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AN OBATA-TYPE THEOREM ON A THREE-DIMENSIONAL CR MANIFOLD

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Abstract. We prove the CR version of the Obata's result for the first eigenvalue of the sub-Laplacian in the setting of a compact strictly pseudoconvex pseudohermitian three-dimensional manifold with non-negative CR-Paneitz operator which satisfies a Lichnerowicz-type condition. We show that if the first positive eigenvalue of the sub-Laplacian takes the smallest possible value, then, up to a homothety of the pseudohermitian structure, the manifold is the standard Sasakian three-dimensional unit sphere.

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1. Introduction. The classical theorems of Lichnerowicz [**22**] and Obata [**23**] give correspondingly a lower bound for the first eigenvalue of the Laplacian on a compact manifold with a lower Ricci bound and characterize the case of equality.

A CR analogue of the Lichnerowicz theorem was found by Greenleaf [**11**] for dimensions $2n + 1 > 5$, while the corresponding results for $n = 2$ and $n = 1$ were achieved later in [**19**] and [**8**] respectively. As a continuation of this line of results in the setting of geometries modelled on the rank one symmetric spaces it was proven in [**13**] a quaternionic contact version of the Lichnerowicz result.

The CR Lichnerowicz-type result states that on a compact $2n + 1$-dimensional strictly pseudoconvex pseudohermitian manifold satisfying a certain positivity condition, the first eigenvalue of the sub-Laplacian is grater than or equal to the first eigenvalue of the standard Sasakian sphere. Greenleaf [**11**] showed the result for $n \geq 3$, while Li and Luk adapted in [**19**] Greenleaf's proof to cover the case $n = 2$. They also gave a version of the case $n = 1$ assuming further a condition on the covariant derivative of the pseudohermitian torsion. Later, Chiu [**8**] found a three-dimensional version, the $n = 1$ case, of Greenleaf's result, where the additional assumption was the CR-invariant condition that the CR-Paneitz operator is non-negative. We remark that if $n > 1$, the CR-Paneitz operator is always non-negative, while in the case $n = 1$ the vanishing of the pseudohermitian torsion implies that the CR-Paneitz operator is non-negative, see [**8**] and [**2**]. Further results in the CR case appeared in [**1**, **3**, **4**, **5**] and [**6**] adding a corresponding inequality for $n = 1$, or characterizing the equality case in the vanishing pseudohermitian torsion case (the Sasakian case).

The problem of the existence of an Obata-type theorem in pseudohermitian manifold was considered in [**4**], where the following CR analogue of Obata's theorem was conjectured.

CONJECTURE 1.1 [**4**]. Let (M, θ) be a closed pseudohermitian (2n+1)-manifold with $n \geq 2$. In addition, we assume the CR-Paneitz operator is non-negative if n = 1. Suppose there is a positive constant k_0 such that the pseudohermitian Ricci curvature Ric and the pseudohermitian torsion A satisfy inequality (1.5). If $\frac{n}{n+1}k_0$ is an eigenvalue of the sub-Laplacian then (M, θ) is the standard (Sasakian) CR structure on the unit sphere in $\mathbb{C}^{n+1}$.

This conjecture was proved in the case of a vanishing pseudohermitian torsion (Sasakian case) in [**4**] for $n \geq 2$ and in [**5**] for $n = 1$. The non-Sasakian case was considered in [**7**] where Conjecture 1.1 was established under some assumptions on the divergence and the second covariant derivative of the pseudohermitian torsion. A dimension independent proof of the results due to Greenleaf, Li, Luk and Chiu based on the non-negativity of the CR-Paneitz operator can be found in the Appendix of [**15**]. The key to this direct exposition of the known results is the last inequality in the proof of [**15**, Theorem 8.8], which states that for any smooth function f on a compact pseudohermitian manifold (M, θ) satisfying (1.5) we have

$$0 \geq \int_M \Big[\Big(-\frac{n+1}{n}\lambda + k_0 \Big) |\nabla f|^2 + \big|(\nabla^2 f)_{[-1]}\big|^2 - \frac{3}{2n} P_f(\nabla f) \Big] \, Vol_\theta. \tag{1.1}$$

Here, following [**10**, **18**] for a given function f, we define the one form,

$$P(X) \equiv P_f(X) = \nabla^3 f(X, e_b, e_b) + \nabla^3 f(JX, e_b, Je_b) + 4nA(X, J\nabla f) \tag{1.2}$$

and also the fourth order differential operator (the so-called CR-Paneitz operator in [**8**]),

$$\begin{aligned} Cf = -\nabla^* P = (\nabla_{e_a} P)(e_a) = \nabla^4 f(e_a, e_a, e_b, e_b) + \nabla^4 f(e_a, Je_a, e_b, Je_b) \\ - 4n\nabla^* A(J\nabla f) - 4n\, g(\nabla^2 f, JA), \end{aligned} \tag{1.3}$$

where $\{e_1, \dots, e_{2n}\}$ is an orthonormal basis, and a summation over repeated indices is understood.

Taking into account the divergence formula on a compact pseudohermitian manifold, the non-negativity condition of the Paneitz operator means that we have

$$\int_M f \cdot Cf \, Vol_\theta = -\int_M P_f(\nabla f) \, Vol_\theta \geq 0$$

for any smooth function f. In the three-dimensional case this condition is a CR invariant since it is independent of the choice of the contact form. This follows from the conformal invariance of C proven in [**12**].

A new method to attack the problem was developed by the authors in [**15**], where Conjecture 1.1 was proved under the additional assumption of a divergence-free torsion. The new approach of [**15**] is based on the explicit form of the Hessian with respect to the Tanaka–Webster connection of an extremal eigenfunction f, i.e. an eigenfunction with eigenvalue $n/(n+1)k_0$, and the formula for the pseudohermitian curvature. Specifically, in the extremal case inequality (1.1) is used in [**15**] to determine, among other things, the horizontal Hessian of an extremal eigenfunction f, $\triangle f = \frac{n}{n+1}k_0 f$, which after a rescaling can be put in the form [**15**]

$$\nabla^2 f(X, Y) = -fg(X, Y) - df(\xi)\omega(X, Y), \qquad X, Y \in H = Ker\,\theta. \tag{1.4}$$

The new idea in [**15**] is to determine the full Hessian with respect to the Tanaka–Webster connection based on (1.4). One of the notable consequences of this is the elliptic equation satisfied by the extremal first eigenfunction, which allows the use of Riemannian unique continuation results. This fact was later used in [**20**] where the divergence-free condition of [**15**] was shown to be superfluous for the results of [**15**] to hold true. In fact, in this very recent paper [**20**] Li and Wang established Conjecture 1.1 for $n > 1$ completing our approach [**15**] with the introduction of a new integration by parts idea, cf. [**20**, Lemma 4], involving suitable powers of an extremal function. One should note that in [**20**] the authors reproved a number of results from [**15**] using complex notation. For example, the formulas in the crucial [**20**, Proposition 4] are stated in [**15**, Lemmas 3.1, 4.3 and Equation (4.4)], and the last equation in the proof of [**15**, Lemma 4.1], while [**20**, Lemma 2] follows directly (is a special case of) Equations (4.2) and (4.3) in [**15**]. After all these results of [**15**], the authors of [**20**] add a new and a very important step, namely [**20**, Lemma 4], which makes possible the above-explained improvement of the result of [**15**] in the case $n > 1$.

In the three-dimensional case, [**20**, Proposition 5] is not correctly proved and a correct proof is contained in Section 6.2 of [**15**]. Furthermore, the proof of Conjecture 1.1 presented in [**20**] for dimension three has a gap, since formula (4.8) in [**20**] does not hold in dimension three, which prevents the use of Lemma 3 and equality (4.3). Therefore, [**20**, Corollary 1] cannot be applied in the three-dimensional case.

The purpose of this paper is to settle Conjecture 1.1 in dimension three where we prove the following theorem.

THEOREM 1.2. *Let (M, θ) be a compact strictly pseudoconvex pseudohermitian CR manifold of dimension three with a non-negative CR-Paneitz operator. Suppose there is a positive constant k_0 such that the pseudohermitian Ricci curvature Ric and the pseudohermitian torsion A satisfy the inequality*

$$Ric(X, X) + 4A(X, JX) \geq k_0 g(X, X), \qquad X \in H = Ker\,\theta. \tag{1.5}$$

If $\lambda = \frac{1}{2}k_0$ is an eigenvalue of the sub-Laplacian, then up to a scaling of θ by a positive constant, (M, θ) is the standard (Sasakian) CR structure on the unit three-dimensional sphere in $\mathbb{C}^2$.

The value of the scaling is determined, for example, by the fact that the standard pseudohermitian structure on the unit sphere has first eigenvalue equal to 2. The corresponding eigenspace is spanned by the restrictions of all linear functions to the sphere.

The proof or Theorem 1.2 is based on the explicit form of the Hessian [**15**] with respect to the Tanaka–Webster connection of an extremal eigenfunction f and the integration by parts involving powers of the extremal eigenfunction introduced in [**20**]. After these initial steps we prove Theorem 1.2 as a consequence of Theorem 5.4 taking into account the already established CR Obata theorem for pseudohermitian manifold with a vanishing pseudohermitian torsion. Thus, the new result here is Theorem 5.4, which shows that if on a three-dimensional compact pseudohermitian manifold satisfying (1.5) and having, in addition, a non-negative Paneitz operator we have an eigenfunction f with a horizontal Hessian given by the above formula (1.4), then the pseudohermitian torsion vanishes, i.e. we have a Sasakian structure. The new idea in dimension three is to compare the calculated in **[15]** Ricci tensor with the Lichnerowicz-type assumption (1.5) which results in the formula for the full Hessian with respect to the Tanaka–Webster connection of an extremal eigenfunction expressed in Lemma 5.1.

REMARK 1.3. Following the preprint **[16]**, which became the current paper, a correction of the results of **[20]** in the three-dimensional case appeared in **[21]**. Despite its priority, the paper **[16]** is not referenced in **[21]** even though a number of results in **[21]** were already proven in **[16]** and **[15]**. Notably, the correction of **[20]** contained in **[21]** uses the above-mentioned idea of **[15]** which allows the 'recovery' of formula (5.2) from **[16]** for the full Hessian of an extremal eigenfunction in dimension three. This is a crucial fact in the proofs of [**21**, Theorems 9 and 10]. In the higher dimensional case, [**21**, Theorems 5 and 8] appeared earlier in **[15]** with the additional assumption that the torsion is divergence-free, see [**15**, Theorem 1.3]. In **[21]** the 'novelty' in the general torsion case is the nontrivial reduction to the zero torsion case, similarly to **[15]** but here using [**20**, Lemma 4], while the remaining parts of the argument are identical to those in **[15]**. These results of **[15]** are not mentioned in **[21]**. Finally, in the three-dimensional case, the first complete proof of [**21**, Theorem 10] is given in the earlier paper **[16]**, while the divergence-free torsion case appeared in **[15]**, but both results are not cited in **[21]**.

CONVENTION 1.4.
(a) We shall use X, Y, Z, U to denote horizontal vector fields, i.e. $X, Y, Z, U \in H = Ker\,\theta$.
(b) $\{e_1, \dots e_{2n}\}$ denotes a local orthonormal basis of the horizontal space H.
(c) The summation convention over repeated vectors from the basis $\{e_1, \dots e_{2n}\}$ will be used. For example, for a (0,4)-tensor P, the formula $k = P(e_b, e_a, e_a, e_b)$ means

$$k = \sum_{a,b=1}^{2} P(e_b, e_a, e_a, e_b).$$

2. Pseudohermitian manifolds and the Tanaka–Webster connection. In this section we will briefly review the basic notions of the pseudohermitian geometry of a CR manifold. Also, we recall some results (in their real form) from **[18, 24–26]**, see also [**9**, **14**, **17**], which we will use in this paper.

A CR manifold is a smooth manifold M of real dimension 2n+1, with a fixed n-dimensional complex sub-bundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C}TM$ satisfying $\mathcal{H} \cap \overline{\mathcal{H}} = 0$ and $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. If we let $H = Re\,\mathcal{H} \oplus \overline{\mathcal{H}}$, the real sub-bundle H is equipped with a formally integrable almost complex structure J. We assume that M is oriented and there exists a globally defined compatible contact form θ such that the *horizontal space* is given by $H = Ker\,\theta$. In other words, the hermitian bilinear form

$$2g(X, Y) = -d\theta(JX, Y)$$

is non-degenerate. The CR structure is called strictly pseudoconvex if g is a positive definite tensor on H. The vector field ξ dual to θ with respect to g satisfying $\xi \lrcorner d\theta = 0$ is called the Reeb vector field. The almost complex structure J is formally integrable in the sense that $([JX, Y] + [X, JY]) \in H$ and the Nijenhuis tensor $N^J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0$.

A CR manifold (M, θ, g) with a fixed compatible contact form θ is called *a pseudohermitian manifold*. In this case the 2-form

$$d\theta_{|_H} := 2\omega$$

is called the fundamental form. Note that the contact form is determined up to a conformal factor, i.e. $\bar{\theta} = \nu\theta$ for a positive smooth function ν, defines another pseudohermitian structure called pseudo-conformal to the original one.

2.1. The Tanaka–Webster connection. The Tanaka–Webster connection [**24–26**] is the unique linear connection ∇ with torsion T preserving a given pseudohermitian structure, i.e. it has the properties

$$\begin{aligned} &\nabla\xi = \nabla J = \nabla\theta = \nabla g = 0, \\ &T(X, Y) = d\theta(X, Y)\xi = 2\omega(X, Y)\xi, \quad T(\xi, X) \in H, \\ &g(T(\xi, X), Y) = g(T(\xi, Y), X) = -g(T(\xi, JX), JY). \end{aligned} \tag{2.1}$$

For a smooth function f on a pseudohermitian manifold M, we denote by ∇f its horizontal gradient,

$$g(\nabla f, X) = df(X). \tag{2.2}$$

The horizontal sub-Laplacian $\triangle f$ and the norm of the horizontal gradient $\nabla f = df(e_a)e_a$ of a smooth function f on M are defined, respectively, by

$$\triangle f \;=\; -\,tr^g_H(\nabla df) \;=\; \nabla^* df = -\,\nabla df(e_a, e_a), \qquad |\nabla f|^2 \;=\; df(e_a)\,df(e_a). \tag{2.3}$$

The function $f \not\equiv 0$ is an eigenfunction of the sub-Laplacian if

$$\triangle f = \lambda f, \tag{2.4}$$

where λ is necessarily a non-negative constant.

It is well known that the endomorphism $T(\xi, .)$ is the obstruction a pseudohermitian manifold to be Sasakian. The symmetric endomorphism $T_\xi : H \longrightarrow H$ is denoted by A,

$$A(X, Y) \overset{def}{=} T(\xi, X, Y),$$

and is called *the (Webster) torsion of the pseudohermitian manifold or pseudohermitian torsion.* It is a completely trace-free tensor of type (2,0) + (0,2),

$$A(e_a, e_a) = A(e_a, Je_a) = 0, \quad A(X, Y) = A(Y, X) = -A(JX, JY). \tag{2.5}$$

Let $R = [\nabla, \nabla] - \nabla_{[,]}$ be the curvature of the Tanaka–Webster connection. The pseudohermitian Ricci tensor Ric, the pseudohermitian scalar curvature S and the pseudohermitian Ricci 2-form ρ are defined by

$$Ric(C, B) = R(e_a, C, B, e_a), \quad S = Ric(e_a, e_a),$$
$$\rho(C, B) = \frac{1}{2}R(C, B, e_a, Ie_a), \ C, B \in \Gamma(TM).$$

As well known, ρ, sometimes called the Webster Ricci tensor, is the (1,1)-part of Ric. In dimension three we have $Ric(.,.) = \rho(J.,.)$. We refer the reader to [**15**] for a quick summary using real expression of the well-known properties of the curvature R of the Tanaka–Webster connection established in [**18**, **25**, **26**], see also [**9**, **14**, **17**].

2.2. The Ricci identities for the Tanaka–Webster connection. We shall repeatedly use the following Ricci identities of orders two and three for a smooth function f, see also [**14**, **15**],

$$\begin{aligned}
\nabla^2 f(X, Y) - \nabla^2 f(Y, X) &= -2\omega(X, Y)df(\xi),\\
\nabla^2 f(X, \xi) - \nabla^2 f(\xi, X) &= A(X, \nabla f),\\
\nabla^3 f(X, Y, Z) - \nabla^3 f(Y, X, Z) &= -R(X, Y, Z, \nabla f) - 2\omega(X, Y)\nabla^2 f(\xi, Z),\\
\nabla^3 f(X, Y, Z) - \nabla^3 f(Z, Y, X) &= -R(X, Y, Z, \nabla f) - R(Y, Z, X, \nabla f)\\
&\quad - 2\omega(X, Y)\nabla^2 f(\xi, Z) - 2\omega(Y, Z)\nabla^2 f(\xi, X)\\
&\quad + 2\omega(Z, X)\nabla^2 f(\xi, Y) + 2\omega(Z, X)A(Y, \nabla f),\\
\nabla^3 f(\xi, X, Y) - \nabla^3 f(X, \xi, Y) &= (\nabla_{\nabla f} A)(Y, X) - (\nabla_Y A)(\nabla f, X) - \nabla^2 f(AX, Y),\\
\nabla^3 f(X, Y, \xi) - \nabla^3 f(\xi, X, Y) &= \nabla^2 f(AX, Y) + \nabla^2 f(X, AY) + (\nabla_X A)(Y, \nabla f)\\
&\quad + (\nabla_Y A)(X, \nabla f) - (\nabla_{\nabla f})A(X, Y).
\end{aligned} \tag{2.6}$$

An important consequence of the first Ricci identity is the following fundamental formula:

$$g(\nabla^2 f, \omega) = \nabla^2 f(e_a, Je_a) = -2n\, df(\xi). \tag{2.7}$$

On the other hand, by (2.3) the trace with respect to the metric is the negative sub-Laplacian

$$g(\nabla^2 f, g) = \nabla^2 f(e_a, e_a) = -\triangle f.$$

We also recall the horizontal divergence theorem allowing 'integration by parts' [**24**]. Let (M, g, θ) be a pseudohermitian manifold of dimension $2n+1$. For a fixed local 1-form θ, the form $Vol_\theta = \theta \wedge \omega^n$is a globally defined volume form, since Vol_θ is independent on the local one form θ. The (horizontal) divergence of a horizontal vector field/one-form $\sigma \in \Lambda^1(H)$ is defined by

$$\nabla^* \sigma = -tr|_H \nabla\sigma = -(\nabla_{e_a}\sigma)e_a.$$

The divergence formula [**24**] gives the 'integration by parts' identity for a one-form of compact support

$$\int_M (\nabla^*\sigma)\, Vol_\theta = 0.$$

3. The vertical Bochner formula. We recall explicitly the vertical Bochner formula from [**15**] since it will provide an important step in the proof of the main result. The proof below is contained in Remark 3.5 of [**15**].

PROPOSITION 3.1 ([**15**], 'vertical Bochner formula'). *For any smooth function f on a pseudohermitian manifold of dimension* $(2n+1)$, *the following identity holds*:

$$-\triangle(\xi f)^2 = 2|\nabla(\xi f)|^2 - 2df(\xi)\cdot\xi(\triangle f) + 4df(\xi)\cdot g(A,\nabla^2 f) - 4df(\xi)(\nabla^* A)(\nabla f). \tag{3.1}$$

Proof. To prove (3.1), we use the last of the Ricci identities (2.6) and the fact that the torsion is trace-free to obtain

$$\begin{aligned}-\frac{1}{2}\triangle(\xi f)^2 &= \nabla^3 f(e_a,e_a,\xi)df(\xi) + \nabla^2 f(e_a,\xi)\nabla^2 f(e_a,\xi)\\ &= \left[\nabla^3 f(\xi,e_a,e_a) + 2g(\nabla^2 f, A) - 2(\nabla^* A)(\nabla f)\right]df(\xi) + |\nabla(\xi f)|^2\\ &= |\nabla(\xi f)|^2 - df(\xi)\cdot\xi(\triangle f) + 2df(\xi)\cdot g(A,\nabla^2 f) - 2df(\xi)(\nabla^* A)(\nabla f),\end{aligned}$$

which completes the proof of (3.1). □

4. The Hessian of an extremal function in the extremal three-dimensional case. In this section we recall some results from [**15**] determining the full Hessian of an '*extremal first eigenfunction*' that is an eigenfunction with the smallest possible eigenvalue assuming the Greenleaf's positivity condition.

Let M be a compact, strictly pseudoconvex CR three-manifold satisfying Greenleaf's positivity condition, which in real notation takes the form

$$Ric(X,X) + 4A(X,JX) \geq k_0\, g(X,X)$$

such that the CR-Paneitz operator is *non-negative on* f. If $\frac{1}{2}k_0$ is an eigenvalue of the sub-Laplacian, $\triangle f = \frac{1}{2}k_0 f$ then the corresponding eigenfunctions satisfy the next identities, cf. Section 3 of [**15**],

$$\nabla^2 f(X,Y) = -\frac{k_0}{4} f g(X,Y) - df(\xi)\omega(X,Y). \tag{4.1}$$

Furthermore, by [**15**, Remark 3.2], we have

$$Ric(\nabla f,\nabla f) + 4A(J\nabla f,\nabla f) = k_0|\nabla f|^2, \qquad \int_M P_f(\nabla f)\, Vol_\theta = 0. \tag{4.2}$$

Since the horizontal space is two dimensional, we can use ∇f, $J\nabla f$ as a basis at the points where $|\nabla f| \neq 0$. In fact, we have $|\nabla f| \neq 0$ almost everywhere. This follows from [**15**, Lemma 5.1] showing that f satisfies a certain elliptic equation which implies that f cannot vanish on any open set since otherwise $f \equiv 0$, which is a contradiction.

The 'mixed' parts of the Hessian are given in the second and the third equations in the proof of Theorem 5.2 in [**15**] as follows:

$$\begin{aligned}&\nabla^2 f(\xi, J\nabla f) = -|\nabla f|^2 + A(J\nabla f,\nabla f) = -\frac{1}{4}Ric(\nabla f,\nabla f) = -\frac{S}{8}|\nabla f|^2,\\ &\nabla^2 f(\xi,\nabla f) = -\frac{1}{3}A(\nabla f,\nabla f), \qquad \nabla^2 f(\nabla f,\xi) = \frac{2}{3}A(\nabla f,\nabla f).\end{aligned} \tag{4.3}$$

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The Ricci identities together with (4.3) imply

$$\nabla^2 f(J\nabla f, \xi) = -|\nabla f|^2 + 2A(J\nabla f, \nabla f). \tag{4.4}$$

Using a homothety we can reduce to the case $\lambda_1 = 2$ and $k_0 = 4$, which are the values for the standard Sasakian round three-sphere. Henceforth, we shall work under these assumptions. Thus, for an extremal first eigenfunction f (by definition $f \not\equiv 0$), we have the equalities

$$\lambda = 2, \qquad \triangle f = 2f, \qquad \int_M (\triangle f)^2 \, Vol_\theta = 2\int_M |\nabla f|^2 \, Vol_\theta. \tag{4.5}$$

In addition, the horizontal Hessian of f satisfies (4.1), which with the assumed normalization takes the form given in equation (1.4).

5. Vanishing of pseudohermitian torsion. In this section we show the vanishing of the pseudo-hermitian torsion. We shall assume, unless explicitly stated otherwise, that M is a compact strictly pseudoconvex pseudohermitian CR manifold of dimension three for which (1.5) holds and f is a smooth function on M satisfying (1.4). In particular, we have done the normalization, if necessary, so that (1.5) holds with $k_0 = 4$.

LEMMA 5.1. *Let f be an extremal eigenfunction of the sublaplacian on a compact strongly pseudoconvex three-dimensional pseudohermitian manifold. Then we have*

$$A(\nabla f, \nabla f) = 0 \tag{5.1}$$

and the 'mixed' derivatives are given by

$$\nabla^2 f(\xi, Y) = df(JY) + A(Y, \nabla f), \qquad \nabla^2 f(Y, \xi) = df(JY) + 2A(Y, \nabla f). \tag{5.2}$$

Proof. Using the 'vertical' Bochner formula (3.1) and taking into account that $g(A, \nabla^2 f) = 0$, after an integration by parts, we obtain

$$\begin{aligned} 0 &= \int_M |\nabla(\xi f)|^2 - df(\xi)\cdot\xi(\triangle f) + 2df(\xi)\cdot g(A, \nabla^2 f) - 2df(\xi)(\nabla^* A)(\nabla f)\, Vol_\theta \\ &= \int_M |\nabla(\xi f)|^2 - 2(\xi f)^2 - 2\nabla^2 f(e_a, \xi)A(e_a, \nabla f)\, Vol_\theta \\ &= \int_M -2(\xi f)^2 + \frac{1}{|\nabla f|^2}\Big[\big(\nabla^2 f(\nabla f, \xi)\big)^2 + \big(\nabla^2 f(J\nabla f, \xi)\big)^2\Big]\, Vol_\theta \\ &\times \int_M \frac{-2}{|\nabla f|^2}\Big[\nabla^2 f(\nabla f, \xi)A(\nabla f, \nabla f) + \nabla^2 f(J\nabla f, \xi)A(J\nabla f, \nabla f)\Big]\, Vol_\theta. \end{aligned} \tag{5.3}$$

using that

$$\frac{1}{|\nabla f|^2}|A(\nabla f, \nabla f)| \le ||A|| \overset{def}{=} \sup_M |A| \quad \text{a.e.}$$

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since $|\nabla f| \neq 0$ almost everywhere. Using (4.3) and (4.4), (5.3) takes the form

$$\int_M \Big[\frac{-8}{9|\nabla f|^2}\Big(A(\nabla f, \nabla f)\Big)^2 + |\nabla f|^2 - 2|\xi f|^2 - 2A(J\nabla f, \nabla f)\Big]\, Vol_\theta = 0. \tag{5.4}$$

Now, we recall [**15**, Lemmas 8.6 and 8.7] implying an identity, which in the case $n = 1$ reduces to

$$2\int_M A(J\nabla f, \nabla f)\, Vol_\theta = \int_M \Big[-\frac{1}{2}g(\nabla^2 f, \omega)^2 + \frac{1}{2}(\triangle f)^2 + \frac{1}{2}P(\nabla f)\Big]\, Vol_\theta. \tag{5.5}$$

Taking into account (2.7), (4.5) and the fact that in the extremal case we have $\int_M P_f(\nabla f) = 0$ by (4.2), (5.4) and (5.5) imply

$$\frac{16}{9}\int_M \Big(\frac{A(\nabla f, \nabla f)}{|\nabla f|}\Big)^2\, Vol_\theta = 0, \tag{5.6}$$

hence the claimed result for torsion A. The formulas for the mixed derivatives follow taking also into account (4.3), and (4.4). □

Lemma 5.1 implies a number of crucial identities, which we record in the following corollaries.

COROLLARY 5.2. *The following identities hold true almost everywhere,*

$$|\nabla f|^2 A(JY, \nabla f) = df(Y)A(\nabla f, J\nabla f), \tag{5.7}$$

$$|\nabla f|^4 |A|^2 = 2\Big(A(\nabla f, J\nabla f)\Big)^2. \tag{5.8}$$

In addition, we have

$$|\nabla f|^2 |A| = -\sqrt{2}A(\nabla f, J\nabla f). \tag{5.9}$$

Proof. Since $\nabla f, J\nabla f$ form the basis of H almost everywhere, then (5.7) follows from (5.1) by a direct verification. Then (5.8) follows since the horizontal space is two-dimensional. Note that Lichnerowicz' condition implies that

$$A(\nabla f, J\nabla f) \leq 0, \tag{5.10}$$

which together with (5.8) imply (5.9). □

The proof of [**15**, Lemma 4.2] shows that (5.2) gives the following fact.

LEMMA 5.3. *Let M be a strictly pseudoconvex pseudohermitian CR manifold of dimension three. If f is an eigenfunction of the sub-Laplacian satisfying* (1.4)*, then the following formula for the third covariant derivative holds*

$$\begin{aligned}\nabla^3 f(X, Y, \xi) = &-df(\xi)g(X, Y) + f\omega(X, Y) - 2fA(X, Y)\\ &- 2df(\xi)A(JX, Y) + 2(\nabla_X A)(Y, \nabla f).\end{aligned} \tag{5.11}$$

We turn to the proof of our main result.

THEOREM 5.4. *Let M be a compact, strictly pseudoconvex pseudohermitian CR manifold of dimension three for which the Lichnerowicz condition* (1.5) *holds and the*

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Paneitz operator is non-negative. If f is an eigenfunction satisfying (1.4)*, then the pseudohermitian torsion vanishes, $A = 0$.*

Proof. First we show

$$g(\nabla f, \nabla |A|^2) = 0.$$

Indeed, Lemma 5.3 gives

$$g(A, \nabla^3 f(.,.,\xi)) = -2f|A|^2 + 2g(A, \nabla A(.,.,\nabla f)). \tag{5.12}$$

Next we compute the above scalar product using the Ricci identities. In fact, in the last Ricci identity we make the substitution

$$\nabla^3 f(\xi, X, Y)) = -df(\xi)g(X, Y) - (\xi^2 f)\omega(X, Y),$$

which follows from the Hessian equation (4.1), to obtain the equation

$$\begin{aligned} \nabla^3 f(X, Y, \xi) = -df(\xi)g(X, Y) - (\xi^2 f)\omega(X, Y) - 2f \cdot A(X, Y) \\ + \nabla A(X, Y, \nabla f) + 2df(\xi)A(JX, Y) - \nabla A(\nabla f, X, Y). \end{aligned} \tag{5.13}$$

Now, (5.13) implies

$$g(A, \nabla^3 f(.,.,\xi)) = -2f|A|^2 + 2g(A, \nabla A(.,.,\nabla f)) - g(A, \nabla A(\nabla f,.,.)). \tag{5.14}$$

Equations (5.12) and (5.14) show that

$$g(A, \nabla A(\nabla f,.,.)) = 0, \qquad g(A, \nabla^3 f(.,.,\xi)) = -2f|A|^2 + 2g(A, \nabla A(.,.,\nabla f)), \tag{5.15}$$

hence

$$g(\nabla f, \nabla |A|^2) = 2g(A, \nabla A(.,.,\nabla f)) = 0. \tag{5.16}$$

Equation (5.16) implies that for any $k > 0$ we have

$$g(\nabla f, \nabla |A|^k) = 0. \tag{5.17}$$

For simplicity, suppose $A \neq 0$ everywhere. For a complete argument, which requires the introduction of a cut-off function, we refer to [**20**, Lemma 4]). Since M is compact, there is a constant $a > 0$ such that at every point of M we have

$$a < |A|, \quad \text{hence} \quad |A|^2 \leq \frac{1}{a}|A|^3. \tag{5.18}$$

The divergence formula gives

$$\begin{aligned} \int_M |A|^3 f^{2(k+1)}\, Vol_\theta &= -\frac{1}{2}\int_M |A|^3 f^{2k+1} \triangle f\, Vol_\theta = \frac{1}{2}\int_M g(\nabla(|A|^3 f^{2k+1}), \nabla f)\, Vol_\theta \\ &= \frac{2k+1}{2}\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta + \frac{1}{2}\int_M f^{2k+1} g(\nabla |A|^3, \nabla f)\, Vol_\theta \\ &= \frac{2k+1}{2}\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta, \end{aligned} \tag{5.19}$$

taking into account (5.17).

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With the help of (5.9), the divergence formula and (5.17) we can compute the last integral as follows:

$$
\begin{aligned}
&\sqrt{2}\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta \\
&\quad = -\int_M |A|^2 f^{2k} A(\nabla f, J\nabla f)\, Vol_\theta \\
&\qquad -\int_M |A|^2 f^{2k} A(e_a, J\nabla f) df(e_a)\, Vol_\theta \\
&\quad = \int_M g(\nabla |A|^2, AJ\nabla f) f^{2k+1}\, Vol_\theta + 2k\int_M |A|^2 f^{2k} A(\nabla f, J\nabla f)\, Vol_\theta \\
&\qquad + \int_M |A|^2 f^{2k+1}\nabla A(e_a, e_a, J\nabla f)\, Vol_\theta + \int_M |A|^2 f^{2k+1} A(e_a, J\nabla_{e_a}(\nabla f))\, Vol_\theta. \qquad (5.20)
\end{aligned}
$$

The last integral equals zero due to (1.4). The first integral is zero due to (5.7) and (5.17). Therefore, the first and the last equality in (5.20) give

$$
\sqrt{2}(2k+1)\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta = \int_M |A|^2 f^{2k+1}\nabla A(e_a, e_a, J\nabla f)\, Vol_\theta. \qquad (5.21)
$$

Therefore, using (5.18) and Hölder's inequality we have

$$
\begin{aligned}
&\sqrt{2}(2k+1)\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta \le ||\nabla^* A|| \int_M |A|^2 f^{2k+1}|\nabla f|\, Vol_\theta \\
&\quad \le \frac{||\nabla^* A||}{a}\int_M |A|^3 f^{2k+1}|\nabla f|\, Vol_\theta \le \frac{||\nabla^* A||}{a}\Big(\int_M |A|^3 f^{2(k+1)}\, Vol_\theta\Big)^{1/2} \\
&\qquad \times\Big(\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta\Big)^{1/2} \\
&\quad = \frac{||\nabla^* A||}{a}\Big(\int_M |A|^3 f^{2(k+1)}\, Vol_\theta\Big)^{1/2}\Big(\frac{2}{2k+1}\int_M |A|^3 f^{2(k+1)}\, Vol_\theta\Big)^{1/2}, \qquad (5.22)
\end{aligned}
$$

using (5.19) in the last equality. Now, equation (5.19) gives

$$
\sqrt{2}(2k+1)\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta \le \frac{||\nabla^* A||}{a}\Big(\frac{2k+1}{2}\Big)^{1/2}\Big(\int_M |A|^3 f^{2k}|\nabla f|^2\, Vol_\theta\Big),
$$

which gives a contradiction by taking k sufficiently large. Therefore, the torsion vanishes, $|A| = 0$.

□

5.1. Proof of Theorem 1.2. We apply Theorem 5.4 to conclude that the pseudohermitian torsion vanishes. The claim of Theorem 1.2 follows by applying the known result in the torsion-free (Sasakian) case, see [**5**] for the original proof or the later proof in [**15**] valid also in certain non-compact cases.

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