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Examples of linear multi-box splines

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Abstract

Let $S_1 = S_1(v_0, \ldots, v_{r+1})$ be the space of compactly supported C^0 piecewise linear functions on a mesh M of lines through $\mathbb{Z}^2$ in directions $v_0, \ldots, v_{r+1}$, possibly satisfying some restrictions on the jumps of the first order derivative. A sequence $\phi = (\phi_1, \ldots, \phi_r)$ of elements of S_1 is called a multi-box spline if every element of S_1 is a finite linear combination of shifts of (the components of) ϕ. We give some examples for multi-box splines and show that they are stable. It is further shown that any multi-box spline is not always symmetric

1. *Introduction*

Multi-box splines were introduced by Goodman [**2, 3, 5**]. They are C^0 piecewise polynomials of degree $n = 1$, which, unlike box splines, allow both stability and reproduction of arbitrary polynomials of degree $n = 1$.

We shall first define the multi-box splines. Let $r \geqslant 1$ and $v_0, \ldots, v_{r+1}$ be pairwise linearly independent vectors in $\mathbb{Z}^2$, where without loss of generality we suppose that for $j = 0, \ldots, r+1$, the components of v_j are coprime. We shall denote by $S_1 = S_1(v_0, \ldots, v_{r+1})$ the space of all functions $f : \mathbb{R}^2 \to \mathbb{R}$ with continuous Fourier transforms of form

$$\widehat{f}(u) = \frac{\sum_{|\alpha|=r-1} P_\alpha(e^{-iu})u^\alpha}{(iuv_0)\ldots(iuv_{r+1})}, \quad u \in \mathbb{R}^2, \tag{1.1}$$

where for any multi-index $\alpha \in \mathbb{N}^2$ of order $|\alpha| = r - 1$, P_α is a Laurent polynomial with real coefficients [**4, 5**]. Here and elsewhere, for $u, v \in \mathbb{R}^2$, uv denotes their scalar product.

Before giving some properties, we discuss the possible symmetry of multi-box splines ϕ for all $r \geqslant 1$. We say that $\phi = (\phi_1, \ldots, \phi_r)$ is *symmetric* if for $j = 1, \ldots, r$, there are $\sigma_j = \pm 1$, that is, ϕ_j is even or odd about $\frac{1}{2}\alpha_j$, $\alpha_j \in \{0, 1\}^2$, with

$$\phi_j(-\cdot) = \sigma_j \phi_j(\cdot + \alpha_j). \tag{1.2}$$

Theorem 1 [**2**]. *In* (1.1), *let*

$$P_\alpha(z) = \sum_{j \in \mathbb{Z}^2} c_{j,\alpha} z^j.$$

Let V denote the set of all non-zero coefficients $c_{j,\alpha}$, $|\alpha| = r - 1$. Then if f is a spline function with compact support and is given by (1.1), *it has a support in the convex hull of V. Conversely if f in S_1 has its support in a convex closed region R and W denotes R intersection with $\mathbb{Z}^2$ (that is all integer points in R), then f has the form* (1.1) *with the set V of non-zero coefficients lying in W.*

Theorem 2 [**5**]. *A function f lies in S_1 if and only if it is a C^0 spline function of degree 1 over $M(v_0, \ldots, v_{r+1})$ with compact support such that the jump of any first order derivative across any line in $M = M(v_0, \ldots, v_{r+1})$ can change only at points of $\mathbb{Z}^2$.*

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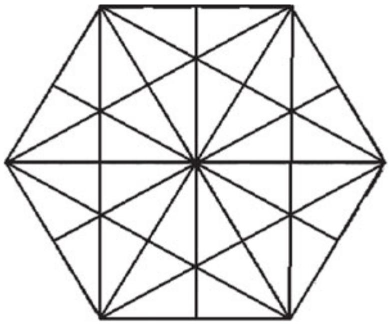

FIGURE 1. *The symmetric hexagonal mesh of the multi-box spline.*

For many choices of $v_0, \dots, v_{r+1}$, Theorem 2 is automatically satisfied and so S_1 comprises all C^0 spline functions of 1st degree $n = 1$, over M with compact support.

THEOREM 3 [5]. *If at most two lines in $M(v_0, \dots, v_{r+1})$ intersect at points not in $\mathbb{Z}^2$, then $S_n(v_0, \dots, v_{r+1})$ comprises all C^0 spline functions of degree $n = 1$ over $M(v_0, \dots, v_{r+1})$ with compact support.*

Now we want to show when, and for which conditions, $\psi \in S_1^r$ can be a local generator for S_1, where S_1^r denotes all row vectors comprising r elements of S_1. So we shall introduce the following theorem.

THEOREM 4 [5]. *The space $S_1 = S_1(v_0, \dots, v_{r+1})$ has a local generator $\phi = (\phi_1, \dots, \phi_r)$. Moreover $\psi = (\psi_1, \dots, \psi_r) \in S_1^r$ is local generator for S_1 if and only if*

$$\widehat{\psi}(u) = \frac{\tilde{u}M(e^{-iu})}{(iuv_0)\dots(iuv_{r+1})}, \quad u \in \mathbb{R}^2, \tag{1.3}$$

where $\tilde{u} := (u_1^{r-1}, u_1^{r-2}u_2, \dots, u_2^{r-1})$ and M is an $r \times r$ matrix of Laurent polynomials with:

$$\det\ M(\underline{z}) = c\underline{z}^k \prod_{j=0}^{r+1}(1 - \underline{z}^{v_j}), \quad \underline{z} = (z, \omega) \in (\mathbb{C}\backslash\{0\})^2, \tag{1.4}$$

for some $k = (k_1, k_2) \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$, where $\underline{z}^k = z^{k_1}\omega^{k_2}$.

For the case $r = 1$, Theorem 4 states that the function ψ is a local generator for S_1 and therefore it is a shift of a multiple of the box spline B_1 (see [5]). For this reason any local generator ϕ for $S_1(v_0, \dots, v_{r+1})$ is called a *multi-box spline* for the case $r = 2$ and extended to $r \geqslant 2$ in [5].

For $r \geqslant 2$ the generator is not unique, because there are many choices of M in (1.4).

The following theorem gives us the conditions of one of the most important properties of the multi-box spline, the stability.

THEOREM 5 [4, 5]. *For the space $S_1 = S_1(v_0, \dots, v_{r+1})$ the following are equivalent.*

(a) *There is a stable local generator $\phi = (\phi_1, \dots, \phi_r)$ of S_1.*
(b) *Every local generator $\phi = (\phi_1, \dots, \phi_r)$ of S_1 is stable.*
(c) *At most r lines in the mesh $M(v_0, \dots, v_{r+1})$ intersect except at points in $\mathbb{Z}^2$.*
(d) *For each $u \in \mathbb{R}^2\backslash 2\pi\mathbb{Z}^2$, there are at most r vectors v_j in $\{v_0, \dots, v_{r+1}\}$ with $e^{iuv_j} = 1$.*

In Section 2 we will construct some linear multi-box spline functions on a six-direction hexagonal mesh as illustrated in [1, p. 101] and as shown in Figure 1. We study the main properties of these newly created functions.

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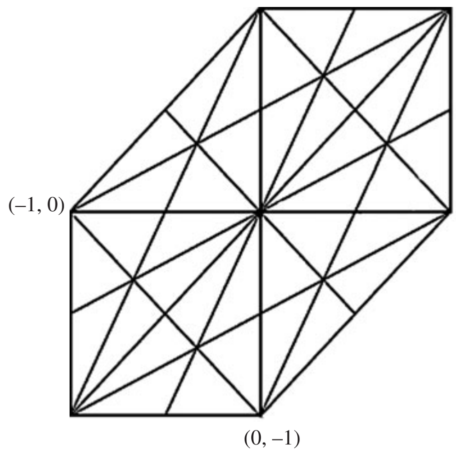

FIGURE 2. *The six-direction mesh of the multi-box spline.*

Section 3 illustrates one example of a linear multi-box spline in a different mesh as shown in Figure 7 and then studies its main properties.

2. *Some linear multi-box splines on a hexagonal mesh*

Take $r = 4$, and the pairwise linearly independent vectors $v_0 = (1, 0)$, $v_1 = (0, 1)$, $v_2 = (1, 1)$, $v_3 = (1, -1)$, $v_4 = (2, 1)$, $v_5 = (1, 2)$.

In this section, we consider two meshes (the six-direction mesh with two coordinate directions as shown in Figure 2, and a symmetric six-direction mesh as seen in Figure 1) obtained from each other by the linear transformation of the plane as in equation (2.3).

Based on Theorem 1, every function f of the space $S_1 = S_1(v_0, \ldots, v_5)$ in the first mesh is to be determined by its continuous Fourier transform $\widehat{f}$ which is defined by equation (1.1). That is

$$\widehat{f}(u, v) = \frac{u^3 P(z, \omega) + u^2 v Q(z, \omega) + uv^2 R(z, \omega) + v^3 S(z, \omega)}{uv(u + v)(u - v)(u + 2v)(2u + v)}. \tag{2.1}$$

We now define four generators $f_1, \ldots, f_4$ on the six-direction mesh and their counterparts $g_1, \ldots, g_4$ on the hexagonal mesh. Their supports are shown in Figures 3 and 4, respectively.

For the first function and because

$$[v(u + v)(u - v), uv(u + v), u(u + 2v)(u + v), uv(2u + v)] = [u^3, u^2 v, uv^2, v^3] * M,$$

where M is the 4×4 matrix and the determinant of $M = 1 \neq 0$, we can replace the numerator of $\widehat{f}$ to give $\widehat{f_1}$ as follows:

$$\widehat{f_1}(u, v) = \frac{v(u + v)(u - v)P(u, v) + uv(u + v)Q(u, v)}{uv(u + v)(u - v)(u + 2v)(2u + v)} + \frac{u(u + v)(u + 2v)R(u, v) + uv(2u + v)S(u, v)}{uv(u + v)(u - v)(u + 2v)(2u + v)}, \tag{2.2}$$

where $z = e^{-iu}$, $\omega = e^{-iv}$. Here P, Q, R and S are Laurent polynomials with real coefficients, defined by

$$P(z, \omega) = a + a_1 z + a_2 z\omega + a_3 \omega + a_4 z^{-1} + a_5 z^{-1}\omega^{-1} + a_6 \omega^{-1},$$
$$Q(z, \omega) = b + b_1 z + b_2 z\omega + b_3 \omega + b_4 z^{-1} + b_5 z^{-1}\omega^{-1} + b_6 \omega^{-1},$$

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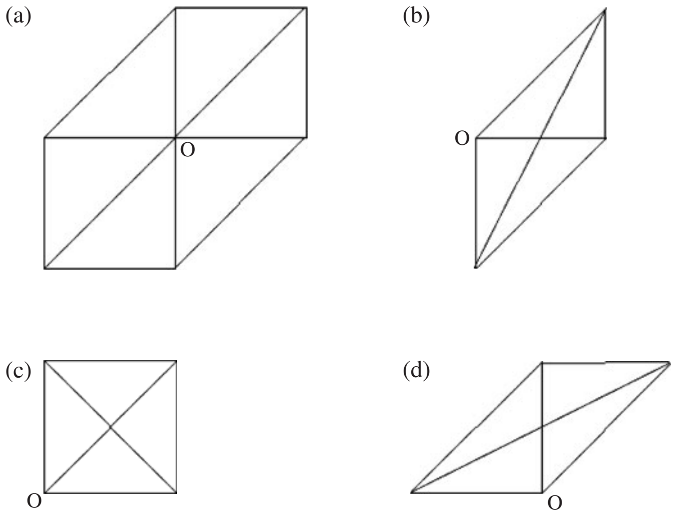

FIGURE 3. *Supports of the functions* $f_i, i = 1, \ldots, 4$.

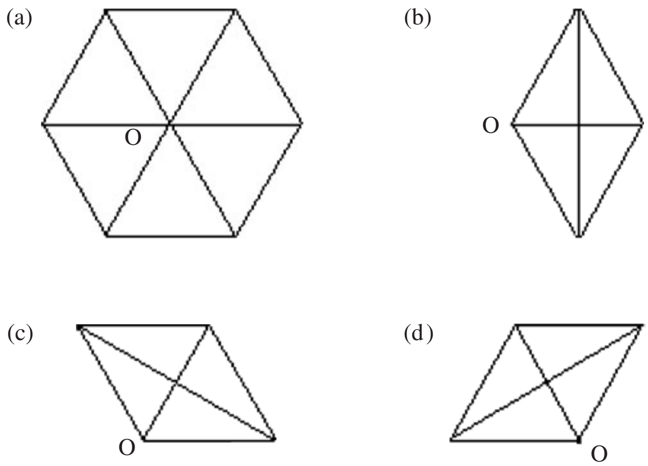

FIGURE 4. *Supports of the functions* $g_i, i = 1, \ldots, 4$.

$$R(z,\omega) = c + c_1 z + c_2 z\omega + c_3\omega + c_4 z^{-1} + c_5 z^{-1}\omega^{-1} + c_6\omega^{-1},$$
$$S(z,\omega) = d + d_1 z + d_2 z\omega + d_3\omega + d_4 z^{-1} + d_5 z^{-1}\omega^{-1} + d_6\omega^{-1}.$$

The coefficients $a, a_1, \ldots, a_6, b, b_1, \ldots, b_6, c, c_1, \ldots, c_6, d, d_1, \ldots, d_6$ will be determined later. In order to have the function g_1 in the symmetric hexagonal mesh as in Figure 1, we use the following transformation

$$g_1(x,y) = f_1(x + y/\sqrt{3}, 2y/\sqrt{3}),$$

and by taking the Fourier transform, we find that

$$\widehat{g}_1(u,v) = \widehat{f}_1(u, \sqrt{3}v/2 - u/2). \tag{2.3}$$

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By substituting the new values of u, v from (2.3) for $\widehat{f}_1(u,v)$ in (2.2), we get

$$\begin{aligned}\widehat{g}_1(u,v) &= \frac{(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)P(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{u(\sqrt{3}v-u)(\sqrt{3}v+u)Q(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{uv(\sqrt{3}v+u)R(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{u(\sqrt{3}v-u)(\sqrt{3}u+v)S(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)},\end{aligned} \tag{2.4}$$

where P takes the following forms in the hexagonal mesh

$$P(u,v) = a + \sum_{j=1}^{6} a_j\, e^{-i\underline{u}\ p_j},$$

with similar formulas for $Q(u,v)$, $R(u,v)$ and $S(u,v)$.

Here and throughout this section, we write $\underline{u} = (u,v)$ and p_j as defined in [1] as follows:

$$p_j = \begin{pmatrix}\cos(j-1)\pi/3 \\ \sin(j-1)\pi/3\end{pmatrix},$$

for $j = 1, \ldots, 6$.

We follow the same steps to construct the second function, to give

$$\begin{aligned}\widehat{f}_2(u,v) &= \frac{uv(u+v)P(z,w) + v(u+v)(u+2v)Q(z,w) + u(u-v)(2u+v)R(z,w)}{uv(u+v)(u-v)(u+2v)(2u+v)} \\ &\quad + \frac{(u-v)(u+2v)(2u+v)S(z,w)}{uv(u+v)(u-v)(u+2v)(2u+v)}.\end{aligned} \tag{2.5}$$

By setting

$$g_2(x,y) = f_2(x + y/\sqrt{3}, 2y/\sqrt{3}),$$

we get

$$\begin{aligned}\widehat{g}_2(u,v) &= \frac{u(\sqrt{3}v-u)(\sqrt{3}v+u)P(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{v(\sqrt{3}v-u)(\sqrt{3}v+u)Q(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{u(\sqrt{3}u+v)(\sqrt{3}u-v)R(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{v(\sqrt{3}u+v)(\sqrt{3}u-v)S(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)}.\end{aligned} \tag{2.6}$$

2.1. *First function*

The function g_1 is symmetric as in

$$g_1(x,y) = g_1(x/2 - \sqrt{3}y/2,\ \sqrt{3}x/2 + y/2),$$

and by using the Fourier transform

$$\widehat{g}_1(u,v) = \widehat{g}_1(u/2 - \sqrt{3}v/2,\ \sqrt{3}u/2 + v/2).$$

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Now by using the above transformation and by replacing the values u, v in (2.4), we get

$$\begin{aligned}\widehat{g}_1(u,v) &= \frac{uv(u+\sqrt{3}v)\widetilde{P}(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{u(u-\sqrt{3}v)(u+\sqrt{3}v)\widetilde{Q}(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{u(u-\sqrt{3}v)(\sqrt{3}u+v)\widetilde{R}(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)} \\ &\quad + \frac{(u-\sqrt{3}v)(u+\sqrt{3}v)(\sqrt{3}u-v)\widetilde{S}(u,v)}{uv(\sqrt{3}v-u)(\sqrt{3}v+u)(\sqrt{3}u-v)(\sqrt{3}u+v)}, \end{aligned} \tag{2.7}$$

where

$$\widetilde{P}(u,v) = a + \sum_{j=1}^{6} a_j e^{-i\underline{u}\ p_{j-1}} = a + \sum_{j=0}^{5} a_{j+1} e^{-i\underline{u}\ p_j},$$

and similarly $\widetilde{Q}(u,v)$, $\widetilde{R}(u,v)$ and $\widetilde{S}(u,v)$.

By comparing (2.4) and (2.7) we get

$$\begin{cases} P = -\widetilde{S}, & Q = -\widetilde{Q}, \\ R = -\widetilde{P}, & S = -\widetilde{R}. \end{cases}$$

We solve in turn each of the above equations.

We shall now apply the continuity conditions for $\widehat{g}_1(u,v)$ in (2.4), which means if the denominator of the fraction is zero, then the numerator must be zero. By applying this observation, we get the following.

- *Condition 1:* $a_3 = -a_2$.
  This is obtained using the fact that $P(0,v) = 0 \quad$ (for all $v \neq 0$).
- *Condition 2:* $3a_1 + 2b_1 - 6a_2 = 0$.
  This is obtained using the fact that $3P(u,0) + 2Q(u,0) + 6S(u,0) = 0$.

Now a_1 and a_2 can be chosen arbitrarily. If we take $a_1 = -a_2 = 1$ then this leads to a particularly simple formula for $\widehat{g}_1$, and $b_1 = -9/2$.

Considering all the previous results we find that $P = R = S$ and

$$Q = -(9/2)P.$$

Finally by substituting the values of P, R, S and Q in (2.4), we find the first multi-box spline in the hexagonal support as

$$\widehat{g}_1(u,v) = \frac{2P_1(u,v)}{u(\sqrt{3}v-u)(\sqrt{3}v+u)}, \tag{2.8}$$

where P_1 takes the following form:

$$P_1(u,v) = e^{-iu} - e^{-i(u/2+\sqrt{3}v/2)} + e^{-i(-u/2+\sqrt{3}v/2)} - e^{iu} + e^{-i(-u/2-\sqrt{3}v/2)} - e^{-i(u/2-\sqrt{3}v/2)}.$$

By changing variables $z = e^{-iu}$, $w = e^{-iv}$ and a simple calculation, we obtain

$$\widehat{f}_1(u,v) = \frac{z^{-1}\omega^{-1}(1-z)(1-\omega)(1-z\omega)}{uv(u+v)}. \tag{2.9}$$

2.2. *Second function*

The function g_2 is symmetric as in

$$g_2(x,y) = g_2(x,-y), \quad g_2(x,y) = g_2(1-x,y),$$

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and by using the Fourier transform

$$\widehat{g}_2(u,v)=\widehat{g}_2(u,-v), \quad \widehat{g}_2(u,v)=e^{-iu}\widehat{g}_2(-u,v).$$

Similar to in the previous section, we find $g_2(u,v)$ as follows:

$$\widehat{g}_2(u,v)=\frac{u\ R(u,v)+\sqrt{3}v\ S(u,v)}{uv(\sqrt{3}v-u)(u+\sqrt{3}v)}, \tag{2.10}$$

where

$$R(u,v)=e^{-i(u/2+\sqrt{3}v/2)}-e^{-i(u/2-\sqrt{3}v/2)}, \quad S(u,v)=1-e^{-iu}.$$

Then

$$\widehat{f}_2(u,v)=\frac{u(z\omega-\omega^{-1})+(u+2v)(1-z)}{uv(u+v)(u+2v)}. \tag{2.11}$$

2.3. *Third function*

The third function, f_3, is defined by its Fourier transform $\widehat{f}_3$, as follows

$$f_3(x,y)=f_2(y,y-x).$$

Now we calculate the function f_3, and by normalization, we get

$$\widehat{f}_3(u,v)=\frac{(u+v)(\omega-z)-(u-v)(1-z\omega)}{uv(u+v)(u-v)}. \tag{2.12}$$

2.4. *Fourth function*

The function f_4 is defined by

$$f_4(x,y)=f_2(y-x,-x).$$

Therefore, we find the Fourier transform of f_4 by normalization as follows:

$$\widehat{f}_4(u,v)=\frac{-v(z^{-1}-z\omega)+(2u+v)(1-\omega)}{uv(u+v)(2u+v)}. \tag{2.13}$$

2.5. *Summary*

We may write $\widehat{f}=(\widehat{f}_1,\widehat{f}_2,\widehat{f}_3,\widehat{f}_4)$ in the following matrix form:

$$\begin{pmatrix}\widehat{f}_1(u,v)\\ \widehat{f}_2(u,v)\\ \widehat{f}_3(u,v)\\ \widehat{f}_4(u,v)\end{pmatrix}=M(z,\omega)\times\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix}\times\frac{1}{uv(u+v)(u-v)(u+2v)(2u+v)},$$

where

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix}=\begin{pmatrix}(u-v)(u+2v)(2u+v)\\ u(u-v)(2u+v)\\ (u+v)(u+2v)(2u+v)\\ v(u-v)(u+2v)\end{pmatrix},$$

and

$$M(z,\omega)=\begin{pmatrix}(z-1)(\omega-1)(z\omega-1) & 0 & 0 & 0\\ 1-z & z\omega-\omega^{-1} & 0 & 0\\ z\omega-1 & 0 & \omega-z & 0\\ 1-\omega & 0 & 0 & z\omega-z^{-1}\end{pmatrix}.$$

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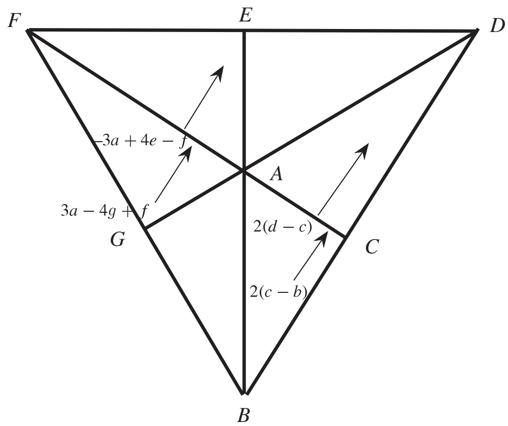

FIGURE 5. *The jump of the derivative of the function in the hexagonal mesh.*

Now we calculate the determinant of M which takes the following form:

$$\det M(z,\omega) = z^{-2}\omega^{-1}(1-z)(1-\omega)(1-z\omega)(1-z\omega^2)(1-z\omega^{-1})(1-z\omega^2).$$

By Theorem 4, the function f is a multi-box spline if the determinant of $M(z,w)$ is in the following form:

$$\det\ M(\underline{z}) = c\underline{z}^k \prod_{j=0}^{r+1}(1-\underline{z}^{v_j}), \quad \underline{z} \in (\mathbb{C}\backslash\{0\})^2,$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$. By comparing our determinant with the above form we find that $c = 1$, $k = (-2,-1)$. So f is a multi-box spline.

Note that three lines in the mesh $M = M(v_0,\ldots,v_5)$ intersect at a point in $\mathbb{R}^2\backslash\mathbb{Z}^2$. Now Theorem 3 says that if at most $n+1=2$ lines in the mesh $M = M(v_0,\ldots,v_5)$ intersect other than in $\mathbb{Z}^2$, then the space $S_1 = S_1(v_0,\ldots,v_5)$ comprises all of the continuous linear splines over $M = M(v_0,\ldots,v_5)$ with compact support. Since this condition is not satisfied, S_1 comprises those continuous linear splines which satisfy the condition that the jump of Df across any line in $M = M(v_0,\ldots,v_5)$ is constant, at any point where three lines in M intersect. See Theorem 2 for the special case of $n = 1$.

In order to derive this jump condition, it will be more convenient to work with the hexagonal mesh. A typical triangle is illustrated in Figure 5, where the jump condition is at the centroid A. The values of the function at the given points $A, B, \ldots, G$ are denoted by $a, b, \ldots, g$.

The value of the jump at the point A is given by this equation:

$$3a = 2g + 2e + 2c - (b + f + d).$$

Now we have to explain where the above condition comes from. Here the jump condition is on the first order derivatives.

We take the triangle $\triangle\, AGF$ with vertices $A(0, \sqrt{3}/3)$, $F(-1/2, \sqrt{3}/2)$, $G(-1/4, \sqrt{3}/4)$, with the corresponding values of our function at these being a, f, g.

Let the direction vector $V = (1/2, \sqrt{3}/2)$, which is parallel to the segment $[BD]$, be defined in terms of the two vectors GA and GF as follows:

$$v_1 = \alpha GA + \beta GF,$$

where

$$GA = (1/4, \sqrt{3}/12), \quad GF = (-1/4, \sqrt{3}/4).$$

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By resolving the above equation, we find that

$$\alpha = 3, \quad \beta = 1,$$

and then

$$v_1 = 3GA + GF.$$

This is equal to

$$v_1 = 3(A - G) + (F - G).$$

So

$$D_v f = 3(a - g) + (f - g) = 3a - 4g + f. \tag{2.14}$$

Similarly on the triangle $\triangle\ AFE$ with vertices $A(0, \sqrt{3}/3)$, $F(-1/2, \sqrt{3}/2)$, $E(0, \sqrt{3}/2)$, and the corresponding values of our function at these being a, f, e, we find that

$$D'_v f = 3(e - a) + (e - f) = -3a + 4e - f. \tag{2.15}$$

So the jump is

$$D_v f - D'_v f = 3a - 4g + f - (3a - 4g + f) = 6a - 4g + 2f - 4e. \tag{2.16}$$

Similarly on the triangles $\triangle\ ABC$ and $\triangle\ ACD$, we find the jump is

$$2(c - b) - 2(d - c) = 4c - 2b - 2d. \tag{2.17}$$

By comparing (2.16) and (2.17), we get

$$6a - 4g + 2f - 4e = 4c - 2b - 2d.$$

This implies that

$$3a = 2(g + e + c) - (b + f + d).$$

By the symmetry of this condition, we will get the same result for the jump condition across the other two mesh lines.

Theorem 5 says that if at most $r = 4$ lines in the mesh $M = M(v_0, \ldots, v_5)$ intersect except at points in $\mathbb{Z}^2$, then f is a stable local generator. Since only three lines in the mesh M intersect other than in $\mathbb{Z}^2$, then f is stable.

The equation (1.2) which defines the symmetry of any function $f = (f_1, f_2, f_3, f_4)$ is satisfied and f is symmetric about (0,0), (1,0), (1,1) and (0,1) respectively, therefore the multi-box spline f is symmetric too.

We can see from (2.9), that the piecewise linear function f_1 is a particular box-spline, that coincides with a hat function known in the finite element method.

The first function f_1 can be replaced by another function ψ_1 which is defined in a smaller support as follows: (see Figure 6)

$$\psi_1(x, y) = f_1(x, y) + f_2(x, y) + f_2((x, y) + (1, 0)), \quad (x, y) \in \mathbb{R}^2.$$

By taking the Fourier transform of the above equation, we get the following:

$$\widehat{\psi}_1(u, v) = \frac{z^{-1}\omega^{-1} + \omega - \omega^{-1} - z\omega}{uv(u + v)} + \frac{z\omega - \omega^{-1} + \omega - z^{-1}\omega^{-1}}{v(u + v)(u + 2v)}. \tag{2.18}$$

We can easily prove that the function $\psi = (\psi_1, f_2, f_3, f_4)$ is a multi-box spline and a local generator for S_1 according to Theorem 4.

3. *Linear multi-box splines on another mesh*

Take $n = 1$, $r = 4$, and the pairwise linearly independent vectors $v_0 = (1, 0)$, $v_1 = (0, 1)$, $v_2 = (2, 1)$, $v_3 = (1, 2)$, $v_4 = (2, -1)$, $v_5 = (1, -2)$, see Figure 7.

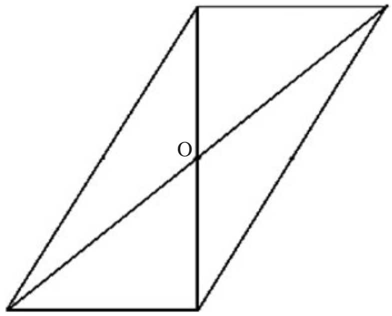

FIGURE 6. *The support of ψ_1 in the new rectangular mesh.*

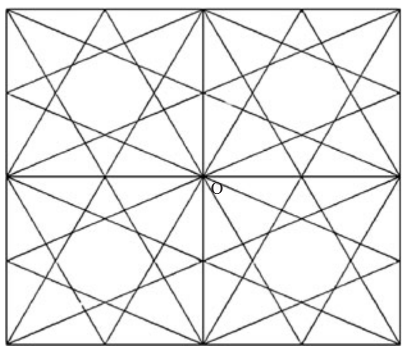

FIGURE 7. *Mesh $M(v_0, \ldots, v_5)$ on $[-1,\ 1]^2$.*

In this case, we firstly need to construct the support of the linear multi-box spline, then form the linear multi-box spline inside it, and show it is a multi-box spline.

By Theorem 4, we have to find generators with good properties like stability and symmetry.

3.1. *The construction algorithm*

We choose five different functions $\phi_1, \phi_2, \psi, \psi_1$ and ψ_2 plus the combination of all of them, this will give us the four functions which form our multi-box spline. Each of the functions ϕ_1, ϕ_2 and ψ take the value zero on $\mathbb{Z}^2$, while ψ_1 and ψ_2 are $\neq 0$ on some points in $\mathbb{Z}^2$.

The combination of ψ_1 and ψ_2 gives us a new function Φ which is zero on $\mathbb{Z}^2$. The function Φ is defined as

$$\Phi = \psi_1 + \psi_1(\cdot - (1,0)) - \psi_2 - \psi_2(\cdot - (0,1)).$$

The function Φ is symmetric under the following conditions

$$\Phi(1-x, y) = \Phi(x, y), \quad \Phi(x, 1-y) = \Phi(x, y), \quad \Phi(x, y) = -\Phi(y, x).$$

We want to express Φ in terms of shifts of ϕ_1, ϕ_2 and ψ, by some unknowns a, b and c, as it will be explained in Step 6. By comparing the two expressions of Φ, we can find the coefficients a, b and c, and then we may write Φ in terms of shifts of ϕ_1, ϕ_2, ψ. Then we will construct another function ϕ_3 as a combination of ψ_1 and shifts of ψ, and then the function ϕ_4 as a linear combination of ψ_2 and shifts of ψ, in such a way that ψ lies in the span of the shifts of $\phi_1, \phi_2, \phi_3, \phi_4$. So $\phi_1, \phi_2, \phi_3, \phi_4$ can be a generator and a multi-box spline for the space S_1, which will be proved later by referring to Theorem 3.

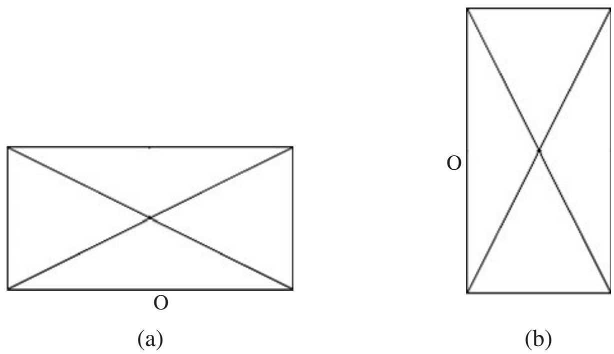

FIGURE 8. *(a) The support of ϕ_1. (b) The support of ϕ_2.*

3.2. *Constructing the linear multi-box spline ϕ*

Step 1. In order to construct the first generator ϕ_1, we choose the support of ϕ_1 as illustrated in Figure 8(a). Let ϕ_1 be defined by its Fourier transform as

$$\widehat{\phi}_1(u,v) = \frac{(2u+v)P_1(z,\omega) + (2u-v)Q_1(z,\omega)}{uv(2u-v)(2u+v)}, \tag{3.1}$$

where $z = e^{-iu}$, $\omega = e^{-iv}$. Here P_1 and Q_1 are Laurent polynomials with real coefficients, defined by

$$P_1(z,\omega) = a + a_1 z + a_2 z\omega + a_3\omega + a_4 z^{-1}\omega + a_5 z^{-1},$$
$$Q_1(z,\omega) = b + b_1 z + b_2 z\omega + b_3\omega + b_4 z^{-1}\omega + b_5 z^{-1}.$$

We require that the function ϕ_1 is symmetric as in

$$\phi_1(x,y) = \phi_1(-x,y), \quad \phi_1(x,y) = \phi_1(x,-y+1),$$

and by using the Fourier transform, we get

$$\widehat{\phi}_1(u,v) = \widehat{\phi}_1(-u,v), \quad \widehat{\phi}_1(u,v) = e^{-iv}\widehat{\phi}_1(u,-v). \tag{3.2}$$

In order to find the coefficients a, a_i, b, b_i, we study the continuity conditions as in the previous section, and by taking $a_1 = -1$, then

$$P_1(z,\omega) = -z + \omega z^{-1}, \quad Q_1(z,\omega) = z\omega - z^{-1}, \quad R_1(z,\omega) = 0, \quad S_1(z,\omega) = 0.$$

Thus

$$\widehat{\phi}_1(u,v) = \frac{(2u+v)(\omega z^{-1} - z) + (2u-v)(z\omega - z^{-1})}{uv(2u-v)(2u+v)}. \tag{3.3}$$

We may write $\widehat{\phi}_1$ in the following matrix form:

$$(\widehat{\phi}_1(u,v)) = M_1(z,\omega) \times \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \times \frac{1}{uv(u+v)(u-v)(u+2v)(2u+v)},$$

where

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \begin{pmatrix}(u-v)(u+2v)(2u+v)\\ u(u-v)(2u+v)\\ (u+v)(u+2v)(2u+v)\\ v(u-v)(u+2v)\end{pmatrix},$$

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and

$$M_1(z,w) = \begin{pmatrix} P_1(z,\omega) & Q_1(z,\omega) & R_1(z,\omega) & S_1(z,\omega) \end{pmatrix},$$

and also

$$\begin{aligned} P_1(z,\omega) &= 4\omega z^{-1}(1 - z^2\omega^{-1}), \\ Q_1(z,\omega) &= -4z^{-1}(1 - z^2\omega), \\ R_1(z,\omega) &= 0, \\ S_1(z,\omega) &= 0. \end{aligned}$$

Step 2. The function ϕ_2 is the transformed image of ϕ_1 by an angle of 90 degrees clockwise,

$$\phi_2(x,y) = \phi_1(-y,x),$$

by taking the Fourier transform

$$\widehat{\phi}_2(u,v) = \widehat{\phi}_1(-v,u).$$

Hence

$$\widehat{\phi}_2(u,v) = \frac{(u-2v)(z\omega - \omega^{-1}) + (u+2v)(\omega - z\omega^{-1})}{uv(u+2v)(u-2v)}, \tag{3.4}$$

see Figure 8(b).

We may write $\widehat{\phi}_2$ in the following matrix form

$$(\widehat{\phi}_2(u,v)) = M_2(z,\omega) \times \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \times \frac{1}{uv(u+v)(u-v)(u+2v)(2u+v)},$$

where

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} (u-v)(u+2v)(2u+v) \\ u(u-v)(2u+v) \\ (u+v)(u+2v)(2u+v) \\ v(u-v)(u+2v) \end{pmatrix},$$

and

$$M_2(z,w) = (P_2(z,\omega) \quad Q_2(z,\omega) \quad R_2(z,\omega) \quad S_2(z,\omega)),$$

and also

$$\begin{aligned} P_2(z,\omega) &= 0, \\ Q_2(z,\omega) &= 0, \\ R_2(z,\omega) &= -4\omega^{-1}(1 - z\omega^2), \\ S_2(z,\omega) &= -4z\omega^{-1}(1 - z^{-1}\omega^2). \end{aligned}$$

Step 3. The function ψ_1 is to be defined by its Fourier transform as shown in Figure 9(a).

We may replace $\widehat{f}(u,v)$ in (2.1) to give $\widehat{\psi}_1$ as follows:

$$\begin{aligned} \widehat{\psi}_1(u,v) = {} & \frac{(2u+v)(u-2v)(2u-v)P(z,\omega) + (u+2v)(u-2v)(2u-v)Q(z,\omega)}{uv(u+2v)(u-2v)(2u-v)(2u+v)}, \\ & + \frac{(u+2v)(2u+v)(2u-v)R(z,\omega) + (u+2v)(2u+v)(u-2v)S(z,\omega)}{uv(u+2v)(u-2v)(2u-v)(2u+v)}, \end{aligned} \tag{3.5}$$

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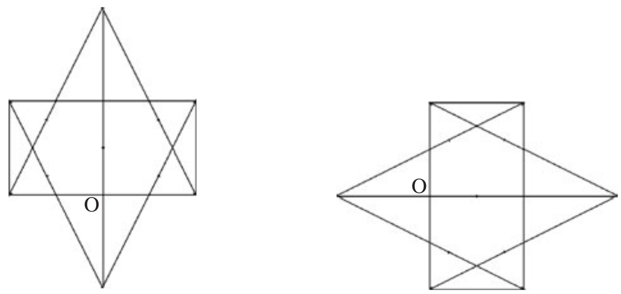

FIGURE 9. *(a) The support of* ψ_1. *(b) The support of* ψ_2.

where $z = e^{-iu}, \omega = e^{-iv}$. Here P, Q, R and S are Laurent polynomials with real coefficients, defined by

$$P(z, \omega) = a_0 + a_1 z + a_2 z\omega + a_3\omega^2 + a_4 z^{-1}\omega + a_5 z^{-1} + a_6\omega^{-1},$$

and similarly for $Q(z, \omega)$, $R(z, \omega)$ and $S(z, \omega)$.

The function ψ_1 is symmetric as in

$$\psi_1(-x, y) = \psi_1(x, y), \quad \psi_1(x, y) = \psi(x, 1 - y),$$

by taking the Fourier transform

$$\widehat{\psi}_1(-u, v) = \widehat{\psi}_1(u, v), \quad \widehat{\psi}_1(u, v) = e^v\widehat{\psi}(u, -v).$$

By applying the continuity conditions, and by taking $a_3 = 1$ and $b_2 = 1$, P, Q, R and S will be defined as

$$P(z, \omega) = z\omega + \omega^2 - z^{-1} - \omega^{-1}, \quad Q(z, \omega) = z\omega - z^{-1},$$
$$R(z, \omega) = \omega^2 + z^{-1}\omega - \omega^{-1} - z, \quad S(z, \omega) = -z + z^{-1}\omega.$$

So ψ_1 is defined as

$$\widehat{\psi}_1(u, v) = \frac{z\omega + \omega^2 - z^{-1} - \omega^{-1}}{uv(u + 2v)} + \frac{z\omega - z^{-1}}{uv(2u + v)} + \frac{\omega^2 + z^{-1}\omega - \omega^{-1} - z}{uv(u - 2v)} + \frac{z^{-1}\omega - z}{uv(2u - v)}. \tag{3.6}$$

Step 4. Now we calculate the function ψ_2, which represents the rotated image of ψ_1 by an angle of 90 degrees clockwise, that is

$$\psi_2(x, y) = \psi_1(y, x),$$

by taking the Fourier transform of this equation

$$\widehat{\psi}_2(u, v) = \widehat{\psi}_1(v, u).$$

We find

$$\widehat{\psi}_2(u, v) = \frac{z\omega + z^2 - \omega^{-1} - z^{-1}}{uv(2u + v)} + \frac{z\omega - \omega^{-1}}{uv(u + 2v)} + \frac{-z^2 - z\omega^{-1} + z^{-1} + \omega}{uv(2u - v)} + \frac{\omega - z\omega^{-1}}{uv(u - 2v)}, \tag{3.7}$$

see Figure 9(b).

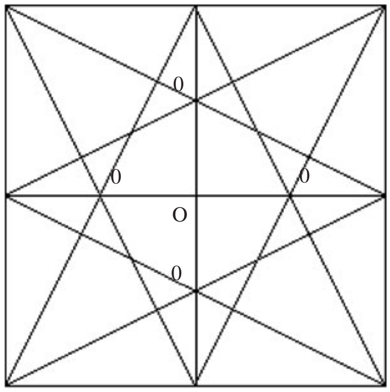

FIGURE 10. *The support of* ψ.

Step 5. We may define ψ as in (3.5), and by its Fourier transform $\widehat{\psi}$:

$$\widehat{\psi}(u,v) = \frac{P(z,\omega)}{uv(u+2v)} + \frac{Q(z,\omega)}{uv(u-2v)} + \frac{R(z,\omega)}{uv(2u-v)} + \frac{S(z,\omega)}{uv(2u+v)}. \tag{3.8}$$

Here P, Q, R and S are Laurent polynomials with real coefficients, defined by

$$P(z,\omega) = a_0 + a_1 z + a_2 z\omega + a_3\omega + a_4 z^{-1}\omega + a_5 z^{-1} + a_6 z^{-1}\omega^{-1} + a_7\omega^{-1} + a_8 z\omega^{-1},$$

and similarly for Q, R and S, see Figure 10.

The function ψ is symmetric as in

$$\psi(-x,y) = -\psi(x,y), \quad \psi(x,-y) = -\psi(x,y), \quad \psi(x,y) = -\psi(y,x),$$

by taking the Fourier transform

$$\widehat{\psi}(-u,v) = -\widehat{\psi}(u,v), \quad \widehat{\psi}(u,-v) = -\widehat{\psi}(u,v), \quad \widehat{\psi}(u,v) = -\widehat{\psi}(v,u).$$

By studying the above, and also the continuity conditions for $\widehat{\psi}(u,v)$ in (3.8), and by taking $a_2 = 1$, $\widehat{\psi}(u,v)$ has the following form:

$$\begin{aligned}\widehat{\psi}(u,v) = {} & \frac{(z^{-1} + z^{-1}\omega - z - z\omega^{-1})}{uv(2u-v)} \\ & + \frac{(z^{-1} + z^{-1}\omega^{-1} - z - z\omega)}{uv(2u+v)} \\ & + \frac{(z\omega + \omega - \omega^{-1} - z^{-1}\omega^{-1})}{uv(u+2v)} \\ & + \frac{(\omega^{-1} + z\omega^{-1} - \omega - z^{-1}\omega)}{uv(u-2v)}.\end{aligned} \tag{3.9}$$

Step 6. Let

$$\Phi(x,y) = \psi_1(x,y) + \psi_1(x-1,y) - \psi_2(x,y) - \psi_2(x,y-1).$$

By taking the Fourier transform

$$\widehat{\Phi}(u,v) = \widehat{\psi}_1(u,v) + e^{-iu}\widehat{\psi}_1(u,v) - \widehat{\psi}_2(u,v) - e^{-iv}\widehat{\psi}_2(u,v),$$

where $z = e^{-iu}$ and $\omega = e^{-iv}$, then

$$\widehat{\Phi}(u,v) = \widehat{\psi}_1(u,v) + z\widehat{\psi}_1(u,v) - \widehat{\psi}_2(u,v) - \omega\widehat{\psi}_2(u,v),$$

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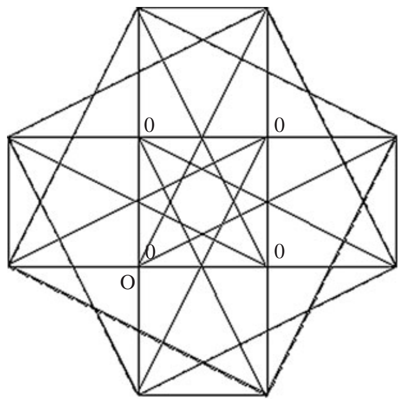

FIGURE 11. *The support of* Φ.

by substituting the values of $\widehat{\psi}_1$ and $\widehat{\psi}_2$, we find that

$$\widehat{\Phi}(u,v) = \frac{z^{-1}\omega - z^2 - z\omega^2 + \omega^{-1}}{uv(2u+v)} + \frac{\omega^2 - z^{-1} + z^2\omega - z\omega^{-1}}{uv(u+2v)} + \frac{z^{-1}\omega - \omega^{-1} + z\omega^2 - z^2}{uv(u-2v)} + \frac{z\omega^{-1} - z^{-1} + z^2\omega - \omega^2}{uv(2u-v)}, \tag{3.10}$$

see Figure 11.

We now show that Φ can be written in the form

$$\Phi(x,y) = aX + bY + cZ,$$

where

$$\begin{aligned} X(x,y) &= \phi_1(x,y-1) + \phi_1(x-1,y-1) + \phi_1(x,y+1) + \phi_1(x-1,y+1) \\ &\quad - \phi_2(x+1,y) - \phi_2(x+1,y-1) - \phi_2(x-1,y) - \phi_2(x-1,y-1), \\ Y(x,y) &= \phi_1(x,y) + \phi_1(x-1,y) - \phi_2(x,y) - \phi_2(x,y-1), \\ Z(x,y) &= \psi(x,y) - \psi(x-1,y) - \psi(x,y-1) + \psi(x-1,y-1). \end{aligned}$$

By taking the Fourier transform

$$\widehat{\Phi} = a\widehat{X} + b\widehat{Y} + c\widehat{Z}, \tag{3.11}$$

by substituting the values of $\widehat{X}$, $\widehat{Y}$ and $\widehat{Z}$, we find that

$$\begin{aligned} \widehat{\Phi}(u,v) &= \frac{(-a-c)(\omega^2 - z^{-1} + z^2\omega - z\omega^{-1}) + (a-c)(z^{-1}\omega^{-1} - \omega - z^2\omega^2 + z)}{uv(u+2v)} \\ &\quad + \frac{b(\omega^{-1} - z\omega - z\omega^2 + 1)}{uv(u+2v)} \\ &\quad + \frac{(-a-c)(z\omega^{-1} - z^{-1} + z^2\omega - \omega^2) + (a-c)(z^{-1}\omega^2 - z\omega + 1 - z^2\omega^{-1})}{uv(2u-v)} \\ &\quad + \frac{b(z^{-1}\omega - z + \omega - z^2)}{uv(2u-v)} \end{aligned}$$

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$$+\frac{(-a-c)(z^{-1}\omega - z^2 - z\omega^2 + \omega^{-1}) + (a-c)(z^2\omega^2 - \omega - z^{-1}\omega^{-1} + z)}{uv(2u+v)}$$
$$+\frac{b(-z^{-1} + z^2\omega - 1 + z\omega)}{uv(2u+v)}$$
$$+\frac{(-a-c)(z^{-1}\omega - \omega^{-1} + z\omega^2 - z^2) + (a-c)(1 - z^{-1}\omega^2 - z\omega + z^2\omega^{-1})}{uv(u-2v)}$$
$$+\frac{b(z\omega^{-1} - \omega - \omega^2 + z)}{uv(u-2v)}. \tag{3.12}$$

By comparing $\widehat{\Phi}(u,v)$ in (3.10) and (3.12) term per term, we find the following equations:

$$-a-c=1, \quad a-c=0, \quad b=0,$$

which give:

$$a=c=-\tfrac{1}{2} \quad \text{and} \quad b=0.$$

Step 7. Let ϕ_3 be defined as

$$\phi_3 = \psi_1 + \frac{3c}{2}\psi - \frac{c}{2}\psi(\cdot - (0,1)),$$

we had $c=-\frac{1}{2}$, so

$$\phi_3(x,y) = \psi_1(x,y) - \tfrac{3}{4}\psi(x,y) + \tfrac{1}{4}\psi(x,y-1). \tag{3.13}$$

By taking the Fourier transform we have

$$\widehat{\phi}_3(u,v) = \widehat{\psi}_1(u,v) - \tfrac{3}{4}\widehat{\psi}(u,v) + \tfrac{1}{4}\omega\widehat{\psi}(u,v). \tag{3.14}$$

By substituting the values of $\widehat{\psi}_1(u,v)$ and $\widehat{\psi}(u,v)$, we get

$$\widehat{\phi}_3(u,v) = \frac{z\omega + 5\omega^2 - 5z^{-1} - \omega^{-1} - 3\omega + 3z^{-1}\omega^{-1} - 1 + z\omega^2}{4uv(u+2v)}$$
$$+\frac{2z^{-1}\omega - 2z - 3z^{-1} + 3z\omega^{-1} - z\omega + z^{-1}\omega^2}{4uv(2u-v)}$$
$$+\frac{6z\omega - 6z^{-1} - 3z^{-1}\omega^{-1} + 3z + z^{-1}\omega - z\omega^2}{4uv(2u+v)}$$
$$+\frac{3\omega^2 + 7z^{-1}\omega - 7\omega^{-1} - 3z - 3z\omega^{-1} + 3\omega + 1 - z^{-1}\omega^2}{4uv(u-2v)}. \tag{3.15}$$

We may write $\widehat{\phi}_3$ in the following matrix form

$$(\widehat{\phi}_3(u,v)) = M_3(z,\omega) \times \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \times \frac{1}{uv(u+v)(u-v)(u+2v)(2u+v)},$$

where

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \begin{pmatrix}(u-v)(u+2v)(2u+v)\\ u(u-v)(2u+v)\\ (u+v)(u+2v)(2u+v)\\ v(u-v)(u+2v)\end{pmatrix},$$

and

$$M_3(z,w) = \begin{pmatrix}P_3(z,\omega) & Q_3(z,\omega) & R_3(z,\omega) & S_3(z,\omega)\end{pmatrix},$$

where

$$P_3(z,\omega) = (1 - z^2\omega^{-1})(2z^{-1}\omega - 3z^{-1} + z^{-1}\omega^2),$$
$$Q_3(z,\omega) = (1 - z^2\omega)(z^{-1}\omega - 3z^{-1}\omega^{-1} - 6z^{-1}),$$
$$R_3(z,\omega) = (1 - z\omega^2)(-5z^{-1} + 3z^{-1}\omega^{-1} - 1 - \omega^{-1}),$$
$$S_3(z,\omega) = (1 - z^{-1}\omega^2)(-7\omega^{-1} + 1 - 3z\omega^{-1} - 3z).$$

Now we calculate $\widehat{\phi}_4(u,v)$ such that

$$\phi_4(x,y) = \phi_3(y,x).$$

By taking the Fourier transform we have

$$\widehat{\phi}_4(u,v) = \widehat{\phi}_3(v,u).$$

By substituting the value of $\widehat{\phi}_3$ we get

$$\begin{aligned}\widehat{\phi}_4(u,v) = {} & \frac{z\omega + 5z^2 - 5\omega^{-1} - z^{-1} - 3z + 3\omega^{-1}z^{-1} - 1 + z^2\omega}{4uv(2u+v)} \\ & + \frac{-2z\omega^{-1} + 2\omega + 3\omega^{-1} - 3z^{-1}\omega + z\omega - z^2\omega^{-1}}{4uv(u-2v)} \\ & + \frac{6z\omega - 6\omega^{-1} - 3z^{-1}\omega^{-1} + 3\omega + z\omega^{-1} - z^2\omega}{4uv(u+2v)} \\ & + \frac{-3z^2 - 7z\omega^{-1} + 7z^{-1} + 3\omega + 3z^{-1}\omega - 3z - 1 + z^2\omega^{-1}}{4uv(2u-v)}. \end{aligned} \tag{3.16}$$

We may write $\widehat{\phi}_4$ in the following matrix form

$$\left(\widehat{\phi}_4(u,v)\right) = M_4(z,\omega) \times \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \times \frac{1}{uv(u+v)(u-v)(u+2v)(2u+v)},$$

where

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \begin{pmatrix}(u-v)(u+2v)(2u+v)\\ u(u-v)(2u+v)\\ (u+v)(u+2v)(2u+v)\\ v(u-v)(u+2v)\end{pmatrix},$$

and

$$M_4(z,w) = (P_4(z,\omega) \quad Q_4(z,\omega) \quad R_4(z,\omega) \quad S_4(z,\omega)),$$

where

$$P_4(z,\omega) = (1 - z^2\omega^{-1})(7z^{-1} - 1 + 3\omega + 3z^{-1}\omega),$$
$$Q_4(z,\omega) = (1 - z^2\omega)(-5\omega^{-1} + 3z^{-1}\omega^{-1} - 1 - z^{-1}),$$
$$R_4(z,\omega) = (1 - z\omega^2)(-6\omega^{-1} - 3z^{-1}\omega^{-1} + z\omega^{-1}),$$
$$S_4(z,\omega) = (1 - z^{-1}\omega^2)(-z^2\omega^{-1} - 2z\omega^{-1} + 3\omega^{-1}).$$

We can confirm that

$$\begin{aligned}&\phi_3 + \phi_3(\cdot - (1,0)) - \phi_4 - \phi_4(\cdot - (0,1)) \\ &\quad = \Phi + 3c\psi - c\psi(\cdot - (1,1)) + c\psi(\cdot - (1,0)) + c\psi(\cdot - (0,1)) \\ &\quad = aX + bY + 4c\psi.\end{aligned}$$

So ψ is in the span of shifts of ϕ_1, ϕ_2, ϕ_3 and ϕ_4. This suggests that $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a multi-box spline.

3.3. *Properties*

Now we show that $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ is a local generator, and study its properties.

We may write $\widehat{\phi}$ in the following form

$$\begin{pmatrix}\widehat{\phi}_1(u,v)\\ \widehat{\phi}_2(u,v)\\ \widehat{\phi}_3(u,v)\\ \widehat{\phi}_4(u,v)\end{pmatrix} = M(z,\omega) \times \begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \times \frac{1}{uv(u+v)(u-v)(u+2v)(2u+v)},$$

where

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} = \begin{pmatrix}(u-v)(u+2v)(2u+v)\\ u(u-v)(2u+v)\\ (u+v)(u+2v)(2u+v)\\ v(u-v)(u+2v)\end{pmatrix},$$

and

$$M(z,w) = \begin{pmatrix} P_1(z,\omega) & Q_1(z,\omega) & R_1(z,\omega) & S_1(z,\omega)\\ P_2(z,\omega) & Q_2(z,\omega) & R_2(z,\omega) & S_2(z,\omega)\\ P_3(z,\omega) & Q_3(z,\omega) & R_3(z,\omega) & S_3(z,\omega)\\ P_4(z,\omega) & Q_4(z,\omega) & R_4(z,\omega) & S_4(z,\omega)\end{pmatrix},$$

where $\widehat{\phi}_1$, $\widehat{\phi}_2$, $\widehat{\phi}_3$ and $\widehat{\phi}_4$ are defined as follows.

For $\widehat{\phi}_1$,

$$\begin{aligned} P_1(z,\omega) &= 4\omega z^{-1}(1 - z^2\omega^{-1}),\\ Q_1(z,\omega) &= -4z^{-1}(1 - z^2\omega),\\ R_1(z,\omega) &= 0,\\ S_1(z,\omega) &= 0. \end{aligned}$$

For $\widehat{\phi}_2$,

$$\begin{aligned} P_2(z,\omega) &= 0,\\ Q_2(z,\omega) &= 0,\\ R_2(z,\omega) &= -4\omega^{-1}(1 - z\omega^2),\\ S_2(z,\omega) &= -4z\omega^{-1}(1 - z^{-1}\omega^2). \end{aligned}$$

For $\widehat{\phi}_3$,

$$\begin{aligned} P_3(z,\omega) &= (1 - z^2\omega^{-1})(2z^{-1}\omega - 3z^{-1} + z^{-1}\omega^2),\\ Q_3(z,\omega) &= (1 - z^2\omega)(z^{-1}\omega - 3z^{-1}\omega^{-1} - 6z^{-1}),\\ R_3(z,\omega) &= (1 - z\omega^2)(-5z^{-1} + 3z^{-1}\omega^{-1} - 1 - \omega^{-1}),\\ S_3(z,\omega) &= (1 - z^{-1}\omega^2)(-7\omega^{-1} + 1 - 3z\omega^{-1} - 3z). \end{aligned}$$

Finally for $\widehat{\phi}_4$,

$$\begin{aligned} P_4(z,\omega) &= (1 - z^2\omega^{-1})(7z^{-1} - 1 + 3\omega + 3z^{-1}\omega),\\ Q_4(z,\omega) &= (1 - z^2\omega)(-5\omega^{-1} + 3z^{-1}\omega^{-1} - 1 - z^{-1}),\\ R_4(z,\omega) &= (1 - z\omega^2)(-6\omega^{-1} - 3z^{-1}\omega^{-1} + z\omega^{-1}),\\ S_4(z,\omega) &= (1 - z^{-1}\omega^2)(-z^2\omega^{-1} - 2z\omega^{-1} + 3\omega^{-1}). \end{aligned}$$

After constructing $M(z,\omega)$, we find the determinant of $M(z,\omega)$ using Maple, as follows:

$$\det\ M(z,\omega) = -2^{10}z^{-2}\omega^{-2}(1-z)(1-\omega) \times (1-z^2\omega^{-1})(1-z^2\omega)(1-z\omega^2)(1-z^{-1}\omega^2).$$

By Theorem 4, the function ϕ is a multi-box spline if the determinant of $M(z,w)$ is in the following form

$$\det\ M(\underline{z}) = c\underline{z}^k \prod_{j=0}^{n+r}(1-\underline{z}^{v_j}), \quad \underline{z} \in (\mathbb{C}\backslash\{0\})^2,$$

for some $k \in \mathbb{Z}^2$, $c \in \mathbb{R}$, $c \neq 0$. By comparing this determinant with the above, we find that $k = (-2,-2)$, $c = -2^{10}$. So ϕ is a multi-box spline.

Note that only three lines in the mesh $M = M(v_0,\ldots,v_5)$ intersect other than in $\mathbb{Z}^2$. So the conditions of Theorem 3 are not satisfied.

Since only three lines in M intersect other than in $\mathbb{Z}^2$, then ϕ is stable as in Theorem 5.

The equation (1.2), which defines the symmetry of any function f_j, shows that ϕ_1 and ϕ_2 are symmetric, however ϕ_3 and ϕ_4 are not symmetric, therefore the multi-box spline $\phi = (\phi_1,\phi_2,\phi_3,\phi_4)$ is not symmetric.

Remark 1. We can see from this case that the multi-box spline is not necessarily symmetric. Our experience suggests that it is not possible to construct a symmetric multi-box spline for this space which, if true, would disprove the conjecture made by Goodman in [**5**] that any space $S_n = S_n(v_0,\ldots,v_{n+r})$ has a symmetric local generator.

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