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ON THE CONSTANT IN THE PÓLYA-VINOGRADOV INEQUALITY

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ABSTRACT. The Pólya-Vinogradov inequality states that for any non-principal character χ modulo q and any $N \geqq 1$,

$$(*) \qquad \left| \sum_{n=1}^{N} \chi(n) \right| \leqq c\sqrt{q} \log q,$$

where c is an absolute constant. We show that (*) holds with $c = 2/(3\pi^2) + o(1)$ in the case χ is primitive and $\chi(-1) = 1$, and with $c = 1/(3\pi) + o(1)$ in the case χ is primitive and $\chi(-1) = -1$. This improves by a factor 2/3 the previously best-known values for these constants.

1. **Introduction.** For a non-principal character χ modulo q let

$$S(\chi) = \max_{N \geqq 1} \left| \sum_{n=1}^{N} \chi(n) \right|.$$

The Pólya-Vinogradov inequality states that

$$(1) \qquad S(\chi) \leqq c\sqrt{q} \log q$$

holds with an absolute constant c. This estimate, first proved independently by Pólya [4] and Vinogradov [7] nearly 70 years ago, still constitutes the best-known unconditional bound for $S(\chi)$. Under the General Riemann Hypothesis, however, (1) can be sharpened to

$$(2) \qquad S(\chi) \leqq c\sqrt{q} \text{ loglog } q,$$

as was shown by Montgomery and Vaughan [3, Theorem 2].

Shortly after the appearance of Pólya's paper in 1918, Landau [2] established for *primitive* characters χ the more precise estimate

$$(3) \qquad \frac{S(\chi)}{\sqrt{q} \log q} \leqq \begin{cases} c_+ + o(1) & \text{if } \chi(-1) = 1 \\ c_- + o(1) & \text{if } \chi(-1) = -1 \end{cases},$$

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as $q \to \infty$, with $c_+ = 1/(2\pi\sqrt{2})$ and $c_- = 1/(2\pi)$. Landau's values for c_+ and c_- have never been improved upon in the literature. However, it has been pointed out by P. T. Bateman that a certain trigonometric sum estimate occurring in Landau's argument is rather crude and can be replaced by a stronger and essentially best-possible one due to W. H. Young (cf. Lemma 4 below). As Bateman has shown, this leads to the improved value π^{-2} for c_+, though it does not result in an improvement of c_-. The purpose of this paper is to further reduce both constants c_+ and c_- by a factor 2/3.

THEOREM. *For primitive characters* χ *modulo* q ($q \geqq 2$), (3) *holds with* $c_+ = 2/(3\pi^2) = 0.067547\ldots$ *and* $c_- = 1/(3\pi) = 0.106103\ldots$. *For arbitrary non-principal characters*, (3) *remains valid, if each of these constants is multiplied by a factor* $4/\sqrt{6}$.

Our proof is based on the classical Fourier analysis approach that leads to exponential sums with coefficients $\chi(n)$. However, instead of estimating these sums trivially, as Landau did, we shall appeal to two rather deep results, namely Burgess' character sum estimate and an estimate for exponential sums with multiplicative coefficients due to Montgomery and Vaughan. In using the latter, we shall follow essentially the argument used by these authors in their conditional proof of (2). Burgess' character sum estimate has not been employed before in this context, and it is somewhat surprising that this estimate, which for "long" character sums is much weaker than the Pólya-Vinogradov inequality (1), plays a key role in sharpening this inequality.

By using more elaborate methods that involve sharp estimates for sums of multiplicative functions, it is possible to further improve the constant c_+, though only by a small margin. We plan to discuss this elsewhere. An improvement of the bound (1) to $o(\sqrt{q}\log q)$ would be highly desirable and have important consequences, but seems to be hopeless at present.

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2. **Lemmas.** We first quote suitable forms of the above-mentioned estimates of Burgess and Montgomery and Vaughan.

LEMMA 1 [1, Theorem A]. *Let* $\epsilon > 0$ *be fixed. Then we have, uniformly for all non-principal characters* χ *modulo* q *and all integers* $N, H \geqq 1$,

$$\left|\sum_{n=N+1}^{N+H} \chi(n)\right| \ll_\epsilon H^{2/3} q^{1/9+\epsilon}.$$

LEMMA 2 [3, Corollary 1]. *Let* N *be a positive integer and* α *a real number, and suppose that* $|\alpha-(a/q)| \leqq q^{-2}$, *where* $(a, q) = 1$ *and* $2 \leqq R \leqq q \leqq N/R$. *Then we have, for any multiplicative function* f *satisfying* $|f| \leqq 1$,

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$$\left|\sum_{n\leqq N} f(n)e(\alpha n)\right| \ll \frac{N}{\log N} + \frac{N(\log R)^{3/2}}{\sqrt{R}},$$

where $e(t) = \exp(2\pi i t)$ *and the implied constant is absolute.*

We use these two lemmas to establish the following result.

LEMMA 3. *Let* $0 < \epsilon < 1/2$ *be fixed. Then we have, uniformly for all primitive characters* χ *modulo* q, $q^{1/3+\epsilon} \leqq x \leqq q$ *and real* α,

$$\left|\sum_{n\leqq x} \chi(n)e(\alpha n)\right| \ll_\epsilon \frac{x}{\log q}.$$

PROOF. The proof is modelled after an argument of Montgomery and Vaughan [3, pp. 78-79]. Let ϵ, χ, q, x and α be fixed as in the lemma and set $N = [x]$, $R = (\log q)^3$. We may clearly assume $q \geqq 10$, so that $2 \leqq R \leqq N$. By Dirichlet's theorem there exist integers r and s, where $(r, s) = 1$ and $1 \leqq s \leqq N/R$, such that

$$\left|\alpha - \frac{r}{s}\right| \leqq \frac{1}{sN/R}. \tag{4}$$

If $s \geqq R$, then the asserted estimate follows from Lemma 2, since

$$\frac{N}{\log N} + \frac{N(\log R)^{3/2}}{\sqrt{R}} \ll \frac{x}{\log q}$$

by the definition of N and R. Suppose therefore that $s < R$. Partial summation with (4) yields

$$\left|\sum_{n\leqq x} \chi(n)e(\alpha n)\right| \ll \left(1 + \left|\alpha - \frac{r}{s}\right| x\right) \max_{u\leqq x} |T(u)| \ll (\log q)^3 \max_{u\leqq x} |T(u)|, \tag{5}$$

where

$$T(u) = \sum_{n\leqq u} \chi(n)e\left(\frac{r}{s}n\right).$$

By grouping in the sum $T(u)$ the terms according to the value of (n, s), we get

$$T(u) = \sum_{dt=s} \sum_{\substack{dm\leqq u\\(m,t)=1}} \chi(md)e\left(\frac{rm}{t}\right) = \sum_{dt=s} \chi(d) \sum_{\substack{1\leqq a\leqq t\\(a,t)=1}} e\left(\frac{ra}{t}\right) \sum_{\substack{m\leqq u/d\\m\equiv a \bmod t}} \chi(m)$$

$$= \sum_{dt=s} \frac{\chi(d)}{\phi(t)} \sum_{\psi \bmod t} \sum_{1\leqq a\leqq t} e\left(\frac{ra}{t}\right)\bar{\psi}(a) \sum_{m\leqq u/d} \chi(m)\psi(m).$$

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The characters $\chi\bar{\psi}$ appearing in the innermost sums are all non-principal, for otherwise χ and $\bar{\psi}$ would be equivalent, i.e., induced by the same primitive character; but this is impossible since χ itself is primitive modulo q and $\bar{\psi}$ is a character to a modulus that is strictly less than q if $q \geqq 10$, as we had assumed at the outset. Therefore, each of the sums over m can be bounded by Lemma 1 by

$$\ll_\epsilon \left(\frac{u}{d}\right)^{2/3}(qt)^{(1+\epsilon)/9} \leqq x^{2/3}(qR)^{(1+\epsilon)/9} \leqq xR^{(1+\epsilon)/9}q^{-2\epsilon/9},$$

and we obtain

$$\max_{u \leqq x} |T(u)| \ll_\epsilon RxR^{(1+\epsilon)/9}q^{-2\epsilon/9} \ll_\epsilon \frac{x}{(\log q)^4}.$$

This, with (5), yields the asserted estimate.

Finally, we need two trigonometric sum estimates due originally to Young [8] which we quote from Pólya-Szegö's book [5]. The second of these estimates was used by Bateman to improve Landau's value for c_+ to π^{-2}, as mentioned in the introduction.

LEMMA 4 [5, Part VI, Problems 28 and 38]. *Uniformly for $x \geqq 1$ and real α we have*

$$\sum_{n \leqq x} \frac{1 - \cos(\alpha n)}{n} \leqq \log x + 0(1)$$

and

$$\sum_{n \leqq x} \frac{|\sin(\alpha n)|}{n} \leqq \frac{2}{\pi} \log x + 0(1).$$

3. **Proof of the theorem.** We consider first the case of primitive characters χ. Following an idea of Schur [6], we start out with the identity

$$\chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^{q} \bar{\chi}(a)e\left(\frac{an}{q}\right) = \frac{1}{\tau(\bar{\chi})} \sum_{0<|a|<q/2} \bar{\chi}(a)e\left(\frac{an}{q}\right),$$

where $\tau(\bar{\chi})$ is the Gaussian sum. Summing over $1 \leqq n \leqq N$, we obtain

$$\sum_{n=1}^{N} \chi(n) = \frac{1}{\tau(\bar{\chi})} \sum_{0<|a|<q/2} \bar{\chi}(a) \sum_{n=1}^{N} e\left(\frac{an}{q}\right)$$

$$= \frac{1}{\tau(\bar{\chi})} \sum_{0<|a|<q/2} \bar{\chi}(a) \frac{e\left(\frac{aN}{q}\right) - 1}{1 - e\left(-\frac{a}{q}\right)}.$$

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Since $|\tau(\bar{\chi})| = \sqrt{q}$ for primitive characters χ and

$$\frac{1}{1 - e\left(-\frac{a}{q}\right)} = \frac{1}{2\pi i \frac{a}{q}}\left(1 + 0\left(\frac{a}{q}\right)\right) = \frac{q}{2\pi i a} + 0(1)$$

for $0 < |a| < q/2$, it follows that

$$\left|\sum_{n=1}^{N} \chi(n)\right| = \frac{\sqrt{q}}{2\pi}\left|\sum_{0<|a|<q/2} \frac{\bar{\chi}(a)\left(e\left(\frac{aN}{q}\right) - 1\right)}{a}\right| + 0\sqrt{q}). \tag{6}$$

If we now would bound the sum on the right of (6) trivially, we would get the Pólya-Vinogradov inequality in the form (1). To obtain the desired more precise estimate, we split this sum into two parts Σ_1 and Σ_2, where in Σ_1 the summation is restricted to the range $0 < |a| \leqq q_1$, and in Σ_2 to $q_1 < |a| < q/2$. We choose $q_1 = q^{1/3+\epsilon}$, ϵ being an arbitrary, but fixed positive number satisfying $0 < \epsilon < 1/2$.

By partial summation and Lemma 3 we have

$$\Sigma_2 \ll (\log q) \max_{q_1 \leqq x \leqq q} \left|\frac{1}{x} \sum_{a \leqq x} \bar{\chi}(a)\left(e\left(\frac{aN}{q}\right) - 1\right)\right| \ll_\epsilon 1.$$

Moreover, noting that

$$\Sigma_1 = \begin{cases} 2i \displaystyle\sum_{1 \leqq a \leqq q_1} \frac{\bar{\chi}(a) \sin\left(\frac{2\pi a N}{q}\right)}{a} & \text{if } \chi(-1) = 1 \\ -2 \displaystyle\sum_{1 \leqq a \leqq q_1} \frac{\bar{\chi}(a)\left(1 - \cos\left(\frac{2\pi a N}{q}\right)\right)}{a} & \text{if } \chi(-1) = -1 \end{cases}$$

and applying Lemma 4, we get

$$|\Sigma_1| \leqq \begin{cases} \frac{4}{\pi} \log q_1 + 0(1) = \left(\frac{4}{3\pi} + \frac{4}{\pi}\epsilon\right) \log q + 0(1) & \text{if } \chi(-1) = 1 \\ 2 \log q_1 + 0(1) = \left(\frac{2}{3} + 2\epsilon\right) \log q + 0(1) & \text{if } \chi(-1) = -1. \end{cases}$$

Altogether, we obtain from (6)

$$\left|\sum_{n=1}^{N} \chi(n)\right| \leqq (c_{\pm} + \epsilon)\sqrt{q} \log q + 0_\epsilon(\sqrt{q})$$

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with the constants c_+ and c_- of the theorem according as $\chi(-1) = +1$ or -1. This proves the asserted estimate (3) for primitive characters χ.

The extension to arbitrary non-principal characters χ is now straightforward, and we follow the argument in [2, p. 86]. Given a non-principal character χ modulo q, let χ^* be the induced primitive character modulo q^*, and set $r = q/q^*$. Then, for any $N \geqq 1$,

$$\sum_{n \leqq N} \chi(n) = \sum_{n \leqq N} \chi^*(n) \sum_{d|(n,r)} \mu(d) = \sum_{d|r} \chi^*(d)\mu(d) \sum_{n \leqq N/d} \chi^*(n),$$

so that

$$S(\chi) \leqq \sum_{d|r} |\mu(d)| S(\chi^*) = 2^{\omega(r)} S(\chi^*),$$

$\omega(r)$ being as usual the number of distinct prime divisors of r. Hence

$$\frac{S(\chi)}{\sqrt{q}\log q} \leqq \frac{2^{\omega(r)}}{\sqrt{r}} \cdot \frac{\log q^*}{\log q} \cdot \frac{S(\chi^*)}{\sqrt{q^*}\log q^*} \leqq \frac{4}{\sqrt{6}}(1 + o(1))c_{\pm},$$

where the last estimate follows from the inequality

$$\frac{2^{\omega(r)}}{\sqrt{r}} \leqq \prod_{p|r} \frac{2}{\sqrt{p}} \leqq \frac{2}{\sqrt{2}}\frac{2}{\sqrt{3}} = \frac{4}{\sqrt{6}}$$

and the already proved case of the theorem. This completes the proof of the theorem.

References

1. D. A. Burgess, *The character sum estimate with $r = 3$*, J. London Math. Soc. **33** (1986), pp. 219-226.

2. E. Landau, *Abschätzungen von Charaktersummen, Einheiten und Klassenzahlen*, Nachrichten Königl. Ges. Wiss. Göttingen (1918), pp. 79-97.

3. H. Montgomery and R. C. Vaughan, *Exponential sums with multiplicative coefficients*, Inventiones Math. **43** (1977), pp. 69-82.

4. G. Pólya, *Über die Verteilung der quadratischen Reste und Nichtreste*, Nachrichten Königl. Ges. Wiss. Göttingen (1918), pp. 21-29.

5. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis II*, 3rd edition, Springer (1964).

6. I. Schur, *Einige Bemerkungen zur vorstehenden Arbeit des Herrn G. Pólya*, Nachrichten Königl. Ges. Wiss. Göttingen (1918), pp. 30-36.

7. I. M. Vinogradov, *Über die Verteilung der quadratischen Reste und Nichtreste*, J. Soc. Phys. Math. Univ. Permi **2** (1919), pp. 1-14.

8. W. H. Young, *On a certain series of Fourier*, Proc. London Math. Soc. (2) **11** (1913), pp. 357-366.

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