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Reverse Hypercontractivity for Subharmonic Functions

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Abstract. Contractivity and hypercontractivity properties of semigroups are now well understood when the generator, A, is a Dirichlet form operator. It has been shown that in some holomorphic function spaces the semigroup operators, e^{-tA}, can be bounded *below* from L^p to L^q when p, q and t are suitably related. We will show that such lower boundedness occurs also in spaces of subharmonic functions.

1 Introduction

A Riemannian manifold, (M, g), equipped with a smooth probability measure μ, posseses a natural second order elliptic operator, d^*d, acting on functions. The adjoint d^* is to be computed here with the help of the measure μ: one regards d as a densely defined operator in $L^2(M, \mu)$ into the space of 1-forms which are square integrable with respect to μ in order to compute its adjoint. Such operators, d^*d, so called Dirichlet form operators, or operators in divergence form, have been studied intensely because they show up in so many different areas of mathematics. Typically, one chooses a self-adjoint version A of the differential operator d^*d on $L^2(M, \mu)$. For example one might impose Dirichlet or Neumann boundary conditions if M is not complete. The nonnegative operator A generates a semigroup e^{-tA} whose properties have been widely explored and already expounded in many texts, *e.g.,* [BH, Da80, Da89, Fu80, HP, MR, Si]. Among other issues are the boundedness properties of the semigroup, not only in $L^2(\mu)$ but also in the L^p spaces and Sobolev spaces. In particular, hypercontractivity is concerned with boundedness of the semigroup operators e^{-tA} from L^p to L^q. When $M = \mathbb{C}^n$ with its standard metric and μ is Gauss measure then the semigroup operator e^{-tA} can actually be bounded *below* from L^p to L^q if p, q and t are properly related and the semigroup is restricted to act only on *holomorphic* functions. This is *reverse* hypercontractivity in these holomorphic function spaces. This phenomenon was discovered by E. Carlen [Ca].

In [GGS] it was shown that Carlen's reverse hypercontractivity in holomorphic function spaces could be understood as a problem in estimating the L^p norm of a Radon–Nikodym derivative $\|d\psi_*\mu/d\mu\|_p$ for some diffeomorphism ψ of M. The reason for this is that over $\mathbb{C}^n$ the second order terms in d^*d amount to the Laplacian, which annihilates holomorphic functions. Thus d^*d reduces to a first order differential operator, X, on holomorphic functions. As a result, the semigroup reduces to a

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composition operator:

$$e^{-tA} f = f \circ \exp(-tX) \tag{1.1}$$

on holomorphic functions. Here $\exp(-tX)$ is the flow induced by the vector field $-X$.

The purpose of this paper is to extend the reverse hypercontractive inequalities of [Ca] and [GGS] from holomorphic functions to subharmonic functions. Of course if f is a harmonic function on $\mathbb{R}^n$ then the heuristic argument leading to (1.1) still applies. But the natural environment for our results seems to be the class of subharmonic functions. In the class of subharmonic functions the reduction of the semigroup to a composition operator as in (1.1) no longer holds. Instead we will establish an inequality

$$e^{-tA} f \geq f \circ \exp(-tX) \tag{1.2}$$

for subharmonic functions f. We will refer to the inequality (1.2) as relative subharmonicity for the reasons explained in the introduction to [G02]. Once the inequality (1.2) is proven, reverse hypercontractive inequalities, typically

$$\|e^{-tA} f\|_q \geq C(t, p, q)\|f\|_p, \tag{1.3}$$

can be deduced from (1.2) by the same method used in [GGS].

The technical problems associated with proving the inequality (1.2) are problems concerning the approximation of subharmonic functions by more regular subharmonic functions. The latter functions must be in the domain of A or of its quadratic form. This requires a regularization method which decreases growth at ∞ and smooths locally while at the same time preserving subharmonicity. The usual methods of dealing with approximation of functions by nice functions, such as functions in $C_c^\infty(M)$, must be modified because functions in $C_c^\infty(M)$ are never subharmonic for the manifolds of interest. We will introduce two methods of regularization. The first applies in one dimension. Here, the subharmonic functions are just the convex functions. The key step in "regularizing" a convex function in one dimension will consist in restricting the convex funtion to a large interval and then extending it linearly to obtain a slower growing convex function. Our main result in one dimension is Theorem 2.8, which will be shown in Section 5 to yield (1.2) and eventually, in Section 6, to yield (1.3). This will be carried out in Section 3.

In the second method of regularization, which is applicable on $\mathbb{R}^n$, we will combine convolution with composition: $f \to (\varphi * f) \circ \exp(-tX)$. Convolution by itself does not act as a bounded operator in $L^2(\mathbb{R}^n, \mu)$ because any translate of a typical measure μ of interest has an unbounded density with respect to μ. But the combined operation does act boundedly on $L^2(\mu)$ under some simple conditions on the relation between X and μ. In this method we will rely on the estimates of $\|d(\exp(tX))_* \mu / d\mu\|_p$ established in [GGS]. We will take advantage here of the fact that these estimates are sensitive to the sign of X. This will be carried out in Section 4.

It has already been observed in [GS] and [So] that the constants $C(t, p, q)$ in (1.3) which were found in [GS] and [GGS], yield reverse hypercontractive inequalities in the holomorphic category which are not saturated by any holomorphic functions. In the present paper we have a larger class of functions than the absolute value of holomorphic functions. Yet we still do not have saturation in this larger class in spite of having the same constants as for the smaller class. In Section 6 we will establish (1.3) with bigger constants in the class of α-subharmonic functions on $\mathbb{R}^n$. These are the functions satisfying $\Delta f \geq \alpha f$ for $\alpha \geq 0$. Yet we still do not have saturation. The optimal constant, $C(t, p, q)$, in (1.3) is therefore still unknown.

2 Notation and Some Statements of Results

Let M be an n dimensional Riemannian manifold with smooth Riemannian metric g. Denote by μ a probability measure on M. We assume throughout that μ has a strictly positive smooth density in each coordinate chart. For $1 \leq p \leq \infty$, $L^p(M, \mu)$ will denote the Banach space of real p-th power integrable functions with norm $\|\cdot\|_p$.

Associated to the triple (M, g, μ) is the pre-Dirichlet form

$$Q_0(f, \varphi) = \int_M h(df, d\varphi)\, d\mu, \quad f \in C^\infty(M),\ \varphi \in C_c^\infty(M), \tag{2.1}$$

where h is the dual Riemannian metric to g on the dual spaces $T^*(M)$. The differential operator

$$d^*d \colon C^\infty(M) \to C^\infty(M)$$

is defined by

$$\int_M (d^*df)\varphi\, d\mu = \int_M h(df, d\varphi)\, d\mu, \quad f \in C^\infty(M),\ \varphi \in C_c^\infty(M). \tag{2.2}$$

Next we pick a version of d^*d which is a nonnegative self-adjoint operator in $L^2(\mu)$. Denote by $(Q, D(Q))$ the closure of the pre-Dirichlet form $(Q_0, C_c^\infty(M))$ in $L^2(\mu)$. Then there is a unique, nonnegative self-adjoint operator $(A, D(A))$ (Friedrichs extension) in $L^2(\mu)$ such that

$$Q(f, g) = \int_M (Af)g\, d\mu, \quad \forall f \in D(A),\ g \in D(Q).$$

If M is not complete then this version of d^*d corresponds to choosing Dirichlet boundary conditions.

$D(Q)$ is a Hilbert space in the energy norm $\sqrt{\|\cdot\|_2^2 + Q}$; see *e.g.*, [BH, Da89, Fu80, MR, Si].

The semigroup $T_t := e^{-tA}, t \geq 0$, is a contraction on $L^2(\mu)$, *i.e.*, $\|T_t f\|_2 \leq \|f\|_2$ for all $f \in L^2(\mu)$. Furthermore, by the Beurling–Deny theorem, T_t has an extension to $L^p(\mu)$ for $p \in [1, 2)$ and $\|T_t f\|_p \leq \|f\|_p$ for all $1 \leq p \leq \infty$. This semigroup is strongly continuous on $L^p(\mu)$ for $1 \leq p < \infty$. Denote by $(A_p, D(A_p))$ the generator of T_t in $L^p(\mu)$ for $1 \leq p < \infty$. Then $T_t f = e^{-tA_p} f$ for all $f \in L^p(\mu)$ and $D(A_p) \subset D(A_q)$ if $\infty > p \geq q \geq 1$. Of course $A_2 = A$.

If μ is the Riemann–Lebesgue measure on M (and therefore not a finite measure in the cases of interest to us) the differential operator d^*d is just the Laplace–Beltrami operator. Otherwise d^*d differs from the Laplace–Beltrami operator by a first order operator.

We are going to choose a decomposition

$$d^*d = -L + X \tag{2.3}$$

in which X is a first order differential operator on M. But L will not necessarily be the Laplace–Beltrami operator. Such a decomposition arose in [G99, G02], where M was a complex manifold and L was the Hermitian Laplacian, which happens to coincide with the Laplace–Beltrami operator only when (M, g) is Kähler. In the present work this decomposition will not be so natural.

In all of our examples the manifold M will be either $\mathbb{R}^n$, a half line or the Riemann surface for $z^{1/n}$. In each case there is a natural Riemannian metric and a natural Laplacian. In all cases we will use a Riemannian metric which is conformal to the standard (or natural) one and we will then take L to be a (positive) multiple, $\sigma(x)\Delta$, of the natural Laplacian. The usual condition for a C^2 function to be subharmonic, namely $\Delta f \geq 0$ (weak derivatives), is then equivalent to $\sigma(x)\Delta f \geq 0$ (weak derivatives), *i.e.,* $Lf \geq 0$ (weak derivatives). The essential feature of the decomposition (2.3) for us is that the flow of the vector field X maps subharmonic functions to subharmonic functions.

Definition 2.1 For any second order differential operator L on M with smooth coefficients, a function $f \in L^1_{\mathrm{loc}}(M, \mu)$ is called L-*subharmonic* if

$$(f, L^*\phi)_{L^2(\mu)} \geq 0, \quad \forall \phi \in C_c^\infty(M)^+, \tag{2.4}$$

where L^* is the $L^2(\mu)$ adjoint of $L|C_c^\infty(M)$. $\mathcal{S}$ will denote the space of L-subharmonic functions on M.

In particular, if $f \in C^2(M)$ and (2.3) holds then (2.4) is equivalent to

$$(X - d^*d)f \geq 0. \tag{2.5}$$

It will be useful to rewrite (2.4) in terms of the operators that we will actually deal with. Aside from domains, the equation (2.3) may be written $L = X - A$, where A is the self-adjoint version of d^*d described above. Thus (2.4) is the weak version of

$$(X - A)f \geq 0. \tag{2.6}$$

That is,

$$(f, (X^* - A)\phi)_{L^2(\mu)} \geq 0 \quad \forall \phi \in C_c^\infty(M)^+. \tag{2.7}$$

In particular, if $f \in D(A) \cap D(X)$ then (2.6) holds when f is L-subharmonic. Here $D(X)$ refers to the domain of $(X^*|C_c^\infty)^*$.

Let us observe immediately that in order for (1.2) to hold for all $t \geq 0$, it is necessary for f to be L-subharmonic. To be specific, assume that $f \in L^p$ for some

$p \in [1, \infty]$ and that (1.2) holds for all $t \geq 0$. If $\phi \in C_c^\infty$ and $\phi \geq 0$ then $(f, (e^{-tA} - (e^{-tX})^*)\phi) = (e^{-tA} f - f \circ e^{-tX}, \phi) \geq 0$. Since this inequality is equality when $t = 0$, we may divide by t and take the limit as $t \downarrow 0$ to find (2.7). Therefore f is L-subharmonic.

We will always assume that the flow of the vector field X, $\exp(tX)$, exists for all $t \in \mathbb{R}$.

We also assume throughout that the flow of X leaves L-subharmonic functions invariant. That is, if f is in $\mathcal{S}$ then so is $f \circ \exp(tX)$ for all $t \in \mathbb{R}$.

Example 2.2 (Measures on $\mathbb{R}^n$) Let $M = \mathbb{R}^n$ and let σ be a strictly positive function in $C^\infty(\mathbb{R}^n)$. We define a Riemannian metric g by

$$g_x(\partial/\partial x_i, \partial/\partial x_j) = \delta_{ij}/\sigma(x). \tag{2.8}$$

Then the dual metric is

$$h_x(dx_i, dx_j) = \delta_{ij}\sigma(x).$$

We take

$$\mu = \rho(x)dx \tag{2.9}$$

where ρ is in $C^\infty(\mathbb{R}^n)$ and is strictly positive.

It will be convenient to express the operator d^*d in terms of ρ and w, where

$$w(x) := \sigma(x)\rho(x), \quad x \in \mathbb{R}^d. \tag{2.10}$$

An integration by parts yields

$$d^*df = -\sigma\Delta f + Xf, \quad f \in C^\infty(\mathbb{R}^n), \tag{2.11}$$

where the smooth vector field X is given by

$$X = -\sum_{i=1}^n \rho^{-1}(\partial w/\partial x_i)\partial/\partial x_i. \tag{2.12}$$

This choice of X gives

$$Lf = \sigma\Delta f, \quad f \in C^\infty(\mathbb{R}^n). \tag{2.13}$$

In this example and also on $(0, \infty)$, we will always choose ρ and σ to be related in such a way that (2.12) reduces to

$$X = c\sum_{i=1}^n x_j\partial/\partial x_j, \quad \text{for some } c > 0. \tag{2.14}$$

Then the flow of X is given by

$$\exp(tX)x = e^{tc}x, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{2.15}$$

By the chain rule,

$$\Delta(f \circ \exp(tX)) = e^{t2c}(\Delta f) \circ \exp(tX).$$

Hence if $Lf \geq 0$ then $L(f \circ \exp(tX)) \geq 0$. So the flow preserves L-subharmonicity.

Here is a class of densities ρ and metrics σ which yield (2.14). Let $\varphi\colon [0, \infty) \to (0, \infty)$ be infinitely differentiable. Assume that $\varphi'(s) < 0$ for all $s \in [0, \infty)$ and that $\varphi(s) \to 0$ as $s \to \infty$. Define

$$w(x) = \varphi(|x|^2), \tag{2.16}$$

$$\rho(x) = -b\,\varphi'(|x|^2), \quad b = \text{constant } > 0, \tag{2.17}$$

$$\sigma(x) = w(x)/\rho(x). \tag{2.18}$$

Then $\partial w/\partial x_j = 2x_j\varphi'(|x|^2) = -x_j(2/b)\rho(x)$. So (2.14) holds with $c = 2/b$. The constant b may be chosen to normalize ρ. In dimension two no further restrictions need be placed on φ, because $\int_{\mathbb{R}^2} \rho(x)\,dx = b\pi\varphi(0) < \infty$. In higher dimensions, $\varphi(s)$ must go to zero fast enough to make ρ integrable. A straightforward computation for $n \geq 3$ shows that $\int_{\mathbb{R}^n} \rho(x)\,dx = \text{const} \int_0^\infty s^{\frac{n-4}{2}} \varphi(s)\,ds$ when $\lim_{s\to\infty} s^{\frac{n-2}{2}} \varphi(s) = 0$. So ρ will be integrable if this limit relation holds and $\int_0^\infty s^{\frac{n-4}{2}} \varphi(s)\,ds < \infty$.

Remark 2.3 The examples for w, ρ, σ in (2.16)–(2.18) are all radial. This is actually forced by (2.14) in dimension $n \geq 2$. Indeed, (2.12) and (2.14) require $\operatorname{grad} w(x) = -cx\rho(x)$, which is orthogonal to any tangent direction to the sphere $\{|x| = r\}$. Hence w is constant on this sphere and so is radial. Writing w in the form (2.16), it now follows that $\operatorname{grad} w(x) = 2x\varphi'(|x|^2)$. The equation $\operatorname{grad} w(x) = -cx\rho(x)$ now forces $\rho(x) = \text{const}\,\varphi'(|x|^2)$, which is also radial. Since $w = \sigma\rho$ we see that σ is also radial. In dimension one ρ and σ need not be symmetric. This will be elaborated in Theorem 2.8.

Example 2.4 (Half line) Let $M = (0, \infty)$. Choose a strictly positive function $\rho \in C^\infty((0, \infty))$ such that $\int_0^\infty \rho(s)\,ds = 1$ and $\int_0^\infty s\rho(s)\,ds < \infty$. Define σ by

$$\sigma(x)\rho(x) = \int_x^\infty s\rho(s)\,ds, \quad x > 0. \tag{2.19}$$

Then $(\sigma\rho)'(x) = -x\rho(x)$ and (2.12) therefore reduces to $X = x\,d/dx$. So $d^*d = -L + X$ with $L = \sigma(x)d^2/dx^2$. The flow of X is $\exp(tX)(x) = e^t x$, which exists for all time. The L-subharmonic functions are those locally L^1 functions on $(0, \infty)$ such that $f''(x) \geq 0$ (in the sense of weak derivatives). After modification on a set of measure zero such a function can be chosen to be continuous, (see Remark 3.1). The condition weak $f'' \geq 0$ is then equivalent to the requirement that f be convex. The flow of X clearly preserves convexity.

Example 2.5 (A Riemann surface, [G02]) Let n be an integer ≥ 2. Denote by M_n the n-sheeted Riemann surface assosiated to $z^{1/n}$ considered as a two-dimensional real manifold. Write $\mathbb{R}^{2^*} = \mathbb{R}^2 - \{0\}$. M_n is a covering space of $\mathbb{R}^{2^*}$ with n leaves.

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Let $\alpha\colon M_n \to \mathbb{R}^{2^*}$ be the natural projection. In small, connected, simply connected coordinate charts on each sheet we can use the coordinates x, y induced by α and also the metric induced on M_n by α from the Euclidean metric on $\mathbb{R}^{2^*}$. Writing $\partial_x = \partial/\partial x$ and $\partial_y = \partial/\partial y$, we have $g(\partial_x, \partial_x) = g(\partial_y, \partial_y) = 1$ and $g(\partial_x, \partial_y) = 0$. Then M_n is an incomplete Riemannian manifold. Let $c > 0$ and define

$$p_c(z) = (2\pi c)^{-1} e^{-|z|^2/2c}, \quad z = (x, y) \in \mathbb{R}^{2^*}.$$

Let μ_n be the measure on M_n whose local density with respect to the Riemannian volume element, $d\,\mathrm{Vol} = dxdy$, is

$$d\mu_n/d\,\mathrm{Vol}(z) = (1/n)p_c(\alpha(z)), \quad z \in M_n.$$

Since $\int_{\mathbb{R}^{2*}} p_c(z)\,dxdy = 1$ we see that μ_n is a probability measure on M_n. Otherwise stated, the probability density is divided equally among the n sheets. Then

$$d^*df = -\Delta f + Xf, \quad f \in C^\infty(M_n),$$

where

$$Xf = x\partial_x f + y\partial_y f.$$

Using local polar coordinates, *i.e.*, $z = r(\sin(\theta), \cos(\theta))$, $r > 0$, the flow corresponding to the vector field X is given by

$$\exp(tX)\big(r(\sin(\theta), \cos(\theta))\big) = e^t r(\sin(\theta), \cos(\theta)).$$

Choosing $L = \Delta$, obviously the flow generated by X preserves L-subharmonicity and exists for all time.

Notation 2.6 We will write

$$\mathcal{S}L^p(\mu) = \mathcal{S} \cap L^p(\mu), \quad 1 \le p < \infty. \tag{2.20}$$

Then $\mathcal{S}L^p(\mu)$ is a closed convex cone in $L^p(\mu)$, as follows from (2.4). Our results are largely concerned with a possibly proper subset of $\mathcal{S} \cap L^p$. Define

$$\begin{aligned} \mathcal{S}^2 &= L^2\text{-closure of } \mathcal{S} \cap D(Q), \\ \mathcal{S}^p &= \mathcal{S}^2 \cap L^p \quad \text{for } 2 < p < \infty \\ \text{and} \qquad \mathcal{S}^p &= \text{ closure of } \mathcal{S}^2 \text{ in } L^p \quad \text{for } 1 \le p < 2. \end{aligned}$$

One always has $\mathcal{S}^2 \subset \mathcal{S}L^2(\mu)$, as is clear from the definitions and the fact that $\mathcal{S}L^2(\mu)$ is closed in $L^2(\mu)$. The question of equality will be a central issue for us.

Overview 2.7 A key result in this paper is that the relative subharmonicity inequality (1.2) holds for sufficiently nice subharmonic functions f. "Nice" means that f must be well related to the domain of A even if it is not actually in the domain of A. In Section 5 we will show that (1.2) holds for all f in $\mathcal{S}^2$, but we will see in Theorem 2.9 that (1.2) can actually fail if f is merely in $\mathcal{S} \cap L^2$. For this reason it is important to know when $\mathcal{S}^2$ and $\mathcal{S} \cap L^2$ coincide. Sections 3 and 4 are devoted to proofs that these spaces coincide under some conditions on M, g, and μ. To begin with, the next theorem shows that they always coincide on the line. On the other hand, Theorem 2.9 shows that they can easily fail to coincide on the half-line and that (1.2) can then fail. This is, of course, a boundary effect.

The next two theorems concern the relation between $\mathcal{S}^2$ and $\mathcal{S} \cap L^2$ in one dimension.

Theorem 2.8 *Let $M = \mathbb{R}$ and let (g, μ) be as in* (2.8) *and* (2.9). *Assume that $\sigma(x)\rho(x) \to 0$ as $x \to \pm\infty$ and that $(\sigma\rho)'(x) = -x\rho(x)$. Then $X = xd/dx$ and*

$$\mathcal{S}^2 = \mathcal{S}L^2(\mu).$$

If $\int_0^\infty x^2\rho(x)\,dx = \int_{-\infty}^0 x^2\rho(x)\,dx = \infty$ then $\mathcal{S}^2 = \{\text{constants}\}$. Otherwise $\mathcal{S}^2$ is infinite dimensional.

This will be proven in Section 3 by a regularization procedure which consists in restricting a convex function to an interval and then extending it linearly to a convex function of possibly slower growth.

In the opposite direction we have the following negative theorem for the half line. It will also be proven in Section 3.

Theorem 2.9 *Take $M = (0, \infty)$. Suppose that g and $\mu := \rho(x)dx$ have the standard form,* (2.8) *and* (2.9) *on* $(0, \infty)$. *(*cf. *Theorem 2.8 for one dimension.). Assume that $\int_0^\infty x^2\rho(x)\,dx < \infty$. Then*

$$\mathcal{S}^2 \subsetneqq \mathcal{S}L^2(\mu).$$

In particular the constant function $f \equiv 1$ is in $\mathcal{S}L^2(\mu)$ but not in $\mathcal{S}^2$. Moreover the inequality (1.2) *fails for this function.*

Remark 2.10 $D(Q)$ is always dense in L^2. But $\mathcal{S} \cap D(Q)$ need not be dense in $\mathcal{S} \cap L^2$. This is what happens in the one dimensional examples described in Theorem 2.9. One sees in these examples how $\mathcal{S}L^2 \ominus \mathcal{S}^2$ detects the boundary of $(0, \infty)$. This sense in which subharmonic functions detect the boundary is similar to that described in [G99, G02] for holomorphic function spaces. For example it was shown in [G02, Theorem 6.2] that for the Riemann surface of Example 2.5, the boundary at $z = 0$ is "detected" in the holomorphic category. We believe that the boundary is also detected in the subharmonic category, *i.e.*, $\mathcal{S}^2 \neq \mathcal{S}L^2$. But we do not have a proof.

Remark 2.11 We are going to prove in Section 4 a theorem of the form

$$\mathcal{S}^2(\mathbb{R}^n, \sigma, \mu) = \mathcal{S}L^2(\mathbb{R}^n, \sigma, \mu)$$

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for a class of measures μ and metrics σ which satisfy some regularity conditions. The proof will use a very different regularization method from the proof of Theorem 2.8.

3 The Space $\mathcal{S}^2$ in One Dimension

Our main objective in this section and the next is to prove that

$$\mathcal{S}^2(M, g, \mu) = \mathcal{S}L^2(M, g, \mu)$$

for several different classes of manifolds without boundary. In each case, we need to use a regularization method which produces a smooth subharmonic approximation to a given subharmonic function in L^2. By smooth we mean slow growth near ∞ as well as locally smooth: the smoothed subharmonic function must lie in $D(Q)$. We are going to describe two different regularization methods. The first will be applied in one dimension, in this section, to prove Theorem 2.8. Here we will take a subharmonic ($\equiv$ convex) function on $\mathbb{R}$, restrict it to an interval, and then extend the restriction linearly to produce a convex support function of linear growth. This method seems limited to one dimension.

At the end of this section we will prove that $\mathcal{S}^2 \neq \mathcal{S}L^2$ on the half line, as asserted in Theorem 2.9.

Remark 3.1 In both of our one dimensional theorems, on the line and on the half-line, L is given as in Examples 2.2 and 2.4. Thus in both cases $Lf(x) = \sigma(x)f''(x)$ for $f \in C^\infty(M)$, where $M = \mathbb{R}$ or $(0, \infty)$. If $f \in L^1_{\text{loc}}(M)$ then, by Definition 2.1, f is L-subharmonic if and only if $\int_M f(x)(d^2/dx^2)(\sigma(x)\rho(x)\phi(x))\,dx \geq 0$ for all $\phi \in C_c^\infty(M)$ with $\phi \geq 0$. Since $\sigma\rho$ is bounded away from zero on compact sets and is itself in $C^\infty(M)$, we see that f is L-subharmonic if and only if

$$\int_M f(x)\psi''(x)\,dx \geq 0 \quad \forall \psi \in C_c^\infty(M) \text{ with } \psi \geq 0. \tag{3.1}$$

This assures that, after modification on a set of measure zero, f is simply a convex function.

Here is a sketch of a proof of this assertion. It suffices in both cases just to take $M := (a', b')$ to be a bounded interval and then to show that if $f \in L^1_{loc}(M)$ (for Lebesgue measure) and satisfies (3.1), then on any compact interval $[a, b] \subset (a', b')$, f can be modified on a set of measure zero so as to become a convex (and therefore continuous) function. To this end choose $\epsilon > 0$ such that $[a - 4\epsilon, b + 4\epsilon] \subset (a', b')$ and choose a sequence $\phi_n \in C^\infty(\mathbb{R})$ such that $\phi_n \geq 0$, $\int_{\mathbb{R}} \phi_n\,dx = 1$ and support $\phi_n \subset [-1/n, 1/n] \cap [-\epsilon, \epsilon]$. Since $f \in L^1([a - 4\epsilon, b + 4\epsilon])$, the convolutions $f_n = \phi_n * f$ are well defined on the interval $I \equiv (a - 3\epsilon, b + 3\epsilon)$. They are in $C^\infty(I)$ and satisfy $f_n'' \geq 0$ on I by (3.1). Moreover, f_n converges to f in $L^1(I)$. Suppose that $a \leq x < y \leq b \leq s \leq b + \epsilon$ and $b + 2\epsilon \leq t \leq b + 3\epsilon$. Since f_n is convex on I we have

$$\frac{f_n(y) - f_n(x)}{y - x} \leq \frac{f_n(t) - f_n(s)}{t - s}.$$

Integrating this inequality with respect to s and t over the two intervals $b \le s \le b+\epsilon$ and $b+2\epsilon \le t \le b+3\epsilon$, we find inequalities whose right-hand sides are integrals that converge to $\int_b^{b+\epsilon} ds \int_{b+2\epsilon}^{b+3\epsilon} dt \frac{f(t)-f(s)}{t-s}$. In particular these integrals are uniformly bounded in n. A similar argument applies to the left of a yielding constants A and B such that

$$A \le \epsilon^2 \frac{f_n(y)-f_n(x)}{y-x} \le B, a \le x < y \le b. \tag{3.2}$$

The functions f_n are therefore uniformly Lip 1 on $[a,b]$. Multiply (3.2) by $y-x$, put $y = b$ and integrate with respect to x over the half interval $[a,(a+b)/2]$ to find that the sequence $f_n(b)$ is uniformly bounded above. A similar argument shows that $\{f_n(a)\}$ is bounded below. Thus the sequence $f_n|[a,b]$ is equicontinuous and uniformly bounded. A subsequence therefore converges uniformly to a continuous, and clearly convex function g on $[a,b]$. Of course $f = g$ a.e. on $[a,b]$.

Proof of Theorem 2.8: Support function regularization in one dimension
Case (i):

$$\int_{-\infty}^{\infty} x^2 \rho(x)\, dx < \infty. \tag{3.3}$$

Let $w(x) = \sigma(x)\rho(x)$ as in (2.10). For $0 \le x \le a$ we have $w(a)-w(x) = -\int_x^a s\rho(s)\, ds$ because $w'(x) = -x\rho(x)$. Since $w(a) \to 0$ as $a \to \infty$, monotone convergence yields $w(x) = \int_x^\infty s\rho(s)\, ds$ for $x \ge 0$ and therefore for all x. Hence $\int_0^\infty w(x)\, dx = \int_0^\infty \int_x^\infty s\rho(s)\, dsdx = \int_0^\infty \int_0^s s\rho(s)\, dxds = \int_0^\infty s^2\rho(s)\, ds < \infty$. A similar argument on $(-\infty,0]$ then gives $\int_{-\infty}^\infty w(x)\, dx < \infty$.

We assert that if $f \in C^\infty(\mathbb{R})$ and has bounded first derivative f', then $f \in D(Q)$. Indeed, choose ψ in $C_c^\infty(\mathbb{R})^+$ with $\psi = 1$ on $[-1,1]$ and $\psi = 0$ outside $[-2,2]$. Let $\psi_n(x) = \psi(x/n)$. Then ψ_n converges to one boundedly as $n \to \infty$ and $\psi_n'(x)$ converges to zero for each x. Moreover, $|x\psi_n'(x)| = |(x/n)\psi'(x/n)| \le 2\sup_{y\in\mathbb{R}} |\psi'(y)|$. So $x\psi_n'(x)$ converges to zero boundedly. Since f' is bounded, $|f(x)|$ has at most linear growth at $\pm\infty$. Hence $f(x)\psi_n'(x)$ converges to zero boundedly. In view of (3.3) and the linear growth of f, we see that $f \in L^2(\mu)$ and $f\psi_n$ converges to f in the $L^2(\mu)$ sense. Moreover,

$$Q(f\psi_n - f\psi_k) = \int_{\mathbb{R}} \bigl[f'(x)\bigl(\psi_n(x)-\psi_k(x)\bigr) + f(x)\psi_n'(x) - f(x)\psi_k'(x)\bigr]^2 \sigma(x)\rho(x)\, dx.$$

But $f'(x)(\psi_n(x)-\psi_k(x))$ goes to zero boundedly, as does also $f\psi_n'$ and $f\psi_k'$, as n and $k \to \infty$. Since $\int_{-\infty}^\infty \sigma(x)\rho(x)\, dx < \infty$, $f\psi_n$ is a Cauchy sequence in Q norm. Hence $f \in D(Q)$.

Now suppose that $f \in \mathcal{S}L^2(\mu)$. Define a sequence of convex functions f_n as follows. Let $f_n(x) = f(x)$ for $|x| \le n$ and let $f_n(x) = f(n) + \alpha(x-n)$ for $x > n$ and $f_n(x) = f(-n) + \beta(x+n)$ for $x < -n$. We may choose α and β so that f_n is convex and $f_n \le f$. For example, choose $\alpha = f'(n)$ if $f'(n)$ exists and otherwise choose α

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to be the slope of any support line of the convex function f at n. Choose β similarly. Then $f_n(x)$ increases to $f(x)$ for all x. Moreover each f_n is linear (actually affine) on the intervals $(-\infty, -n]$ and $[n, \infty)$. We assert that $f_n \in \mathcal{S}^2$. To see this, choose an approximate identity $\varphi_k \in C_c^\infty(\mathbb{R})^+$ with supp $\varphi_k \subset [-k^{-1}, k^{-1}]$. It is straightforward to verify that $\varphi_k * f_n$ is affine on each interval $(-\infty, -(n+1)]$ and $[n+1, \infty)$ using the identity

$$(3.4) \qquad (\varphi_k * (ay + b))(x) = ax + b - a \int_{-k^{-1}}^{k^{-1}} \varphi_k(s) s\, ds.$$

So $\varphi_k * f_n$ has bounded derivative and is therefore in $D(Q)$ by what was proved in the previous paragraph. Moreover $\varphi_k * f_n$ converges to f_n in $L^2(\mu)$, as is clear from (3.4) and the fact that f_n is continuous on $[-(n+1), n+1]$. So $f_n \in \mathcal{S}^2$. Finally, the inequality $|f - f_n|^2 \le |f - f_1|^2$ shows that $f_n \to f$ in $L^2(\mu)$ by dominated convergence. Hence $f \in \mathcal{S}^2$. Therefore $\mathcal{S}^2 = \mathcal{S}L^2$ and in particular the functions $a|x - b|$ are in $\mathcal{S}^2$ for all $a > 0$ and $b \in \mathbb{R}$. So $\mathcal{S}^2$ is infinite dimensional.

Case (ii):

$$(3.5) \qquad \int_{-\infty}^{0} x^2 \rho(x)\, dx = \int_{0}^{\infty} x^2 \rho(x)\, dx = \infty.$$

Suppose that $f \in \mathcal{S}L^2(\mu)$. Since f is L-subharmonic, it has a version which is convex and continuous. We pick this version and denote it also by f. If f is not constant, then there are two distinct points x_0 and x_1 with $x_0 < x_1$ such that $f(x_0) \ne f(x_1)$. Let $\alpha = (f(x_1) - f(x_0))/(x_1 - x_0)$. Assume first that $\alpha > 0$. Since f is convex, we have $f(x) \ge f(x_1) + \alpha(x - x_1)$ for $x \ge x_1$. The right side is positive for large positive x and is not in $L^2(\mu)$ by (3.5). Hence f is not in $L^2(\mu)$. Similarly, if $\alpha < 0$ then $f(x) \ge f(x_0) + \alpha(x - x_0)$ for $x \le x_0$, and the right side is positive for large negative x and is also not in $L^2(\mu)$. So f is constant.

Conversely, if f is constant then clearly $f \in \mathcal{S}L^2(\mu)$. To show that $f \in \mathcal{S}^2$, choose ψ_n as in Case (i). Then ψ_n converges to one boundedly and further $x\psi_n'(x) \le C$ for some constant C and all x and n. Consequently $(\psi_n'(x) - 0)^2 w(x) \le Cw(x)/x^2$ for $|x| \ge 1$. The left side goes to zero uniformly on $\mathbb{R}$ and the right side is integrable on $\{|x| \ge 1\}$ because w is bounded. Hence $Q(\psi_n - 1) \to 0$. So the constant functions are in $D(Q)$.

Case (iii):

$$(3.6) \qquad \int_{-\infty}^{0} x^2 \rho(x)\, dx < \infty \quad \text{and} \quad \int_{0}^{\infty} x^2 \rho(x)\, dx = \infty.$$

Suppose that $f \in \mathcal{S}L^2$. If $x_0 < x_1$, then the argument in Case (ii) shows that $f(x_1) - f(x_0) \le 0$ because otherwise f exceeds a linearly increasing function as $x \to +\infty$ and is therefore not in L^2 by (3.6). So f is nonincreasing. Let $B = \inf f$, where B might be finite or $-\infty$. Let $c > B$ and define $g_c(x) = \max(c, f(x))$. Then g is convex, bounded below and constant on a half line to the right, say on $[b, \infty)$. Moreover f is

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the L^2 limit of g_c as $c \downarrow B$, as is clear if $B > -\infty$ because of uniform convergence and also if $B = -\infty$ by dominated convergence, since f is in L^2. So it suffices to show that g_c is in $\mathcal{S}^2$. To keep the notation simple, we will just assume that f itself is in $\mathcal{S} \cap L^2$, is constant on a half ray to the right and we will show that it is in $\mathcal{S}^2$. Let m be large and positive and let h_m be a convex function which agrees with f to the right of $-m$, is linear to the left of $-m$, and is dominated above by f everywhere. Such a function was constructed in the proof of Case (i). We have already seen in the proof of Case (i) that, as $m \to \infty$, h_m converges to f in L^2 norm. Thus it suffices to show that h_m is in $\mathcal{S}^2$ for each m. Again, to keep the notation simple, we may assume that f itself is in $\mathcal{S} \cap L^2$, is consant on a half line to the right, say on $[b, \infty]$, and is linear on $(-\infty, a]$ with $a < b$. Now choose an approximate identity, φ_k, as in Case (i), and let $f_k = \varphi_k * f$. Then f_k is infinitely differentiable, is constant to the right of $b + 1$, is linear to the left of $a - 1$ as already discussed in Case (i), and converges to f in L^2 as $k \to \infty$, as already explained in Case (i). So it suffices to show that f is in $\mathcal{S}^2$ if it is in $C^\infty(\mathbb{R})$, is constant on an interval $[b, \infty)$, linear on an interval $(-\infty, a]$, and is in $\mathcal{S} \cap L^2$. To this end, choose ψ_n as in Case (i) and observe that $\psi_n(x) f(x)$ is in $C_c^\infty(\mathbb{R})$. Moreover this sequence is Cauchy in energy norm as one sees by using the argument in Case (i) on the left half line and the argument in Case(ii) on the right half line. Hence f is in $D(Q)$ and is therefore in $\mathcal{S}^2$.

Case (iv): The remaining case is equivalent to Case (iii) upon replacing $f(x)$ by $f(-x)$. ∎

We will need the following lemma to prove Theorem 2.9.

Lemma 3.2 *Under the hypotheses of Theorem 2.9 there holds*

$$f(x) \leq x \|f\|_{L^2(\mu)} \Big(\int_1^\infty y^2 \rho(y)\, dy \Big)^{-1/2} \quad \text{for } 0 < x \leq 1 \tag{3.7}$$

for any function $f \in \mathcal{S}^2$

Proof As in Case (i) of the proof of Theorem 2.8 we may write $w(x) = \int_x^\infty s\rho(s)\, ds$. So w is bounded away from 0 on $(0, 1]$. For $f \in C_c^\infty(0, \infty)$ and $x \in (0, 1]$ we have

$$|f(x)| \leq \int_0^x |f'(s)|\, ds \leq \Big(\int_0^x |f'(s)|^2 w(s)\, ds \Big)^{1/2} \Big(\int_0^1 w(s)^{-1}\, ds \Big)^{1/2}.$$

Since the first factor is dominated by $Q(f)^{1/2}$, the inequality holds for all $f \in D(Q)$. It follows that $f(x) \to 0$ as $x \downarrow 0$ for all $f \in D(Q)$.

It suffices to prove (3.7) for any function $f \in \mathcal{S} \cap D(Q)$ because any L^2 limit of such functions is again subharmonic and has a continuous version for which (3.7) continues to hold.

Suppose then that $f \in \mathcal{S} \cap D(Q)$. As we have seen, $\lim_{x \downarrow 0} f(x) = 0$. Defining $f(0) = 0$ makes f convex on $[0, \infty)$. Fix $x \in (0, 1]$ and let $y \geq 1$. Since x is in the segment $[0, y]$, convexity of f on $[0, y]$ gives

$$f(x) = f\Big(\frac{y - x}{y} 0 + \frac{x}{y} y\Big) \leq \frac{y - x}{y} f(0) + \frac{x}{y} f(y) = \frac{x}{y} f(y).$$

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Hence

$$y f(x) \le x f(y) \quad \text{for } 0 < x \le 1 \text{ and } y \ge 1. \tag{3.8}$$

Now if $f(x) < 0$, then (3.7) clearly holds. If $f(x) \ge 0$, then (3.8) shows that $f(y) \ge 0$ also. So we may square both sides of (3.8) and integrate with respect to y over $[1, \infty)$ to find $f(x)^2 \int_1^\infty y^2 \rho(y)\, dy \le x^2 \int_0^\infty f(y)^2 \rho(y)\, dy$. Taking the square root gives (3.7). ■

Proof of Theorem 2.9 The constant function equal to one is in $\mathcal{S}L^2$ but does not satisfy (3.7) and so is not in $\mathcal{S}^2$. Hence $\mathcal{S}^2 \neq \mathcal{S}L^2$. This completes the proof of the first assertion of Theorem 2.9. But it may be interesting to observe that the functions $f(x) = x^{-a}$ are also in $\mathcal{S}L^2$, for $0 < a < 1/2$, but are not in $\mathcal{S}^2$. Now if $f \equiv 1$ then $f \circ \exp(-tX) \equiv 1$ also. But, for any $t > 0$, $e^{-tA} f$ is in the domain of A, and therefore goes to zero as $x \downarrow 0$. So (1.2) cannot hold for small x. ■

4 $\mathcal{S}^2(\mathbb{R}^n)$: Second Regularization Method for Subharmonic Functions

Convolution by a fixed function $\varphi \in C_c^\infty(\mathbb{R}^n)$ is not, by itself, a good smoothing operation in $L^2(\mathbb{R}^n, \mu)$ because it is an unbounded operator for most measures of interest, including Gauss measure. This results from the fact that a translate of μ has an unbounded density with respect to μ in these important cases. We will instead combine convolution with dilation. The success of this method depends on the fact that the transform of μ by a combined translation and dilation (by the flow $\exp(-tX)$) can have a bounded density with respect to μ. To carry this out, we begin by recalling some results from [GGS] in the next definition and proposition.

Definition 4.1 (μ-divergence) Let V be a vector field on a manifold M. Let μ be a probability measure on M with a smooth, strictly positive density in each coordinate chart. The *μ-divergence* of V is the unique function W on M satisfying

$$\int_M V\varphi \, d\mu = \int_M \varphi W \, d\mu, \quad \forall \varphi \in C_c^\infty(M). \tag{4.1}$$

For example, if $M = \mathbb{R}^n$, $d\mu = \rho dx$ and $V = \sum_{j=1}^n a_j(x)\partial/\partial x_j$, then (4.1) gives $\mu\text{-div}\, V = -\operatorname{div} V - V \log \rho$, which reduces to the *negative* of the usual divergence of V if μ is Lebesgue measure.

Define

$$B(s) := \log\Big(\int_M e^{\mu\text{-div}\, V/s} \, d\mu\Big)^s, \qquad 0 < s < \infty, \tag{4.2}$$

and assume that

$$\kappa := \inf\{s > 0 | B(s) < \infty\} < \infty. \tag{4.3}$$

We will use the following result from [GGS, Theorem 2.14].

Proposition 4.2 (L^p norm of a Radon–Nikodym derivative induced by a flow) *Assume that the flow* $\exp(tV)$ *exists for all* $t \in \mathbb{R}$, *that* μ-div V *is in* $L^1(\mu)$ *and that* (4.3) *holds. Then for all* $t \geq 0$

$$\|d(\exp(tV)_*\mu)/d\mu\|_{r'} \leq e^{\Lambda_\kappa(r,t)}, \quad e^{\kappa t} < r < \infty, \tag{4.4}$$

where r' *is the conjugate index to* r *and*

$$\begin{aligned} \Lambda_\kappa(r,t) &= \frac{1}{b}\int_{b/r}^{b} \frac{B(\kappa+y)}{(\kappa+y)}\,dy, \quad \kappa > 0, \\ \Lambda_0(r,t) &= \frac{t}{r-1}\int_{\frac{r-1}{tr}}^{\frac{r-1}{t}} \frac{B(y)}{y}\,dy, \quad \kappa = 0, \end{aligned} \tag{4.5}$$

where $b = \frac{\kappa(r-e^{\kappa t})}{(e^{\kappa t}-1)}$.

Corollary 4.3 *Suppose that* μ-div V *is in* $L^1(\mu)$ *and* μ-div $V \leq C < \infty$. *Then*

$$\|d(\exp(tV)_*\mu)/d\mu\|_\infty \leq e^{tC} \quad \forall t \geq 0. \tag{4.6}$$

Moreover, for $1 \leq p \leq \infty$, *and for all* $f \in L^p$,

$$\|f \circ \exp(tV)\|_p \leq e^{tC/p}\|f\|_p \quad \forall t \geq 0. \tag{4.7}$$

Proof For $t = 0$, (4.6) is obvious. Suppose that $t > 0$. From (4.2) we find $B(s) \leq \log(e^{C/s})^s = C$ for all $s > 0$. Hence $\kappa = 0$ by (4.3). Now (4.5) yields $\Lambda_0(r,t) \leq (tC/(r-1))\log r$ for any $r > 1$. Hence by (4.4)

$$\|d(\exp(tV)_*\mu)/d\mu\|_{r'} \leq \exp\Big(tC\frac{\log(r)}{r-1}\Big), \quad \forall r > 1.$$

L'Hôpital's rule shows that the limit on the right-hand side as $r \downarrow 1$ is e^{tC}, which proves (4.6). Now (4.7) follows, for $1 \leq p < \infty$, from

$$\|f \circ \exp(tV)\|_p^p \leq \int |f(x)|^p e^{tC}\,d\mu(x).$$

Clearly (4.7) is correct also for $p = \infty$ if one interprets $e^{tC/p} = 1$. ■

Example 4.4 Let $c > 0$ and let

$$d\gamma_c(x) = (2\pi c)^{-n/2}e^{-|x|^2/(2c)}dx$$

denote the Gauss measure on $\mathbb{R}^n$ with covariance $c > 0$. Let g be the standard metric on $\mathbb{R}^n$. With $\mu = \gamma_c$, (2.12) reduces to

$$X = c^{-1}\sum_{i=1}^n x_i\partial/\partial x_i. \tag{4.8}$$

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Then (2.13) yields $L = \Delta$. Furthermore, an integration by parts shows that

$$\gamma_c\text{-div}\, X = c^{-1}(c^{-1}|x|^2 - n)$$

and

$$\int_{\mathbb{R}^n} e^{\gamma_c\text{-div}\, X/s}\, d\gamma_c = e^{-n/(sc)}(1 - 2/(sc))^{-n/2}, \quad sc > 2.$$

This integral is infinite if $sc \leq 2$. Hence $\kappa = 2/c$. The inequality (4.4) was shown in [GGS, Example 2.16] to be equality when $V = X$. Hence the estimate (4.4) in Proposition 4.2 is sharp in this Gaussian case. Specifically, it was shown in [GGS, Equation (2.47)] that

$$\|d(\exp(tX)_*\gamma_c/d\gamma_c\|_{L^{r'}} = \left(e^{-2t/(cr)}\left(\frac{r-1}{r - e^{2t/c}}\right)^{1-r^{-1}}\right)^{n/2}. \tag{4.9}$$

We will use this later.

On the other hand, γ_c-div $X \geq -n/c$. Hence from Corollary 4.3 we can infer that $\|d(\exp(-tX)_*\gamma_c)/d\gamma_c\|_\infty \leq e^{tn/c}$ for all $t \geq 0$. An easy calculation shows that equality holds here also. Thus Corollary 4.3 is also sharp for Gauss measure and X.

Now consider the vector field $a \cdot \nabla$ that generates translations in the direction of the vector $a \in \mathbb{R}^n$. An integration by parts shows that γ_c-div$(a \cdot \nabla) = c^{-1}a \cdot x$, which is unbounded above and below. But γ_c-div$(X + a \cdot \nabla) = c^{-1}(c^{-1}|x|^2 + a \cdot x - n)$, which is bounded below. Corollary 4.3 therefore shows that the density $d(\exp(-t(X+a \cdot \nabla))_*\gamma_c)/d\gamma_c$ is bounded. This will be a key element in the regularization method of Theorem 4.5.

Theorem 4.5 (Joint dilational and translational regularization) *Let μ be a smooth measure on $\mathbb{R}^n$. For each vector $a \in \mathbb{R}^n$ assume that μ-div$(X + a \cdot \nabla)$ is bounded below uniformly on compact a sets. Explicitly, we assume that for each compact set K in $\mathbb{R}^n$, there is a constant C_K such that*

$$\mu\text{-div}(X + a \cdot \nabla) \geq -C_K, \quad \forall a \in K. \tag{4.10}$$

Assume also that μ-div$(X + a \cdot \nabla) \in L^1(\mu)$ for all $a \in \mathbb{R}^n$. Let $t > 0$ and let $y \in \mathbb{R}^n$. For any function $f \in L^p(\mu)$ define $f_{t,y}(x) = f(e^{-t}x - y)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$. Then:

(i) *The map $f \mapsto f_{t,y}$ is a bounded operator from $L^p(\mu)$ into $L^p(\mu)$.*

(ii) *For fixed f in L^p and fixed $t > 0$, the map $y \to f_{t,y}$ is continuous on $\mathbb{R}^n$ to $L^p(\mu)$.*

(iii) *The contracted convolution operator*

$$f \mapsto (f * \varphi)(e^{-t}\,\cdot\,)$$

is a bounded operator from $L^p(\mu)$ into $L^p(\mu)$ for each $t > 0$.

Proof: (i) The flow generated by the vector field $V_a \equiv X + a \cdot \nabla$ is given by the solution of the differential equation $dx(t)/dt = x(t) + a$ with initial condition $x(0) = x$. Thus

$$\exp(tV_a)(x) = e^t x + a(e^t - 1). \tag{4.11}$$

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By Corollary 4.3 applied to $-V_a$, we have

$$\|f \circ \exp(-tV_a)\|_p \le e^{tC_K/p}\|f\|_p, \qquad \text{if } a \in K. \tag{4.12}$$

Now

$$f(\exp(-tV_a)x) = f(e^{-t}x - a(1 - e^{-t})). \tag{4.13}$$

So given $y \in \mathbb{R}^n$, choose $a = (1 - e^{-t})^{-1}y$ and K large enough to contain a. Then (4.12) holds and so $f \mapsto f((e^{-t}\cdot) - y)$ is bounded on $L^p(\mu)$.

(ii) The standard proof of this for Lebesgue measure requires a minor modification: given $\epsilon > 0$, choose a function $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\|f - \psi\|_p < \epsilon$. Let $f_t(u) = f(e^{-t}u)$. Then we may write (4.13) as

$$f(\exp(-tV_a)x) = f_t(x - a(e^t - 1)).$$

If $y_k \to y_0$ in $\mathbb{R}^n$, choose a compact set K in $\mathbb{R}^n$ such that $(e^t - 1)^{-1}y_k \in K$ for all k. Then (4.12) yields

$$\|f_t(\cdot - y_k) - \psi_t(\cdot - y_k)\|_p \le e^{tC_K/p}\|f - \psi\|_p \le \epsilon e^{tC_K/p}.$$

So writing $f_t^y(x) = f_t(x - y)$, we have

$$\|f_t^{y_k} - f_t^{y_0}\|_p \le \|f_t^{y_k} - \psi_t^{y_k}\|_p + \|\psi_t^{y_k} - \psi_t^{y_0}\|_p + \|\psi_t^{y_0} - f_t^{y_0}\|_p.$$

Since the middle term goes to zero as $k \to \infty$, we have $\limsup_{k\to\infty}\|f_t^{y_k} - f_t^{y_0}\|_p \le 2\epsilon e^{tC_K/p}$, which completes the proof of part (ii).

(iii) If $K \supset (1 - e^{-t})^{-1}\operatorname{supp}\varphi$, then (4.12) yields

$$\|(f * \varphi)(e^{-t}\cdot)\|_p \le \|\varphi\|_{L^1(\mathbb{R}^n,dx)}e^{tC_K/p}\|f\|_p. \qquad \blacksquare$$

Remark 4.6 The inequality (4.10) can be used to give a lower bound on a “perturbed Hamiltonian operator” on $L^2(\mu)$. Ignoring technical domain issues, write X^* for the adjoint of the differential operator X in $L^2(\mu)$. Define $H_0 = X + X^*$ and similarly define $V(a) = (a\cdot\nabla) + (a\cdot\nabla)^*$. If $f \in C_c^\infty(\mathbb{R}^n)$ and μ-div$(X + a\cdot\nabla) \ge -C(a)$, *cf.* (4.10), then

$$\big((H_0 + V(a))f, f\big)_{L^2(\mu)} \ge -C(a)\|f\|^2_{L^2(\mu)}.$$

This follows from the identities

$$\begin{aligned}\Big((H_0 + V(a))f, f\Big)_{L^2(\mu)} &= \int_{\mathbb{R}^n} (Xf)\overline{f} + f\overline{(Xf)} + (a\cdot\nabla f)\overline{f} + f\overline{(a\cdot\nabla f)}\,d\mu\\ &= \int_{\mathbb{R}^n} (X + a\cdot\nabla)|f|^2\,d\mu\\ &= \int_{\mathbb{R}^n} |f|^2\,\mu\text{-div}(X + a\cdot\nabla)\,d\mu.\end{aligned}$$

So (4.10) is a sufficient condition for semiboundedness of the operator $H_0 + V(a)$ acting in $L^2(\mathbb{R}^n, \mu)$. The operator $H_0 + V(a)$ actually does reduce to a standard kind of perturbation of the harmonic oscillator Hamiltonian if one takes $\mu =$ Gauss measure γ on $\mathbb{R}^{2m}$, identifies $\mathbb{R}^{2m}$ with $\mathbb{C}^m$, and restricts the operator H_0 and $V(a)$ to the space $\mathcal{H}^2(\gamma)$ of holomorphic functions in $L^2(\mathbb{C}^m, \gamma)$. As is well known (see *e.g.,* [GM]), the Segal–Bargmann transform takes the harmonic oscillator Hamiltonian from the Schrödinger representation (*i.e.,* $L^2(\mathbb{R}^m, dx)$) to the above operator H_0 in the holomorphic function representation. In the Schrödinger representation the operator $V(a)$ is given by multiplication by the linear functional $a \cdot x$ when $a \in \mathbb{R}^m$. If one carries out the computation of the constant $C(a)$ in the Gaussian case, one finds that it is strongly dimension dependent and therefore of questionable usefulness for quantum field theory.

The following theorem is the motivation for the previous regularization theorem.

Theorem 4.7 ($\mathcal{S}^2 = \mathcal{S}L^2(\mathbb{R}^n)$) *Let* $(\mathbb{R}^n, g, \mu)$ *be as in Example 2.2. Assume that* μ-div$(X + a \cdot \nabla)$ *is bounded below uniformly on compact sets as in Theorem 4.5 and*

$$E(c) = \sup_{x \in \mathbb{R}^n} \frac{\sigma(cx)\rho(cx)}{\rho(x)} < \infty \qquad \textit{for each} \quad c > 1. \tag{4.14}$$

Then

$$\mathcal{S}^2 = \mathcal{S}L^2(\mu).$$

Proof Suppose first that v is in $C^\infty(\mathbb{R}^n) \cap L^2(\mu)$ and that the first derivatives $\partial v/\partial x_j$ are also in $L^2(\mu)$. We assert that for any $s > 0$, $v_s \equiv v \circ e^{-sX} \in D(Q)$. To this end write $|dv(y)|^2 = \sum_{j=1}^n |\partial v/\partial x_j|^2$ and note that

$$\begin{aligned}
\int_{\mathbb{R}^n} |dv_s(x)|^2_{T_x^*}\, d\mu &= \int_{\mathbb{R}^n} |dv_s(x)|^2 \sigma(x)\rho(x)\, dx \\
&= \int_{\mathbb{R}^n} e^{-2s} |(dv)(e^{-s}x)|^2 \sigma(x)\rho(x)\, dx \\
&= \int_{\mathbb{R}^n} e^{(n-2)s} |dv(y)|^2 \frac{\sigma(e^s y)\rho(e^s y)}{\rho(y)} \rho(y)\, dy \\
&\le e^{(n-2)s} E(e^s) \| |dv| \|_2^2.
\end{aligned}$$

Choose a function $\psi \in C_c^\infty(\mathbb{R}^n)^+$ such that $\psi(x) = 1$ for $|x| \le 1$ and put $\psi_n(x) = \psi(x/n)$. Then $\psi_n \to 1$ boundedly and $|d\psi_n(x)| \le (1/n)|(d\psi)(x/n)|$, which goes to zero uniformly on $\mathbb{R}^n$. Writing $\psi_n^s(x) = \psi_n(e^s x)$, the previous derivation shows that

$$\int_{\mathbb{R}^n} |d(v_s(x) - \psi_n(x)v_s(x))|^2_{T_x^*}\, d\mu \le e^{(n-2)s} E(e^s) \| |d(v - \psi_n^s v)| \|^2_{L^2(\mu)}.$$

A double use of the dominated convergence theorem now shows that the right side goes to zero as $n \to \infty$. To show that $v_s \in D(Q)$, it only remains to show that

$\psi_n v_s \to v_s$ in $L^2(\mu)$ which clearly holds if $v_s \in L^2(\mu)$. But Corollary 4.3 shows that v_s is indeed in $L^2(\mu)$ and therefore $v \circ \exp(-sX) \in D(Q)$.

Next suppose that $f \in \mathcal{S}L^2(\mu)$. If $\varphi \in C_c^\infty(\mathbb{R}^n)^+$ then $f * \varphi$ is again L-subharmonic and is also in $C^\infty(\mathbb{R}^n)$. Now let $\varphi^t(z) = \varphi(e^{-t}z)e^{-nt}$. Then

$$\begin{aligned}(f * \varphi)_t &\equiv (f * \varphi)(e^{-t}x)\\ &= \int_{\mathbb{R}^n} f(e^{-t}x - y)\varphi(y)\,dy = \int_{\mathbb{R}^n} f(e^{-t}(x-z))\varphi(e^{-t}z)e^{-tn}\,dz.\end{aligned}$$

That is,

$$(f * \varphi)_t = f_t * \varphi^t. \tag{4.15}$$

By Theorem 4.5(iii), the function $v \equiv (f * \varphi) \circ \exp(-tX) = (f * \varphi)_t$ is back in $L^2(\mu)$ for any $t > 0$ and v is also L-subharmonic. Since $v = f_t * \varphi^t$, it is also infinitely differentiable and, using (4.15) repeatedly, one sees that all its derivatives are in $L^2(\mu)$. By what has been proved in the first paragraph, v_s is in $D(Q)$. That is, $(f * \varphi) \circ \exp(-(t+s)X) \in D(Q)$ for any $t > 0$ and $s > 0$. So $(f * \varphi) \circ \exp(-tX) \in \mathcal{S} \cap D(Q)$ for any $t > 0$. We are going to leave t fixed and let φ run through an approximate identity. For this purpose we must interchange the order of composition and convolution as in (4.15). Note that $\int_{\mathbb{R}^n} \varphi^t(z)\,dz = \int_{\mathbb{R}^n} \varphi(y)\,dy$. So if φ_k is a sequence in $C_c^\infty(\mathbb{R}^n)^+$ such that $\int_{\mathbb{R}^n} \varphi_k(y)\,dy = 1$ and $\operatorname{supp} \varphi_k \downarrow \{0\}$ then the sequence in φ_k^t has the same properties. By Theorem 4.5, Parts (i) and (iii) f_t is not only in $L^2(\mu)$ but its translates, $f_t(\cdot - z)$, are also in $L^2(\mu)$ and the map $z \to f_t(\cdot - z)$ is continuous from $\mathbb{R}^n$ into $L^2(\mu)$, respectively. It now follows that $(f * \varphi_k)_t$, which equals $f_t * \varphi_k^t$, converges to f_t in $L^2(\mu)$. So $f_t \in \mathcal{S}^2$ for each $t > 0$. Now, for $h \in C_c(\mathbb{R}^n)$, $\lim_{t \downarrow 0} h \circ \exp(-tX) \to h$ in $L^2(\mu)$. By (4.12) (with $a = 0$) the map $h \to h \circ \exp(-tX)$ is uniformly bounded from L^2 to L^2 for $0 \le t \le 1$. Hence $f_t \to f$ in L^2 as $t \downarrow 0$. Therefore $f \in \mathcal{S}^2$. ∎

5 Relative Subharmonicity

In this section we are going to prove the inequality (1.2) in two circumstances. In Theorem 5.1 we will prove it over a general Riemannian manifold, but only for functions in $\mathcal{S}^p$. We will need to assume that X is reasonably related to the measure, as in (5.1), and to the metric, as in (5.2).

In Theorem 5.7 we will prove that (1.2) holds over $\mathbb{R}^n$ for all functions in $\mathcal{S}L^p(\mathbb{R}^n, \mu)$ under some mild restrictions on g and μ. We do not know whether $\mathcal{S}^2 = \mathcal{S}L^2$ in the presence of these restrictions.

5.1 Relative Subharmonicity for Functions in $\mathcal{S}^p(M)$.

Theorem 5.1 *Let M be a Riemannian manifold with Riemannian metric g and let μ be a smooth probability measure on M with strictly positive density as in Section* 2. *We suppose, as usual, that in the decomposition* (2.3) *the flow of X leaves L-subharmonic functions invariant. Denote by A the self-adjoint version of d^*d defined in Section* 2.

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Assume that there is a constant C such that

$$\mu\text{-div}\, X \geq -C \tag{5.1}$$

and that the left side is in $L^1(\mu)$. Assume also that for each $t \geq 0$ there is a constant B_t such that

$$|\exp(-tX)_*|_{T_x \to T_{\exp(-tX)x}} \leq B_t, \quad \forall x \in M. \tag{5.2}$$

Let $f \in \mathcal{S}^2$. Then

$$e^{-tA} f \geq f \circ \exp(-tX), \quad \mu\text{-a.e.}, \quad \forall t \geq 0. \tag{5.3}$$

Corollary 5.2 *Under the hypotheses of Theorem 5.1 the inequality* (5.3) *holds for all $f \in \mathcal{S}^p$ with $1 \leq p < \infty$.*

The proof of Theorem 5.1 will be broken into several lemmas.

Lemma 5.3 *Suppose that*

(i) $f \in \mathcal{S} \cap D(Q)$,
(ii) $Xf \in L^2(\mu)$ *and*
(iii) $\psi \geq 0$ *and* $\psi \in D(A)$.

Then

$$(Xf, \psi) - (f, A\psi) \geq 0. \tag{5.4}$$

Proof If $\psi \in C_c^\infty(M)^+$ then (5.4) obviously holds, see Definition 2.1. Let $\psi \in D(Q)^+$. There exists a sequence of functions $\phi_n \in C_c^\infty(M)$ which converges to ψ in energy norm. We may choose the sequence of functions ϕ_n to converge to ψ a.e. Since the functions ϕ_n may not be nonnegative we will modify them as follows. Let u be in $C^\infty(\mathbb{R})$ and be such that

$$u \geq 0,\ \forall s; \quad u(s) = 0,\ \forall s \leq 0; \quad u(s) = s,\ \forall s \geq 1; \quad \text{and} \quad u' \geq 0.$$

Clearly $u \circ \phi_n \in C_c^\infty(M)^+$, and $u \circ \phi_n - u \circ \psi \to 0$ in $L^2(\mu)$ because $|u(\phi_n(x)) - u(\psi(x))| \leq \sup_{s \in \mathbb{R}} |u'(s)|\, |\phi_n(x) - \psi(x)|$. Moreover $u \circ \phi_n - u \circ \psi \to 0$ in energy norm because

$$\begin{aligned} d(u \circ \phi_n)(x) - d(u \circ \psi)(x) &= u'(\phi_n(x)) d\phi_n(x) - u'(\psi(x)) d\psi(x) \\ &= u'(\phi_n(x))(d\phi_n(x) - d\psi(x)) \\ &\quad + \big(u'(\phi_n(x)) - u'(\psi(x))\big)\, d\psi(x) \end{aligned}$$

which gives

$$\begin{aligned} \sqrt{Q(u \circ \phi_n - u \circ \psi)} &\leq \sup_{s \in \mathbb{R}} |u'(s)| \sqrt{Q(\phi_n - \psi)} \\ &\quad + \sqrt{\textstyle\int_M |u'(\phi_n(x)) - u'(\psi(x))|^2 |d\psi|^2_{T_x^*}\, d\mu(x)}. \end{aligned}$$

But $Q(\phi_n - \psi) \to 0$ by assumption while the coefficient of $|d\psi|^2_{T_x^*}$ in the last term goes to zero a.e. and boundedly. So the last term goes to zero by the dominated convergence theorem. Now

$$(Xf, u \circ \phi_n) - (f, A(u \circ \phi_n)) \geq 0.$$

Since f has finite energy, we may write this as

$$(Xf, u \circ \phi_n) - Q(f, u \circ \phi_n) \geq 0.$$

Since $u \circ \phi_n \to u \circ \psi$ in both $L^2(\mu)$ norm and Q norm we find

$$(Xf, u \circ \psi) - Q(f, u \circ \psi) \geq 0. \tag{5.5}$$

Now let $0 < \epsilon < 1$ and define $u_\epsilon(s) = \epsilon u(s/\epsilon)$ for all $s \in \mathbb{R}$. Then we see that $u_\epsilon(s) = 0$ if $s \leq 0$, $u_\epsilon(s) = s$ if $s \geq \epsilon$, $u_\epsilon \geq 0$ and moreover $u'_\epsilon = u'(\,\cdot\,/\epsilon)$ which is nonnegative and bounded, uniformly in ϵ. Say $u'_\epsilon \leq b$ for all ϵ in $(0, 1)$. Clearly (5.5) holds for u_ϵ instead of u. Now $\|u_\epsilon \circ \psi - \psi\|_2^2 = \int_{0<\psi(x)<\epsilon} |u_\epsilon(\psi(x)) - \psi(x)|^2 \, d\mu(x) \leq \int_{0<\psi(x)<\epsilon} (2\epsilon)^2 \, d\mu(x) \to 0$ as $\epsilon \to 0$. Furthermore,

$$\begin{aligned}
Q(u_\epsilon \circ \psi - \psi) &= \int_M |u'_\epsilon(\psi(x)) d\psi - d\psi|^2_{T_x^*} \, d\mu(x) \\
&= \int_{0<\psi(x)<\epsilon} |u'_\epsilon(\psi(x)) - 1|^2 |d\psi|^2_{T_x^*} \, d\mu(x) \\
&\leq (b+1)^2 \int_{0<\psi(x)<\epsilon} |d\psi|^2_{T_x^*} \, d\mu(x) \to 0
\end{aligned}$$

as $\epsilon \to 0$ because $Q(\psi) < \infty$. Hence (5.5) yields

$$(Xf, \psi) - Q(f, \psi) \geq 0.$$

Since $D(A) \subset D(Q)$ this proves (5.4). ∎

Lemma 5.4 *Let (M, g, μ) be a Riemannian manifold with smooth probability measure μ as in Section 2. Suppose that X is a smooth vector field on M whose flow $\exp(-tX)$ exists for all $t \geq 0$. Suppose also that μ-div X is in $L^1(\mu)$ and is bounded below. Specifically assume that (5.1) holds. Define, for any function f on M,*

$$V_t f = f \circ \exp(-tX).$$

Then,

(i) *for $1 \leq p < \infty$ and $f \in L^p(\mu)$, the function $[0, \infty) \ni t \to V_t f$ is continuous into L^p and*

$$\|V_t f\|_p \leq e^{tC/p} \|f\|_p. \tag{5.6}$$

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(ii) *If $1 \le p < \infty$ and $u \in C_c^\infty(\mathbb{R})$ with support in $[0,\infty)$, let*

$$v \equiv \int_0^\infty (V_t f) u(t)\, dt$$

(which can simply be interpreted as a Riemann integral with a continuous $L^p(\mu)$ valued integrand). Then the map $s \to V_s v$ from $(0,\infty)$ into $L^p(\mu)$ is infinitely differentiable. If u_n is a sequence of such functions satisfying $u_n \ge 0$, $\int u_n(s)ds = 1$, and support $u_n \subset [0, 1/n]$, then the corresponding sequence v_n converges to f in L^p.

(iii) *Assume further that for each $t \ge 0$ there is a constant B_t such that* (5.2) *holds. Let $f \in D(Q)$. Then $V_t f$ is also in $D(Q)$ and, as a function from $[0,\infty)$ into the Hilbert space $D(Q)$, is continuous. Moreover,*

$$Q(V_t f) \le B_t^2 e^{tC}\, Q(f), \quad \forall f \in D(Q). \tag{5.7}$$

Defining v_n as in (ii)*, one has $v_n \in D(Q)$ and $v_n \to f$ in Q norm.*

Proof The inequality (5.6) is a restatement of (4.7). If $f \in C_c(M)$, then $t \to V_t f$ is clearly continuous into L^p. The uniform bound (5.6) now yields the strong continuity of V_t on L^p. The assertion (ii) follows from strong continuity in the standard manner for bounded semigroups. To prove (iii) fix $t > 0$ and let $\psi = \exp(-tX)$. If $f \in C_c^\infty(M)$, then so is $f \circ \psi$. Moreover, in view of (4.6), we find

$$\begin{aligned} Q(f\circ\psi) &= \int_M |d(f\circ\psi)|^2_{T_x^*}\, d\mu(x) = \int_M |\psi^*(df)|^2_{T_x^*}\, d\mu(x) \\ &\le \int_M |\psi^*|^2_{T^*_{\psi(x)} \to T_x^*} |df|^2_{T^*_{\psi(x)}}\, d\mu(x) \le B_t^2 \int_M |df|^2_{T^*_{\psi(x)}}\, d\mu(x) \\ &= B_t^2 \int_M |df|^2_{T_y^*} J_\psi(y)\, d\mu(y) \le B_t^2 e^{tC} \int_M |df|^2_{T_y^*}\, d\mu(y). \end{aligned}$$

This proves (5.7) for $f \in C_c^\infty(M)$.

Now to say that f in $D(Q)$ means that there exists a sequence $f_n \in C_c^\infty(M)$ which is Cauchy in Q norm and which converges to f in $L^2(\mu)$. By (5.7) $f_n \circ \psi$ is then also Cauchy in Q norm and by (5.6) $f_n \circ \psi$ converges to $f \circ \psi$ in $L^2(\mu)$. Hence $f \circ \psi$ is in $D(Q)$ and (5.7) holds for f. The strong continuity of V_t in energy norm now follows in a standard manner. ∎

Lemma 5.5 *Suppose that*

(i) *$f \in \mathcal{S} \cap D(Q)$ and*
(ii) *$Xf \in L^2(\mu)$.*

Then (5.3) *holds.*

Proof Let $t > 0$ and define $f_s(x) = f(\exp((s-t)X)x)$ for $0 \leq s \leq t$. The map $s \to f_s$ is a continuous function into $L^2(\mu)$ on $[0,t]$ by Lemma 5.4. By condition (ii) it is differentiable on $(0,t)$ with derivative

$$df_s/ds = (Xf) \circ \exp((s-t)X) = Xf_s.$$

Let $\phi \in C_c^\infty(M)^+$ and define, for $0 \leq s \leq t$, $v(s) = (e^{-sA}f_s, \phi)$. So $v(s) = (f_s, e^{-sA}\phi)$, which is clearly continuous on $[0,t]$ and differentiable on $(0,t)$. Moreover

$$dv(s)/ds = (Xf_s, e^{-sA}\phi) - (f_s, Ae^{-sA}\phi).$$

Now $f_s \in D(Q)$ by condition (i) and Lemma 5.4(iii). Since $e^{-sA}\phi$ is nonnegative and in $D(A)$ we may apply (5.4) with $\psi = e^{-sA}\phi$, to conclude that $dv/ds \geq 0$. Hence $v(t) \geq v(0)$. That is

$$(e^{-tA}f, \phi) \geq (f \circ \exp(-tX), \phi).$$

Since ϕ is arbitrary in $C_c^\infty(M)^+$, (5.3) follows under the hypotheses of this lemma. ■

Lemma 5.6 *If $f \in \mathcal{S} \cap D(Q)$, then* (5.3) *holds.*

Proof If $f \in \mathcal{S} \cap D(Q)$, the functions v_n, defined in Lemma 5.4(ii), are L-subharmonic because the functions u_n are assumed nonnegative. But each v_n satisfies the hypotheses of Lemma 5.5 by Lemma 5.4(ii) and (iii). Hence $e^{-tA}v_n \geq v_n \circ \exp(-tX)$ a.e. Both sides converge in $L^2(\mu)$ as $n \to \infty$, by Lemma 5.4(i) and (ii), to $e^{-tA}f$ and $f \circ \exp(-tX)$ respectively. Dropping to a subsequence we may assume pointwise convergence a.e. Thus (5.3) holds when f satisfies the hypotheses of this lemma. ■

Proof of Theorem 5.1 If $f \in \mathcal{S}^2$, then there exists a sequence of functions $f_n \in \mathcal{S} \cap D(Q)$ which converge to f in L^2. Since (5.3) holds for each f_n by Lemma 5.6, we may repeat the limiting procedure of Lemma 5.6 to conclude that (5.3) holds for f. ■

Proof of Corollary 5.2 For $p \geq 2$ the statement is clear since then $\mathcal{S}^p \subset \mathcal{S}^2$.

If $f \in \mathcal{S}^p$, $p < 2$, then there exists a sequence f_n in $\mathcal{S}^2$ which converges to f in L^p. Since (5.3) holds for each f_n by Theorem 5.1, we may repeat the limiting procedure of Lemma 5.6 in $L^p(\mu)$ to conclude that (5.3) holds for f. Of course, one should now replace A by A_p in (5.3). ■

5.2 Relative Subharmonicity for Functions in $\mathcal{S}L^p(\mathbb{R}^n)$

In the following theorem we will prove the relative subharmonic inequality (1.2) for all $f \in \mathcal{S}L^p(\mathbb{R}^n)$. We do not know if this space actually coincides with $\mathcal{S}^p$ in these cases.

Theorem 5.7 *Let $(\mathbb{R}^n, g, \mu)$ and X be as in Example 2.2. Assume that for each compact set $K \subset \mathbb{R}^n$ there is a constant C_K such that*

$$\mu\text{-div}(X + a \cdot \nabla) \geq -C_K \quad \forall a \in K \tag{5.8}$$

and that the left-hand side is in $L^1(\mu)$ for all $a \in \mathbb{R}^n$. Assume also that $\sigma(\,\cdot\,)$ is in $L^r(\mathbb{R}^n, \mu)$ for all $r < \infty$. Let $1 < p < \infty$. Then

$$e^{-tA_p} f \geq f \circ \exp(-tX), \quad \mu\text{-a.e}, \quad \forall\, f \in \mathcal{S}L^p(\mu) \text{ and } \forall t \geq 0. \tag{5.9}$$

Lemma 5.8 *Let $1 < q < \infty$ and let $f \in \mathcal{S}L^q(\mu)$. Suppose that*

(i) *$f \circ \exp(-tX) \in D(A_q)$, for all $t \geq 0$ and*
(ii) *$f \circ \exp(-tX)$ is a differentiable function of t into $L^q(\mu)$ for $t \geq 0$.*

Then

$$e^{-tA_q} f \geq f \circ \exp(-tX), \quad \mu\text{-a.e}, \ \forall t \geq 0. \tag{5.10}$$

Proof The proof differs only slightly from that of Lemma 5.5. Fix $t > 0$ and define $f_s = f \circ \exp((s-t)X)$ for $0 \leq s \leq t$. Then by condition (ii), f_s is a differentiable function from $(0, t)$ into $L^q(\mu)$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)^+$ and let q' be the conjugate index to q. Let $v(s) = (e^{-sA_q} f_s, \varphi) = (f_s, e^{-sA_{q'}}\varphi)$. Since $\varphi \in D(A_{q'})$ and $f_s \in \mathcal{S} \cap D(A_q)$ we have, for $0 < s < t$,

$$\begin{aligned} \frac{d}{ds} v_s &= \frac{d}{ds}\big(f_s, e^{-sA_{q'}}\varphi\big) \\ &= \big(Xf_s, e^{-sA_{q'}}\varphi\big) + \big(f_s, -A_{q'}e^{-sA_{q'}}\varphi\big) = \big((X - A_q)f_s, e^{-sA_{q'}}\varphi\big) \\ &\geq 0. \end{aligned} \tag{5.11}$$

Here we have used the fact $(X - A_q)f_s \geq 0$ and $e^{-tA_{q'}}\varphi \geq 0$. So $v(t) \geq v(0)$ and (5.10) now follows as in Lemma 5.5. ∎

Lemma 5.9 *Suppose that $1 < q < p$ and that $\sigma(\,\cdot\,) \in \bigcap_{r<\infty} L^r(\mu)$. Let $f \in C^\infty(\mathbb{R}^n)$ and assume that all its partial derivatives up to the second order are in $L^p(\mu)$. If, in addition, $Xf \in L^p(\mu)$, then $f \in D(A_q)$.*

Proof Recall that for $f \in C^\infty(\mathbb{R}^n)$ we have

$$d^* df(x) = -\sigma(x)\Delta f(x) + (Xf)(x), \quad x \in \mathbb{R}^n.$$

Choose $\psi \in C_c^\infty(\mathbb{R}^n)$ with $0 \leq \psi \leq 1$ and $\psi(x) = 1$ for $|x| \leq 1$. Define $\psi_m(x) = \psi(x/m)$. Then each derivative of ψ_m goes to zero uniformly as $m \to \infty$ while $\psi_m(x) \to 1$ pointwise and boundedly. Define $f_m := \psi_m f$, which is in $C_c^\infty(\mathbb{R}^n)$. By the chain rule we obtain

$$\begin{aligned} \partial_i f_m &= \partial_i \psi_m f + \psi_m \partial_i f \\ \partial_j \partial_i f_m &= \partial_j \partial_i \psi_m f + \partial_i \psi_m \partial_j f + \partial_j \psi_m \partial_i f + \psi_m \partial_j \partial_i f, \quad 0 \leq i, j \leq n. \end{aligned}$$

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Now $X\psi_m \to 0$ boundedly by the same argument as in the proof of Theorem 2.8, Case (i). Moreover it follows by dominated convergence that

$$\lim_{m\to\infty} f_m = f, \quad \lim_{m\to\infty} \partial_i f_m = \partial_i f, \quad \lim_{m\to\infty} \partial_j\partial_i f_m = \partial_j\partial_i f$$

in $L^p(\mu)$. Since the function $\sigma(\,\cdot\,)$ is in $L^r(\mu)$ for all $r < \infty$, we can conclude that

$$\lim_{m\to\infty} d^* d f_m = -\sigma\Delta f + Xf$$

in $L^q(\mu)$. Thus $f \in D(A_q)$ because $C_c^\infty(\mathbb{R}^n) \subset D(A_q)$ and A_q is closed. ■

Proof of Theorem 5.7 Suppose that $f \in \mathcal{S}L^p(\mu)$. If $\varphi \in C_c^\infty(\mathbb{R}^n)^+$ then $f * \varphi$ is again L-subharmonic and is also in $C^\infty(\mathbb{R}^n)$. By Theorem 4.5(iii), the function $v_s \equiv (f * \varphi) \circ \exp(-sX)$ is back in $L^p(\mu)$ for any $s > 0$ and v_s is also L-subharmonic. Choose $u \in C_c^\infty(\mathbb{R})^+$ as in Lemma 5.4(ii), and let

$$\zeta = \int_0^\infty v_s \circ \exp(-rX) u(r)\, dr. \tag{5.12}$$

Then ζ is in L^p by Lemma 5.4(ii), $\zeta \circ \exp(-tX)$ is an infinitely differentiable function of $t \in [0,\infty)$ into L^p, and $\zeta \in \mathcal{S}$ because $u \geq 0$. For $t \geq 0$, and in view of (2.15),

$$\zeta(\exp(-tX)x) = \int_0^\infty \int_{\mathbb{R}^n} f(z)\varphi(e^{-c(t+s+r)}x - z) dz\, u(r)\, dr, \tag{5.13}$$

which is clearly in $C^\infty(\mathbb{R}^n)$ and all its derivatives are in $L^p(\mu)$. So if $1 \leq q < p$ then by Lemma 5.9, $\zeta \circ \exp(-tX) \in D(A_q)$. As a function of $t \in [0,\infty)$ it is also infinitely differentiable into L^q because $q < p$. So we may apply Lemma 5.8 to conclude that (5.10) holds for ζ. To remove the regularizations, first choose u_n as in Lemma 5.4(ii) and use the boundedness of e^{-tA_q} and $\circ \exp(-tX)$ in L^q to conclude that (5.10) holds for v_s. That is,

$$e^{-tA_q} v_s \geq v_s \circ \exp(-tX) \ a.e. \tag{5.14}$$

Now we let ϕ run through an approximate identity φ_k and obtain as in the proof of Theorem 4.7 that $v_{s,k} \equiv (f * \varphi_k) \circ \exp(-sX)$ converges to $f_s \equiv f \circ \exp(-sX)$ in $L^q(\mu)$. Hence, by continuity of e^{-tA_q} and $\exp(-tX)$ in $L^q(\mu)$ (see Lemma 5.4(i)) (5.14) holds for f_s instead of v_s. Finally, by Lemma 5.4(i) again, we can let $s \downarrow 0$, obtaining (5.9) with A_q instead of A_p. But, since $f \in L^p$, (5.9) holds as written. ■

6 Reverse Hypercontractivity

In Section 5 we proved relative subharmonicity in two circumstances, (*cf.* Theorems 5.1 and 5.7). In this section we will show how relative subharmonicity yields reverse hypercontractivity.

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Theorem 6.1 *Let M be a Riemannian manifold with Riemannian metric g and let μ be a smooth probability measure on M with strictly positive density as in Section 2. We suppose, as usual, that in the decomposition (2.3) the flow of X leaves L-subharmonic functions invariant. Denote by A the self-adjoint version of d^*d defined in Section 2. We continue the notation of Proposition 4.2.*

Suppose that $T > 0$, $p \geq 1$ and that $q > pe^{\kappa T}$. Let $r = q/p$. Let $0 \leq f \in L^p(\mu)$ and assume that, for some constant $\beta > 0$,

$$e^{-TA_p} f \geq \beta f \circ \exp(-TX), \quad \mu\text{-a.e.} \tag{6.1}$$

Then

$$\|e^{-TA_p} f\|_q \geq \beta \|f\|_p e^{-\Lambda_\kappa(r,T)/p}. \tag{6.2}$$

Proof The proof is similar to the proof of Theorem 3.4 in [GGS]. Let $h := f \circ \exp(-TX)$. So $f = h \circ \exp(TX)$. Write $J = d(\exp(TX)_* \mu)/d\mu$. Then

$$\|f\|_p^p = \|h \circ \exp(TX)\|_p^p = \int h^p J \, d\mu \leq \Big(\int h^q \, d\mu \Big)^{1/r} \|J\|_{r'}.$$

So $\|h\|_q \geq \|f\|_p \|J\|_{r'}^{-1/p}$. Since $f \geq 0$, both sides of (6.1) are nonnegative. Hence

$$\|e^{-TA_p} f\|_q \geq \beta \|f \circ \exp(-TX)\|_q = \beta \|h\|_q \geq \beta \|f\|_p \|J\|_{r'}^{-1/p}. \tag{6.3}$$

The inequality (4.4) completes the proof. ∎

Corollary 6.2 *Under the hypotheses of Theorem 5.1 the inequality (6.2) holds, with $\beta = 1$, for all nonnegative functions $f \in \mathcal{S}^p(M)$ if $1 \leq p < \infty$.*

Corollary 6.3 *Under the hypotheses of Theorem 5.7 the inequality (6.2) holds, with $\beta = 1$, for all nonnegative functions $f \in \mathcal{S}L^p(\mathbb{R}^n, \mu)$ if $1 < p < \infty$.*

The following corollary was first proved in [GS] for holomorphic functions. See also [GGS, Corollary 3.5].

Corollary 6.4 (Gauss measure) *Take $\mu = \gamma_c$ on $\mathbb{R}^n$ with the standard metric as in Example 4.4. Suppose that $t > 0$, $p \geq 1$ and $q > pe^{2t/c}$. Then*

$$\|e^{-tA_p} f\|_q \geq \|f\|_p \Big(e^{-2t/(cq)} \Big(\frac{q-p}{q - e^{2t/c} p} \Big)^{(1/p - 1/q)} \Big)^{-n/2} \tag{6.4}$$

for $0 \leq f \in \mathcal{S}L^p(\mathbb{R}^n, \gamma_c)$.

Proof Apply (6.2) with $\beta = 1$ and use (4.9). ∎

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7 Saturation and Alpha Subharmonicity

If ϕ is a holomorphic function on $\mathbb{C}^n$ and $f(z) = |\phi(z)|$, then f is subharmonic. Taking $n = 2m$ in Corollary 6.4, the inequality (6.4) reduces to the corresponding inequality obtained in the holomorphic category in [GGS, (3.9)]. F. Galaz-Fontes and S. Sontz have shown [GS, So] that the inequality (6.4) is not saturated by $|\phi|$ for any holomorphic function ϕ. The question arises, therefore, as to whether the inequality (6.4) is saturated in the larger class of nonnegative subharmonic functions. We think not, but we do not have a proof. Thus we do not know whether the coefficient of $\|f\|_p$ in (6.4) is the largest possible in the category of nonnegative subharmonic functions. With the hope of shedding some light on this we will explore in this section a smaller class of subharmonic functions for which the coefficient of $\|f\|_p$ in (6.4) can actually be increased.

Denote by γ_c the Gauss measure on $\mathbb{R}^n$ with the standard metric, as in Example 4.4.

Definition 7.1 Let $\alpha \geq 0$. A nonnegative function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n, \gamma_c)$ is called α-*subharmonic*, if $(\Delta - \alpha)f \geq 0$ in the weak sense. That is,

$$(f, (L^* - \alpha)\phi)_{L^2(\gamma_c)} \geq 0, \quad \forall \phi \in C_c^\infty(M)^+, \tag{7.1}$$

where $L = \Delta$, $L^* = \Delta - 2X - n + x^2$ is the formal adjoint of L in $L^2(\gamma_c)$ and X is given by (4.8), namely, $X = c^{-1}\sum_{j=1}^n x_j \partial/\partial x_j$.

Notation 7.2 $\mathcal{S}_\alpha$ will denote the space of α-subharmonic functions on $\mathbb{R}^n$. Note that $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$ if $\alpha \geq \beta$.

Example 7.3 Let $f_u(x) = e^{(u,x)}$, $u, x \in \mathbb{R}^n$. Then $f_u \in \mathcal{S}_{|u|^2}$ because $\Delta f_u = |u|^2 f_u$.

Theorem 7.4 *Let $f \in \mathcal{S}_\alpha \cap L^p(\gamma_c)$ for some $p > 1$. Then*

$$\exp(-tA_p)f \geq e^{(c\alpha/2)(1-e^{-2t/c})} f \circ \exp(-tX), \qquad \gamma_c\text{-}a.e, \quad \forall t \geq 0. \tag{7.2}$$

Proof First observe that if $f \in \mathcal{S}_\alpha$ then $f \circ \exp(-tX) \in \mathcal{S}_{\alpha e^{-2t/c}}$ because

$$\begin{aligned}\Delta[f \circ \exp(-tX)] &= \Delta[f(e^{-t/c}\cdot)] = e^{-2t/c}(\Delta f)(e^{-t/c}\cdot)\\ &\geq \alpha e^{-2t/c} f(e^{-t/c}\cdot).\end{aligned} \tag{7.3}$$

Now we are going to ignore domain issues in this simple Gaussian setting because these have been addressed in great generality in [G02, Section 4]. Let $0 \leq s \leq t$ and define $F(s) = e^{-sA}(f \circ \exp((s-t)X))$. Then

$$\begin{aligned}F'(s) &= e^{-sA}(-A + X)[f \circ \exp((s-t)X)] = e^{-sA}\Delta[f \circ \exp((s-t)X)]\\ &\geq \alpha e^{2(s-t)/c}F(s).\end{aligned} \tag{7.4}$$

Thus if we define $v(s) = e^{-(c\alpha/2)e^{2(s-t)/c}}F(s)$, then we find $v'(s) \geq 0$. Hence $v(t) \geq v(0)$. Inserting the definition of v into this inequality yields (7.2) ■

Corollary 7.5 (Gauss measure) *Suppose that $t > 0$, $p \geq 1$ and $q > pe^{2t/c}$. Then*

$$\|e^{-tA_p} f\|_q \geq \|f\|_p e^{(c\alpha/2)(1-e^{-2t/c})} \Big(e^{-2t/(cq)} \Big(\frac{q-p}{q-e^{2t/c}p} \Big)^{(1/p-1/q)} \Big)^{-n/2}, \tag{7.5}$$

for $0 \leq f \in \mathcal{S}_\alpha \cap L^p$.

Proof Apply (6.2) with $\beta = e^{(c\alpha/2)(1-e^{-2t/c})}$ and use (4.9) just as in the proof of (6.4). ■

Example 7.6 (Non-saturation for the exponential function) Let $f_u(x) = e^{ux}$ with u and $x \in \mathbb{R}$. Then $f_u \in \mathcal{S}_\alpha$ for $\alpha = u^2$, as in Example 7.3. Since f_u is an eigenfunction for the Laplacian, the inequalities (7.3) and (7.4) are both equalities when $f = f_u$. As a result the inequality (7.2) is also an equality in this case. Inserting then the flow $\exp(-tX)x = e^{-t/c}x$ into that equality, one finds

$$\exp(-tA) f_u(x) = e^{c\alpha\delta/2} e^{ue^{-t/c}x}, \quad x \in \mathbb{R} \tag{7.6}$$

where $\delta = 1 - e^{-2t/c}$. Now a Gaussian integration shows that $\|f_v\|_p = e^{cv^2 p/2}$ for all $v \in \mathbb{R}$. Hence, by (7.6), we have $\|e^{-tA} f_u\|_q = e^{(c\alpha/2)(\delta + qe^{-2t/c})}$. Thus

$$\|e^{-tA} f_u\|_q = \|f_u\|_p \, e^{c\alpha\delta/2} e^{(c\alpha/2)(qe^{-2t/c}-p)}.$$

While the first factor $e^{c\alpha\delta/2}$ agrees with the first factor in the coefficient of $\|f\|_p$ in (7.5), the second factor is much larger than the second factor in (7.5). This example shows, therefore, that even though the coefficient on the right of (7.5) is larger than that in (6.4), the inequality (7.5) may still not be saturated in the class of α-subharmonic functions, even asymptotically as $\alpha \to \infty$.

8 Reverse Logarithmic Sobolev Inequalities

Although hypercontractivity and logarithmic Sobolev inequalities are more or less equivalent [G75], reverse hypercontractivity and reverse logarithmic Sobolev inequalities seem to follow from a more or less common hypothesis rather than from each other. This was the case in the holomorphic setting [GGS] and we will see that this is the case in the subharmonic setting also. The following theorem is similar in its statement and proof to that in [GGS, Corollary 4.3] after changing from the holomorphic to the subharmonic categories. We will sketch the proof by referring to the corresponding steps in [GGS].

Theorem 8.1 (Reverse logarithmic Sobolev inequality) *Let (M, g, μ) be as in Section 2. Assume that there is a constant C such that*

$$\mu\text{-div}\, X \geq -C,$$

and that the left side is in $\bigcap_{1\le p<\infty} L^p(\mu)$*. Assume also that* (4.3) *holds. Let* $s > \kappa$*. Suppose that* $0 \le f \in \mathcal{S} \cap D(A)$ *and that* $Xf \in L^2(\mu)$*. Then*

$$\int_M |\nabla f|^2 \, d\mu \le s \int_M f^2 \log(f/\|f\|_2) + (1/2)\|f\|_2^2 B(s). \tag{8.1}$$

The proof depends on the following lemma, which itself follows immediately from two lemmas in [GGS]. We want to emphasize that subharmonicity is not involved in this lemma.

Lemma 8.2 ([GGS, Lemmas 2.15, 4.4]) *Suppose that* μ-div X *is in* $\bigcap_{1\le p<\infty} L^p(\mu)$ *and that* (4.3) *holds. Assume that* $0 \le h \in L^1$ *and that* Xh *is in* L^1*. Then, for any* $s > \kappa$*,*

$$\int_M Xh \, d\mu \le s\Big(\int_M h \log(h/\|h\|_1)\, d\mu\Big) + \|h\|_1 B(s). \tag{8.2}$$

Proof Combine [GGS, Lemmas 2.15, 4.4]. In those lemmas it is also assumed that $h \in L^q$ for some $q > 1$. But here we can allow the right side of (8.2) to be infinite. A smooth truncation of h yields the extension of [GGS, Lemmas 2.15, 4.4] to (8.2): replace h by $\phi \circ h$ where $\phi(x) = x$ for $x < n$, is constant for $x \ge n+1$ and is smooth on $\mathbb{R}$. Then (8.2) holds for $\phi \circ h$ and one can let $n \to \infty$ to prove (8.2) in the generality stated. ∎

Proof of Theorem 8.1 If $0 \le f \in \mathcal{S} \cap D(A)$ and $Xf \in L^2$ then

$$\int_M |\nabla f|^2 \, d\mu = (Af, f) \le (Xf, f) = (1/2) \int_M X(f^2)\, d\mu \tag{8.3}$$

Combine this with Lemma 8.2 using $h = f^2$. ∎

Remark 8.3 A similar derivation yields an L^p version of (8.1), namely, assuming $0 \le f$ is subharmonic,

$$\begin{aligned}\int_M |\nabla f^{p/2}|^2 \, d\mu \le \frac{p^2}{4(p-1)} s\Big(\int_M f^p \log(f/\|f\|_p)\, d\mu\Big) \\ + \frac{p}{4(p-1)}\|f\|_p^p B(s).\end{aligned} \tag{8.4}$$

One uses the standard identities

$$\begin{aligned}\int |\nabla f^{p/2}|^2 d\mu &= (p/2)^2(p-1)^{-1} \int \nabla f \cdot \nabla f^{p-1} d\mu \\ &= (p/2)^2(p-1)^{-1} \int (A_p f) f^{p-1} d\mu \\ &\le (p/2)^2(p-1)^{-1} \int (Xf) f^{p-1} d\mu \\ &= (p/4)(p-1)^{-1} \int X(f^p) d\mu.\end{aligned}$$

An application of Lemma 8.2 with $h = f^p$ now yields (8.4). We will not state precise regularity conditions on f needed to justify this computation.

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