma)|d\beta(u) \leqslant \int_0^\infty d\beta(u) \int_0^\infty e^{-(u+\sigma)t}(\phi(t))^{-\frac{1}{2}}|G(t)|dt$$
$$= \int_0^\infty e^{-\sigma t}|G(t)|dt = g(\sigma).$$

Hence

$$\int_0^\infty |E(\sigma t, \gamma)| \int_0^\infty |f(u+\sigma)|d\beta(u)d\sigma \leqslant \int_0^\infty |E(\sigma t, \gamma)g(\sigma)|d\sigma < \infty$$

and the interchanges are justified.

For example, if $\alpha(x) = [x]$, then an easy calculation shows that

$$\beta(x) = \sum_{0 \leqslant n < x - \frac{1}{2}} \frac{(2n)!}{2^{2n}(n!)^2} H(x-(n+\tfrac{1}{2}))$$

and

$$E_\alpha(s, t, \gamma) = \sum_{0 \leqslant n \leqslant s - \frac{1}{2}} \frac{(2n)!}{2^{2n}(n!)^2} E((s-(n+\tfrac{1}{2}))t, \gamma).$$

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