

## ON GILP'S GROUP-THEORETIC APPROACH TO FALCONER'S DISTANCE PROBLEM

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**Abstract.** In this paper, we follow and extend a group-theoretic method introduced by Greenleaf–Iosevich–Liu–Palsson (GILP) to study finite points configurations spanned by Borel sets in  $\mathbb{R}^n$ ,  $n \geq 2$ ,  $n \in \mathbb{N}$ . We remove a technical continuity condition in a GILP's theorem in [*Revista Mat. Iberoamer* **31** (2015), 799–810]. This allows us to extend the Wolff–Erdogan dimension bound for distance sets to finite points configurations with  $k$  points for  $k \in \{2, \dots, n + 1\}$  forming a  $(k - 1)$ -simplex.

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**1. Introduction.** This paper is a self-contained sequel of [18] on dimension result for triangle sets and more generally on higher order finite points configurations. Let  $n \geq 2$  and  $k \in \{2, \dots, n + 1\}$  be integers. In this paper, we study  $k$  points configurations spanned by a subset  $F \subset \mathbb{R}^n$ . We start with some definitions.

**DEFINITION 1.1.** Let  $n \geq 2$  and  $2 \leq k \leq n + 1$  be integers. Given a set  $F \subset \mathbb{R}^n$ , define 
$$\Delta_k(F) = \{(r_{ij}, 1 \leq i < j \leq k) \in \mathbb{R}^{k(k-1)/2} : x_1, \dots, x_k \in F, |x_i - x_j| = r_{ij}, 1 \leq i < j \leq k\}.$$

A special case is when  $k = 2$ . In this case, we write  $D(F) = \Delta_2(F)$  and call it the distance set of  $F$ . If  $F$  is a finite set in  $\mathbb{R}^2$ , by a result in [10] we have

$$\#D(F) \gtrsim \#F / \log \#F.$$

Here for a set  $A$ , we use  $\#A$  to denote the cardinality of  $A$ . The above result settled the challenging Erdős distance conjecture. For  $F$  being a compact set with positive Hausdorff dimension, we are interested in whether  $D(F)$  has full Hausdorff dimension or even positive Lebesgue measure. In this direction, we have the following conjecture.

**CONJECTURE** (Falconer's distance conjecture). *Let  $n \geq 2$  be an integer. Let  $F \subset \mathbb{R}^n$  be a compact set with  $\dim_{\text{H}} F > n/2$ . Then,  $D(F)$  has positive Lebesgue measure.*

See [9, 11, 14, 15] for some recent progresses towards the above conjecture for  $n = 2$ . For  $n \geq 3$ , see [1] and [2].

A natural generalisation of the distance set problem is to consider finite points configurations with more than two points, see [7] and the references therein. When  $n = 2$ ,  $k = 3$ , we meet the problems considering ‘triangle sets in the plane’, see [18]. In particular, if  $\dim_{\text{H}} F = s \in (0, 2)$ , one can show that the lower box dimension of  $\Delta_3(F)$  is at least  $3s/2$ .

Unlike most of the results which follow from harmonic analytic methods, this  $3s/2$  bound holds for  $s < 1$  as well. For distance sets ( $k = 2, n \geq 2$ ), one can obtain a similar result which says that the upper box dimension of  $D(F)$  is at least  $s/n$ . We note here that the  $s/n$  bound is often strict, see [6].

Following the approach in [7], we make the following definition.

DEFINITION 1.2. Let  $F \subset \mathbb{R}^n$  be a compact set and let  $\mu$  be a probability measure supported on  $F$ . For  $g \in \mathbb{O}(n)$ , the orthogonal group on  $\mathbb{R}^n$ , we construct a measure  $\nu_g$  as follows,

$$\int_{\mathbb{R}^n} f(z) d\nu_g(z) = \int_F \int_F f(u - gv) d\mu(u) d\mu(v), \forall f \in C_0(\mathbb{R}^n),$$

here  $C_0(\mathbb{R}^n)$  is the space of continuous functions with compact support on  $\mathbb{R}^n$ . In other words,  $\nu_g = \mu * g\mu$ . We also construct a measure  $\nu$  on  $\Delta_k(F) \subset \mathbb{R}^{k(k-1)/2}$  by

$$\int f(t) d\nu(t) = \int f(|x_1 - x_2|, \dots, |x_i - x_j|, \dots, |x_{k-1} - x_k|) d\mu(x_1) \dots d\mu(x_k),$$

$$\forall f \in C_0(\mathbb{R}^{k(k-1)/2}),$$

where  $t$  is a  $k(k - 1)/2$ -vector with entries  $|x_i - x_j|$  for  $1 \leq i < j \leq k$ .

In this way, we see that  $\nu$  is ‘the natural measure’ supported on  $\Delta_k(F)$ . In particular, we have  $\dim_H \nu \leq \dim_H \Delta_k(F)$ . We will introduce some notions of dimensions in the following section. Notice that our definitions are slightly different than those in [7]. Here, we use  $k$  to denote the number of vertices of the ‘simplex structures’ we want to count in  $F$  while in [7],  $k$  is the order of the simplices. For example, when  $k = 2$ , our definition gives distance sets while the definitions in [7] gives triangle sets.

In this paper, we prove the following result which extends [7, Theorem 1.3]. The  $L^2$  function part was essentially proved in [7] with an additional condition that  $\nu_g$  needs to be absolutely continuous with respect to the Lebesgue measure for almost all  $g \in \mathbb{O}(n)$ . In what follows, see Section 3.2 for the definition of Frostman’s measures.

THEOREM 1.3. Let  $\mu$  be a  $s$ -Frostman measure with compact support on  $\mathbb{R}^n$ . Let  $k \in \{2, \dots, n + 1\}$  be an integer and  $\nu_g, \nu$  be as in Definition 1.2. We write  $\hat{\nu}$  for the Fourier transform of  $\nu$ . Then for each  $\epsilon > 0$ , there are constants  $C, C_\epsilon > 0$  such that for all  $\delta > 0$  we have

$$\int_{B_{\delta^{-1}}(0)} |\hat{\nu}(\omega)|^2 d\omega \leq C \delta^{-n(k-1)} \int \int \nu_g^{k-1}(B_{2.5\delta}(z)) d\nu_g(z) dg$$

$$\leq C_\epsilon \max\{\delta^{-((n-s)(k-1)-\gamma_s+\epsilon)}, 1\}.$$

If  $-((n - s)(k - 1) - \gamma_s + \epsilon) > 0$ , then  $\nu$  can be viewed as an  $L^2$  function. Here,  $\gamma_s$  can be chosen as follows:

$$\gamma_s = \begin{cases} s & s \in (0, (n - 1)/2]; \\ (n - 1)/2 & s \in [(n - 1)/2, n/2]; \\ (n + 2s - 2)/4 & s \in [n/2, (n + 2)/2]; \\ s - 1 & s \in [(n + 2)/2, n). \end{cases}$$

The above result generalises [7, Theorem 1.3] in two ways. First, it provides us a good estimate of the growth of  $\|\nu_\delta\|_2^2$  with respect to  $\delta \rightarrow 0$ , which in turn allows us to estimate

the Hausdorff dimension of  $\Delta_k(F)$ . Here,  $\nu_\delta$  represents a  $\delta$ -scale smooth approximation of the measure  $\nu$ . More precisely, it is  $\nu * \phi_\delta$  where  $\psi_\delta = \delta^{-k(k-1)/2} \phi(\cdot/\delta)$  for a smooth cut-off function  $\phi$  on  $\mathbb{R}^{k(k-1)/2}$ . See Section 2 for more details. Second, we can drop the technical continuity condition mentioned above. In this way, the above theorem can be seen as an alternative approach to the dimension results of distance sets discussed in [13, Chapter 15]. We record the Hausdorff dimension estimate as a corollary.

COROLLARY 1.4. *Let  $F \subset \mathbb{R}^n$ ,  $n \geq 2$ ,  $n \in \mathbb{N}$  be a compact set with  $\dim_H F = s$ . Then for each  $k \in \{2, \dots, n + 1\}$ , we have*

$$\dim_H \Delta_k(F) \geq \min \left\{ \frac{k(k-1)}{2} - n(k-1) + s(k-1) + \gamma_s, \frac{k(k-1)}{2} \right\},$$

where  $\gamma_s$  is the same quantity as in the statement of Theorem 1.3.

We will prove the above result in Section 4.2. For example, when  $k = 3$ ,  $n = 2$  we have

$$\dim_H \Delta_3(F) \geq \begin{cases} 3s - 1 & s \in [1/3, 1/2]; \\ 2s - 0.5 & s \in [1/2, 1]; \\ \min\{2.5s - 1, 3\} & s \in [1, 2]. \end{cases}$$

If  $2.5s - 1 > 3$ , that is,  $s > 8/5$ , then  $\Delta_3(F)$  would have positive Lebesgue measure. This is a result proved in [7].

We have another consequence from Theorem 1.3. We can cover  $\mathbb{R}^n$  with closed  $\delta$ -cubes  $\mathcal{K}_\delta$  with disjoint interiors. For each  $K \in \mathcal{K}_\delta$ , we use  $2K$  to denote the  $2\delta$ -cube with the same centre as  $K$ . Observe that

$$\begin{aligned} \int v_g^{k-1}(B_\delta(z)) \nu_g(B_\delta(z)) dz &\leq \sum_{K \in \mathcal{K}_\delta} \int_K v_g^k(B_\delta(z)) dz \\ &\leq \sum_{K \in \mathcal{K}_\delta} \delta^{-n} \sup_{z^* \in K} v_g^k(B_\delta(z^*)) \\ &\leq \delta^{-n} \sum_{K \in \mathcal{K}_\delta} \int_{2K} v_g^{k-1}(B_{2\sqrt{n}\delta}(z)) d\nu_g(z). \end{aligned}$$

Since  $\{2K\}_{K \in \mathcal{K}_\delta}$  covers  $\mathbb{R}^n$  with maximal multiplicity  $2^{n+1}$ , we see that

$$\int v_g^{k-1}(B_\delta(z)) \nu_g(B_\delta(z)) dz \leq \delta^{-n} 2^{n+1} \int v_g^{k-1}(B_{2\sqrt{n}\delta}(z)) d\nu_g(z).$$

By Theorem 1.3 and the argument above, we see that if  $(n - s)(k - 1) - \gamma_s < 0$ ,

$$\delta^{-kn} \int \int v_g^k(B_\delta(z)) dz dg \lesssim 1.$$

From here, we deduce the following corollary.

COROLLARY 1.5. *Let  $\mu$  be a  $s$ -Frostman measure with compact support on  $\mathbb{R}^n$ . Let  $k \in \{2, \dots, n + 1\}$  be an integer and  $\nu_g, \nu$  be as in Definition 1.2. If  $-((n - s)(k - 1) - \gamma_s) > 0$ , then for almost all  $g \in \mathbb{O}(n)$ ,  $\nu_g$  is an  $L^k(\mathbb{R}^n)$  function. In particular, for such  $g \in \mathbb{O}(n)$ ,  $\nu_g$  is absolutely continuous with respect to the Lebesgue measure. Here  $\gamma_s$  can be chosen as in Theorem 1.3.*

If  $s = n$ , then  $\nu_g$  is an  $L^\infty$ -function for almost all  $g \in \mathbb{O}(n)$ . For  $k = 2$ , we see that the positivity criterion happens when

$$s > \frac{n}{2} + \frac{1}{3}. \tag{1.1}$$

This recovers a result stated at the end of [13, Section 15.5]. In Section 7, we give some sketched discussions in this situation. In [13, Section 7.3], it was asked whether the following conjecture is true.

**CONJECTURE 1.6.** Let  $\mu$  be a  $s$ -Frostman measure with compact support on  $\mathbb{R}^n$ . If  $s > n/2$  then for almost all  $g \in \mathbb{O}(n)$ ,  $\nu_g$  is absolutely continuous with respect to the Lebesgue measure.

Now there are better results than (1.1), see [9] and the references therein for more details.

**2. Notation.** 1. Let  $f$  be a function on  $\mathbb{R}^n$ , we write  $\hat{f}$  for its Fourier transform,

$$\hat{f}(\omega) = \int f(x)e^{-2\pi i(\omega,x)} dx,$$

where  $\omega \in \mathbb{R}^n$  and  $(\omega, x)$  is the Euclidean inner product between  $\omega$  and  $x$ . Let  $\mu$  be a probability measure on  $\mathbb{R}^n$  we also write  $\hat{\mu}$  for its Fourier transform,

$$\hat{\mu}(\omega) = \int e^{-2\pi i(\omega,x)} d\mu(x).$$

2. For each integer  $n \geq 1$ , we will often need to find a smooth cut-off function  $\phi_n$  on  $\mathbb{R}^n$ . More precisely, we define  $\phi$  to be 1 on the unit ball and 0 outside the ball of radius 2 centred at the origin. Then, we can smoothly construct this function  $\phi_n$ . Let  $\delta > 0$  we write  $\phi_{\delta,n}$  to be the function

$$x \in \mathbb{R}^n \rightarrow \delta^{-n} \phi_n(x\delta^{-1}).$$

Throughout this paper, when the ambient space of  $\psi_n$  is clear, we will just write it as  $\phi$ . Let  $f$  be a function on  $\mathbb{R}^n$  we write  $f_\delta = f * \phi_\delta$ . Similarly for a measure  $\mu$ , we write  $\mu_\delta = \mu * \phi_\delta$ .

3. It is convenient to introduce notions  $\approx, \lesssim, \gtrsim$  for approximately equal, approximately smaller and approximately larger. As our estimates always involve scales, we use  $1 > \delta > 0$  to denote a particular scale. Then for two quantities  $f(\delta), g(\delta)$ , we define the following:

$$f \lesssim g \iff \exists M > 0, \forall \delta > 0, f(\delta) \leq Mg(\delta).$$

$$f \gtrsim g \iff g \lesssim f.$$

$$f \approx g \iff f \lesssim g \text{ and } g \lesssim f.$$

We will use the same symbols for scales tending to  $\infty$  as well. More precisely, for  $R \in (0, \infty)$ , and quantities  $f(R), g(R)$ , we write

$$f(R) \lesssim g(R)$$

if there is a constant  $C > 0$  such that  $f(R) \leq Cg(R)$  for all  $R > 0$ . Similar meanings can be given to symbols  $\gtrsim$  and  $\approx$ .

**3. Preliminaries.**

**3.1. Hausdorff dimension for sets.** Let  $n \geq 1$  be an integer. Let  $F \subset \mathbb{R}^n$  be a Borel set. For any  $s \in \mathbb{R}^+$  and  $\delta > 0$ , define the following quantity

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{i=1}^\infty (\text{diam}(U_i))^s : \bigcup_i U_i \supset F, \forall i \geq 1, U_i \subset \mathbb{R}^n, \text{diam}(U_i) < \delta \right\}.$$

The  $s$ -Hausdorff measure of  $F$  is

$$\mathcal{H}^s(F) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(F).$$

The Hausdorff dimension of  $F$  is

$$\dim_H F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

More details about the Hausdorff dimension can be found in [5] and [12].

**3.2. Frostman's measure.** It is known (e.g., see [13, Theorem 2.7]) that if  $F$  is a Borel subset of  $\mathbb{R}^n$  with  $\dim_H F = s$ , then for any  $\epsilon > 0$  there is a measure  $\mu$  supported in  $F$  such that for all  $x \in F$  and  $r > 0$  we have  $\mu(B(x, r)) \leq r^{s-\epsilon}$ . Such a measure  $\mu$  is usually called a  $(s - \epsilon)$ -Frostman measure.

**3.3. Energy integrals and Hausdorff dimension for measures.** Let  $\mu \in \mathcal{P}(\mathbb{R}^n)$ , the space of Borel probability measures on  $\mathbb{R}^n$ . For each positive number  $t > 0$ , we define the  $t$ -energy of  $\mu$  to be

$$I_t(\mu) = \int \int \frac{d\mu(x)d\mu(y)}{|x - y|^t}.$$

Through Fourier transform, it can be shown that

$$I_t(\mu) = \gamma(n, s) \int |\hat{\mu}(\omega)|^2 |\omega|^{t-n} d\omega,$$

where  $\gamma(n, s) = \pi^{s-n/2} \Gamma((n-s)/2) / \Gamma(s/2)$  and when  $s \in (0, n)$  we have  $\gamma(n, s) \in (0, \infty)$ , see [13, Sections 3.4 and 3.5]. We define the Hausdorff dimension of  $\mu$  as follows:

$$\dim_H \mu = \sup\{t > 0 : I_t(\mu) < \infty\}.$$

Let  $F \subset \mathbb{R}^n$  be a Borel set, then we have

$$\dim_H F = \sup\{t > 0 : \exists \mu \in \mathcal{P}(F), I_t(\mu) < \infty\}.$$

This implies that if  $\mu \in \mathcal{P}(F)$ , we have  $\dim_H \mu \leq \dim_H F$ .

**3.4. Spherical averages and Wolff–Erdogan's estimate.** Let  $\mu \in \mathcal{P}(\mathbb{R}^n)$ . We define the following spherical average for  $\hat{\mu}$ ,

$$S(\mu, R) = \int_{S^{n-1}} |\hat{\mu}(R\sigma)|^2 d\sigma,$$

where  $d\sigma$  is the normalised Lebesgue measure on  $S^{n-1}$ . We have the following deep result on the decay rate of  $S(\mu, R)$  as  $R \rightarrow \infty$ , see [4, 17]. The following version is taken from [13, Theorem 15.7].

**THEOREM 3.1 (Wolff–Erdogan estimate).** *Let  $\mu \in \mathcal{P}(\mathbb{R}^n)$  with compact support, for each  $s \geq n/2$ ,  $\epsilon > 0$ , there is a positive constant  $C(n, s, \epsilon)$  and for all  $R > 0$ , we have*

$$S(\mu, R) \leq C(n, s, \epsilon)R^{\epsilon-\gamma_s}I_s(\mu).$$

Here  $\gamma_s$  can be chosen as follows:

$$\gamma_s = \begin{cases} s & s \in (0, (n-1)/2]; \\ (n-1)/2 & s \in [(n-1)/2, n/2]; \\ (n+2s-2)/4 & s \in [n/2, (n+2)/2]; \\ s-1 & s \in [(n+2)/2, n). \end{cases}$$

Thus, if  $\dim_{\text{H}} \mu > s$ , then we see that  $I_s(\mu) < \infty$  and  $S(\mu, R) \lesssim R^{-\gamma_s}$ .

**3.5. Orthogonal group, Haar measure.** For each integer  $n \geq 2$ , we denote  $\mathbb{O}(n)$  the orthogonal group of order  $n$  over  $\mathbb{R}$ . It can be represented by  $n \times n$  real matrices  $A$  with  $A^T A = I$ .  $\mathbb{O}(n)$  is a real compact Lie group of algebraic dimension  $n(n-1)/2$ . We associate  $\mathbb{O}(n)$  with the normalised Haar measure and we often write

$$\int dg$$

instead of

$$\int_{\mathbb{O}(n)} dg$$

for simplicity.

**3.6. Group-theoretic energy.** Let  $n \geq 2$  be an integer and  $k \in \{2, \dots, n+1\}$ . Let  $\mu \in \mathcal{P}(\mathbb{R}^n)$ . For each  $\delta > 0$ ,  $g \in \mathbb{O}(n)$ , we define  $k$ -group-theoretic energy for  $\mu$  at scale  $\delta$  with respect to  $g$  to be

$$E^k(\mu, g, \delta) = \mu^{2k} \{(x_1, \dots, x_k, y_1, \dots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leq \delta\}.$$

Often, we can write  $E(\mu, g, \delta)$  for  $E^k(\mu, g, \delta)$  as the dependence on  $k$  will be always assumed. If  $A \subset \mathbb{R}^n$  is a finite set and  $\mu$  is the normalised counting measure on  $A$ . Let  $k = 2$ ,  $\delta = 0$ , we see that

$$E(\mu, g, 0) = \mu^4 \{(x_1, x_2, y_1, y_2) \in A^4 : x_1 - x_2 = g(y_1 - y_2)\},$$

which counts the number of quadruples  $(x_1, x_2, y_1, y_2)$  of  $A$  such that  $x_1 - x_2 = g(y_1 - y_2)$ . This idea was introduced in [3] and it played a crucial role in Guth–Katz’s proof of Erdős’ distance problem, see [10] and [8, Section 9].

### 4. An $L^2$ approach to the Hausdorff dimension.

**4.1. Some general results.** In this section, we discuss a simple method for estimating the Hausdorff dimension of a compactly supported Borel probability measure  $\mu$  in  $\mathcal{P}(\mathbb{R}^n)$ . We denote its Fourier transform as  $\hat{\mu}$ . It is a continuous function as  $\mu$  is compactly supported. In general it is not  $L^2$ , for otherwise  $\mu$  is in fact an  $L^2$  function. To measure how

far away it is from being  $L^2$ , we take the following ball average, see also [13, Section 3.8],

$$A(\mu, R) = \int_{B_R(0)} |\hat{\mu}(\omega)|^2 d\omega.$$

If  $\lim_{R \rightarrow \infty} A(\mu, R) < \infty$ , then  $\mu$  can be viewed as an  $L^2$  function. In general, we expect that  $A(\mu, R)$  tends to  $\infty$  at a certain speed. If there is a constant  $C > 0$  and a number  $s > 0$  such that

$$A(\mu, R) \leq CR^s$$

for all  $R > 0$ , then we see that for  $t \in (0, n)$

$$I_t(\mu) = \int |\hat{\mu}(\omega)|^2 |\omega|^{t-n} d\omega = \int_{|\omega| \leq 1} |\hat{\mu}(\omega)|^2 |\omega|^{t-n} d\omega + \sum_{j \geq 0} \int_{2^j \leq |\omega| \leq 2^{j+1}} |\hat{\mu}(\omega)|^2 |\omega|^{t-n} d\omega.$$

Since  $\mu$  is a probability measure,  $\hat{\mu}$  is bounded on unit ball. Therefore, we see that

$$\int_{|\omega| \leq 1} |\hat{\mu}(\omega)|^2 |\omega|^{t-n} d\omega < \infty.$$

For each  $j \geq 0$ , we have

$$\int_{2^j \leq |\omega| \leq 2^{j+1}} |\hat{\mu}(\omega)|^2 |\omega|^{t-n} d\omega \leq A(\mu, 2^{j+1}) 2^{j(t-n)} \leq C 2^{(j+1)s} 2^{j(t-n)} = C 2^s 2^{j(s+t-n)}.$$

If  $s + t - n < 0$ , the sum with respect to  $j$  converges and we have

$$I_t(\mu) < \infty.$$

Therefore,  $\dim_H \mu \geq t$  whenever  $t < n - s$ . This implies that

$$\dim_H \mu \geq n - s. \tag{4.1}$$

In order to study this  $L^2$  phenomena more systematically, we introduce the following notion of dimension,

$$\dim_{L^2} \mu = n - \limsup_{R \rightarrow \infty} \frac{\log A(\mu, R)}{\log R}.$$

There are several other ways of doing this  $L^2$  approach. For example, we can define

$$A(\mu, R, h) = \int_{B(0,R)} |\hat{\mu}(\omega)|^2 h(\omega) d\omega$$

for a weight function  $h$  on  $\mathbb{R}^n$ . For example, if we choose  $h(\omega) = |\omega|^{-t}$  for a number  $t \geq 0$  we see that

$$\begin{aligned} A(\mu, R, h) &\leq A(\mu, 1, h) + \sum_{j \geq 0} \int_{|\omega| \in [2^j, 2^{j+1}]} |\hat{\mu}(\omega)|^2 |\omega|^{-t} d\omega \leq A(\mu, 1, h) \\ &+ \sum_{j \geq 0, 2^j \leq 2R} 2^{-jt} A(\mu, 2^{j+1}). \end{aligned}$$

Thus, if  $A(\mu, 2^j) \lesssim 2^{uj}$  then we see that

$$\sum_{j \geq 0, 2^j \leq 2R} 2^{-jt} A(\mu, 2^{j+1}) \lesssim \sum_{j \geq 0, 2^j \leq 2R} 2^{-jt} 2^{ju} \lesssim R^{u-t}$$

if  $u - t > 0$  or else the above sum is bounded uniformly for all  $R$ . In terms of the  $L^2$ -dimension, we see that if  $\dim_{L^2} \mu > n - t$  then

$$\sup_R A(\mu, R, h) < \infty,$$

otherwise

$$A(\mu, R, h) \lesssim R^{n - \dim_{L^2} \mu - t}.$$

In general,  $A(\mu, R, |\cdot|^{-t})$  could have a smaller growth exponent. It is interesting to find the infimum among all possible values  $s$  such that

$$A(\mu, R, |\cdot|^{-t}) \lesssim R^s$$

holds for all  $R > 0$ . More precisely, we consider the following quantity

$$\dim_{L^2, t} \mu = n - t - \limsup \frac{\log A(\mu, R, |\cdot|^{-1})}{\log R}.$$

For  $t \geq 0$ , we have

$$\dim_{L^2, t} \mu \leq \dim_{L^2} \mu. \tag{4.2}$$

In general, it is possible that the above inequality is strict.

By collecting the results (4.1) and (4.2), we have shown the following result.

**THEOREM 4.1.** *Let  $n \geq 1$  be an integer and  $\mu \in \mathcal{P}(\mathbb{R}^n)$  be a Borel probability measure. Then, we have*

$$\dim_{\text{H}} \mu \geq \dim_{L^2} \mu.$$

The function  $t \geq 0 \rightarrow \dim_{L^2, t} \mu$  is non-increasing and bounded from above by  $\dim_{L^2} \mu$ .

In most cases, it is difficult to estimate  $A(\mu, R)$  directly. A useful method is to consider the  $L^2$ -norm of  $\mu_\delta = \mu * \phi_\delta$ . Notice that  $\mu_\delta$  is a Schwartz function taking non-negative values. Since  $\hat{\mu}_\delta = \hat{\mu} \hat{\phi}_\delta$  and  $\hat{\phi}_\delta$  decays very fast outside the ball  $B_{\delta^{-1}}(0)$ , we see that

$$A(\mu, \delta^{-1}) \lesssim \|\mu_\delta\|_2^2 = \int \mu_\delta^2(x) dx,$$

where the implicit constant in  $\lesssim$  depends only on the choice of the cut-off function  $\phi$ .

**4.2. Wolff–Erdogan bound for finite points configurations: Proof of Corollary 1.4.**

Before we prove Theorem 1.3, let us see how to obtain a Hausdorff dimension estimate. Let  $F \subset \mathbb{R}^n$  and  $\dim_{\text{H}} F = s$ . Then, we can choose  $(s - \epsilon)$ -Frostman measure on  $F$  for each  $\epsilon > 0$ . Then by Theorem 1.3 together with the discussions above, we see that

$$\dim_{\text{H}} \Delta_k(F) \geq \frac{k(k-1)}{2} - (n-s)(k-1) + \gamma_s,$$



provided that the RHS is not greater than  $k(k - 1)/2$ , otherwise,  $\Delta_k(F)$  has positive Lebesgue measure. For  $k = 2$ , this result revisits the Wolff–Erdogan–Mattila's bound for the Hausdorff dimension of distance set.

**5. GILP's lemma and an energy integral estimate.** First, we introduce a lemma obtained in [7].

LEMMA 5.1. *Let  $n \geq 2$  and  $k \in \{2, \dots, n + 1\}$  be integers. Let  $\mu \in \mathcal{P}([0, 1]^n)$  and  $v_g, \nu$  as defined before. Then there is a constant  $C > 0$  and we have for all  $\delta > 0$*

$$\int v_\delta^2(z) dz \leq C\delta^{-n(k-1)} \int E(\mu, g, \delta) dg,$$

where  $v_\delta = \nu * \phi_\delta$  is the smoothed version of  $\nu$  with scale  $\delta > 0$  and  $E(\mu, g, \delta)$  is the group-theoretic energy of  $\mu$  with scale  $\delta > 0$ .

*Proof.* A proof can be found in [7, Section 2]. □

LEMMA 5.2. *Let  $n \geq 2$  and  $k \in \{2, \dots, n + 1\}$  be integers. Let  $\mu \in \mathcal{P}[0, 1]^n$  and  $v_g, \nu$  as defined before. Then for each  $\delta > 0, g \in \mathbb{O}(n)$ , we have*

$$E(\mu, g, \delta) \leq \int v_g^{k-1}(B_{2.5\delta}(z)) dv_g(z).$$

*Proof.* By putting in definitions, we see that the statement of this lemma is equivalent to

$$\begin{aligned} &\mu^{2k}\{(x_1, \dots, x_k, y_1, \dots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leq \delta, 1 \leq i < j \leq k\} \\ &\leq \int v_g^{k-1}(B_{2\delta}(z)) dv_g(z). \end{aligned}$$

To prove this, let  $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$  be fixed, consider the following section

$$\{(x_k, y_k) : |(x_i - gy_i) - (x_j - gy_j)| \leq \delta, 1 \leq i < j \leq k\}.$$

It is easy to see that the above section is contained in

$$E = \{(x_k, y_k) : |(x_k - gy_k) - (x_1 - gy_1)| \leq \delta\}.$$

We see that the  $\mu^{2k}$  measure is now bounded from above by

$$\begin{aligned} &\mu^{2(k-1)}\{(x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}) \in \mathbb{R}^{2(k-1)n} : |(x_i - gy_i) - (x_j - gy_j)| \leq \delta, 1 \leq i < j \\ &\leq k - 1\} \times \int 1_E(x_k, y_k) d\mu(x_k) d\mu(y_k). \end{aligned}$$

Observe that  $1_E(x_k, y_k) = f(x_k - gy_k)$  for  $f : z \in \mathbb{R}^n \rightarrow f(z) = 1_{\{|a - (x_1 - gy_1)| \leq \delta\}}(z)$ . By the definition of  $v_g$ , we see that

$$\int 1_E(x_k, y_k) d\mu(x_k) d\mu(y_k) \leq v_g(B_{2\delta}(x_1 - gy_1)).$$

If  $v_g$  does not give positive measure on any spheres, then we would get

$$\int 1_E(x_k, y_k) d\mu(x_k) d\mu(y_k) = v_g(B_\delta(x_1 - gy_1)).$$

However, we do not assume this continuity of  $v_g$  and we only have an upper bound. We can do the above step  $k - 1$  times and by Fubini's theorem we see that

$$\begin{aligned} &\mu^{2k}\{(x_1, \dots, x_k, y_1, \dots, y_k) \in \mathbb{R}^{2kn} : |(x_i - gy_i) - (x_j - gy_j)| \leq \delta, 1 \leq i < j \leq k\} \\ &\leq \int v_g^{k-1}(B_{2\delta}(x_1 - gy_1))d\mu(x_1)d\mu(y_1) \leq \int v_g^{k-1}(B_{2.5\delta}(z))dv_g(z). \end{aligned}$$

If  $v_g(B_{2\delta}(\cdot))$  would be continuous, then we would have

$$\int v_g^{k-1}(B_{2\delta}(x_1 - gy_1))d\mu(x_1)d\mu(y_1) = \int v_g^{k-1}(B_{2\delta}(z))dv_g(z).$$

In general, we choose a continuous function sandwiched by  $v_g(B_{2\delta}(\cdot))$  and  $v_g(B_{2.5\delta}(\cdot))$  (by taking convolution with a suitable smooth cut-off function), and then apply the definition of  $v_g$  to arrive at the above inequality. □

**6. The main result.** In this section, we give a detailed proof of Theorem 1.3. We note that in [7], a proof is given under the condition that  $v_g$  is absolutely continuous for almost all  $g \in \mathbb{O}(n)$ . In [18], a sketched proof is given for the case when  $k = 3$  and we note that the same strategy works for general cases  $k \geq 2$  as well and here we will provide more details.

*Proof of Theorem 1.3.* By Lemmas 5.1 and 5.2, we see that as  $\delta \rightarrow 0$ ,

$$\int v_\delta^2(z)dz \leq C\delta^{-n(k-1)} \int \int v_g^{k-1}(B_{2.5\delta}(z))dv_g dg,$$

where  $C > 0$  is a constant. The situation would be simple if  $v_g(B_{2.5\delta}(z))$  would be continuous with respect to  $z$ . However, we cannot assume this continuity condition. To deal with this issue, let  $\phi^{DD}(\cdot)$  be a radial Schwartz function such that  $\hat{\phi}^{DD}$  is real-valued, non-negative, vanishes outside the ball of radius  $0.5c'' > 0$  around the origin and is equal to a positive number  $c > 0$  on a ball of radius  $c' > 0$  around the origin. Now we take the square  $\phi^D = (\phi^{DD})^2$  and see that

$$\hat{\phi}^D = \hat{\phi}^{DD} * \hat{\phi}^{DD}.$$

We see that  $\hat{\phi}^D$  is radial, real-valued, non-negative, vanishes outside the ball of radius  $c''$  around the origin. Unlike  $\hat{\phi}^{DD}$ ,  $\hat{\phi}^D$  is no longer a constant function on any ball centred at the origin. By further rescaling if necessary, we may assume that  $\phi^D(x) \geq 1$  for  $x \in B_{2.5}(0)$ . This can be done because  $\phi^D$  is real-valued, Schwartz and  $\phi^D(0) > 0$ . Since  $\hat{\phi}^D$  is compactly supported, we can denote  $c''' = \|\hat{\phi}^D\|_\infty$ . Then we write  $h_{g,\delta} = v_g * \phi^D(\delta^{-1}\cdot)$ . We see that

$$v_g(B_{2.5\delta}(z)) = \int_{B_{2.5\delta}(z)} dv_g(x) \leq \int \phi^D((z-x)/\delta)dv_g(x) = h_{g,\delta}(z).$$

Now we write  $f_{g,\delta}(\cdot) = \delta^{-n}h_{g,\delta}(\cdot)$ , as a result we see that

$$\int v_\delta^2(z)dz \lesssim \int \int f_{g,\delta}^{k-1}(z)dv_g(z)dg.$$

Let  $\psi$  be a smooth cut-off function supported in  $\{\omega \in \mathbb{R}^n : |\omega| \in [0.5, 4]\}$  and identically equal to 1 in  $\{\omega \in \mathbb{R}^n : |\omega| \in [1, 2]\}$ . We can also require that  $\sum_{j \in \mathbb{Z}} \psi(2^{-j}\omega) = 1$

and this is the starting point of the Littlewood–Paley decomposition. Let  $f_{g,\delta,j}, \nu_{g,j}$  be the  $j$ -th Littlewood–Paley piece of  $f_{g,\delta}, \nu_g$ , respectively, namely,  $\hat{f}_{g,\delta,j}(\omega) = \hat{f}_{g,\delta}(\omega)\psi(2^{-j}\omega)$  and similarly for  $\nu_{g,j}$ . We need to bound  $\|f_{g,\delta,j}\|_\infty$  as well as  $\|\nu_{g,j}\|_\infty$ . The later can be bounded by  $C'2^{j(n-s)}$  for any  $s < \dim_H F$  with a constant  $C'$  depending on the function  $\psi$ . This was shown in [7, p. 805]. For the former, we will be interested in estimating  $\|f_{g,\delta,j}\|_\infty$  when  $2^j$  is not as large as  $\delta^{-1}$ . In this case, recall that  $f_{g,\delta} = \nu_g * \phi_\delta^D$  and in terms of Fourier transform we have

$$\hat{f}_{g,\delta,j} = \hat{\nu}_g \hat{\phi}_\delta^D \psi(2^{-j}\cdot)$$

Recall that  $\phi_\delta^D(\cdot) = \delta^{-n}\phi^D(\cdot/\delta)$ , therefore we have  $\hat{\phi}_\delta^D(\cdot) = \hat{\phi}^D(\delta\cdot)$ . Then we see that

$$\begin{aligned} \|f_{g,\delta,j}\|_\infty &\leq \|\hat{f}_{g,\delta,j}\|_1 \\ &\leq c''' \int |\hat{\nu}_g(\omega)\psi(2^{-j}\omega)|d\omega \\ &\leq c''' \int_{B_{2^{j+2}}(0)} |\hat{\mu}(\omega)\hat{\mu}(g\omega)|d\omega \\ &\leq c''' \sqrt{\int_{B_{2^{j+2}}(0)} |\hat{\mu}(\omega)|^2d\omega \int_{B_{2^{j+2}}(0)} |\hat{\mu}(g\omega)|^2d\omega}. \end{aligned}$$

By the discussion in [13, Section 3.8], we see that

$$\int_{B_{2^{j+1}}(0)} |\hat{\mu}(\omega)|^2d\omega \lesssim 2^{(j+2)(n-s)}.$$

The same estimate holds for  $\int_{B_{2^{j+2}}(0)} |\hat{\mu}(g\omega)|^2d\omega$  as well. Therefore, we see that

$$\|f_{g,\delta,j}\|_\infty \leq C'2^{j(n-s)}$$

where  $C' > 0$  is a constant which does not depend on  $g, j, \delta$ . Observe that if  $2^{j-1} > c''\delta^{-1}$ , then  $f_{g,\delta,j} = 0$  and this is the reason for considering  $2^j$  to be not much larger than  $\delta^{-1}$ . Thus, we have obtained a complete estimate for  $\|f_{g,\delta,j}\|_\infty$ .

In what follows, we want to estimate the following integral:

$$\int f_{g,\delta}^{k-1}(z)d\nu_g(z). \tag{6.1}$$

We want to apply the argument in [7, Section 3] and we provide details depending on whether  $k = 2$  or  $k \geq 3$ . We note here that the argument in [7, Section 3] works only for  $k \geq 3$  but we shall extend it to the case when  $k = 2$ .

**6.1. Case  $k = 2$ .** In this situation, the equation (6.1) can be written as

$$\int f_{g,\delta}d\nu_g.$$

We can apply [13, Formula (3.27)] and as a result we see that

$$\int f_{g,\delta}(z)d\nu_g(z) = \int \hat{\nu}_g(\omega)\overline{\hat{f}_{g,\delta}}(\omega)d\omega.$$

We note here that  $\widehat{f}_{g,\delta}(\omega) = \widehat{f}_{g,\delta}(-\omega)$ . Therefore, we see that

$$\int \int f_{g,\delta}(z) dv_g(z) dg = \sum_{j \in \mathbb{Z}} \int \int \widehat{f}_{g,\delta}(-\omega) \widehat{v}_g(\omega) \psi(2^{-j}\omega) d\omega dg.$$

Recall that  $f_{g,\delta} = v_g * \phi_\delta^D$ , we see that

$$\widehat{f}_{g,\delta} = \widehat{v}_g \widehat{\phi}_\delta^D.$$

Then since  $v_g = \mu * g\mu$ , we see that

$$\widehat{v}_g(\omega) = \widehat{\mu}(\omega) \widehat{\mu}(g\omega).$$

As a result, we see that

$$\int \int \widehat{f}_{g,\delta}(-\omega) \widehat{v}_g(\omega) \psi(2^{-j}\omega) d\omega dg = \int \int |\widehat{\mu}(\omega)|^2 |\widehat{\mu}(g\omega)|^2 \widehat{\phi}_\delta^D(\omega) \psi(2^{-j}\omega) d\omega dg.$$

Observe that  $\widehat{\phi}_\delta^D$  is a cut-off function at scale  $\delta^{-1}$ . More precisely, for  $|\omega| > c''\delta^{-1}$ , we have

$$\widehat{\phi}_\delta^D(\omega) = 0.$$

By integrating first with respect to  $dg$  and then  $d\omega$ , we see that

$$\begin{aligned} & \int \int |\widehat{\mu}(\omega)|^2 |\widehat{\mu}(g\omega)|^2 \widehat{\phi}_\delta^D(\omega) \psi(2^{-j}\omega) d\omega dg \\ &= C(n) \int \left( \int_{S^{n-1}} |\widehat{\mu}(t\sigma)|^2 d\sigma \right)^2 \psi(2^{-j}t) \widehat{\phi}_\delta^D(t) t^{n-1} dt, \end{aligned} \tag{6.2}$$

where  $d\sigma$  is the Lebesgue probability measure on  $S^{n-1}$ . We write  $\widehat{\phi}_\delta^D(t) = \widehat{\phi}_\delta^D(\omega)$  for  $|\omega| = t$  and similarly for  $\psi(2^{-j}t)$ . Since  $\psi$  and  $\phi_\delta^D$  are radial functions, the above step is well defined. The constant  $C(n)$  is a positive number which depends only on  $n$ .

We need to sum (6.2) over  $j \in \mathbb{Z}$ . Because of the cut-off property of  $\phi_\delta^D$ , we only need to consider the sum up to  $\sum_{j:2^j \leq 2c''\delta^{-1}}$ . More precisely, there is a positive constant  $C''' > 0$  and we have

$$\int \int f_{g,\delta}(z) dv_g(z) dg \leq C'' \sum_{j:2^j \leq 2c''\delta^{-1}} \int \left( \int_{S^{n-1}} |\widehat{\mu}(t\sigma)|^2 d\sigma \right)^2 \psi(2^{-j}t) t^{n-1} dt.$$

In fact when  $2^j > 2c''\delta^{-1}$ , then  $\psi(2^{-j}\omega) \widehat{\phi}_\delta^D(\omega) = 0$ . Therefore, we do not need to sum larger values of  $j$ . This is because we can choose a special cut-off function  $\phi^D$  whose Fourier transform is compactly supported. This makes  $\phi^D$  not compactly supported but we do not need this. We still need to sum negative values of  $j$  but as  $\mu$  is a probability measure we have

$$\sum_{j \leq 0} \int \left( \int_{S^{n-1}} |\widehat{\mu}(t\sigma)|^2 d\sigma \right)^2 \psi(2^{-j}t) t^{n-1} dt \leq C''' \int_{[0,2]} t^{n-1} dt < \infty,$$

for a positive constant  $C''' > 0$ . We can use Theorem 3.1 (Wolff–Erdogan). For all  $\epsilon > 0$ , there is a constant  $C_\epsilon > 0$  such that for each  $t > 0$  we have

$$\int_{S^{n-1}} |\hat{\mu}(t\sigma)|^2 d\sigma \leq C_\epsilon t^{-\gamma_s + \epsilon}.$$

We can insert one of the factors  $\int |\hat{\mu}(t\sigma)|^2 d\sigma$  into (6.2) and we see that for each  $\epsilon > 0$ ,

$$\int \left( \int_{S^{n-1}} |\hat{\mu}(t\sigma)|^2 d\sigma \right)^2 \psi(2^{-j}t)t^{n-1} dt \lesssim \int_{2^{j-1}}^{2^{j+2}} \left( \int |\hat{\mu}(t\sigma)|^2 d\sigma \right) t^{n-1} t^{-\gamma_s + \epsilon} dt.$$

Then we see that

$$\sum_{j:1 \leq 2^j \leq 2c''\delta^{-1}} \int \left( \int_{S^{n-1}} |\hat{\mu}(t\sigma)|^2 d\sigma \right)^2 \psi(2^{-j}t)t^{n-1} dt \lesssim \int_0^{4c''\delta^{-1}} \left( \int |\hat{\mu}(t\sigma)|^2 d\sigma \right) t^{n-1} t^{-\gamma_s + \epsilon} dt.$$

Up to a multiple constant, the RHS above is equal to

$$\int_{|\omega| \leq 4c''\delta^{-1}} |\hat{\mu}(\omega)|^2 |\omega|^{-\gamma_s + \epsilon} d\omega.$$

If  $I_{n-\gamma_s-\epsilon}(\mu) < \infty$ , then the above integral is bounded uniformly for  $\delta \rightarrow 0$ , in this case  $D(F)$  would have positive Lebesgue measure. Therefore, we consider the case when  $n - \gamma_s - s + \epsilon > 0$ . By the discussions in Section 4, we see that

$$\int_{|\omega| \leq 4c''\delta^{-1}} |\hat{\mu}(\omega)|^2 |\omega|^{-\gamma_s + \epsilon} d\omega \lesssim \delta^{-\dim_{L^2, \gamma_s + \epsilon} \mu} \leq \delta^{-(n-\gamma_s-s+\epsilon)}.$$

For the rightmost inequality, we need the fact that  $\mu$  is an  $s$ -Frostman measure. Thus, we showed that

$$\int v_\delta^2(z) dz \lesssim \int \int f_{g,\delta}(z) dv_g(z) dg \lesssim \delta^{-(n-\gamma_s-s+\epsilon)}.$$

This concludes the case when  $k = 2$ .

**6.2. Case  $k \geq 3$ .** We need to estimate the following integral

$$\int f_{g,\delta}^{k-1}(z) dv_g(z).$$

We see that

$$\begin{aligned} \int f_{g,\delta}^{k-1}(z) dv_g(z) &\stackrel{[13, \text{Formula (3.27)}]}{=} \int \hat{v}_g(\omega) \overline{\hat{f}_{g,\delta}^{k-1}(\omega)} d\omega \\ &= \int (\hat{f}_{g,\delta} \overset{(k-1)\text{-times}}{*} \hat{f}_{g,\delta})(-\omega) \hat{v}_g(\omega) d\omega \end{aligned}$$

We write the Littlewood–Paley decompositions  $\hat{v}_g = \sum_{j \in \mathbb{Z}} \hat{v}_{g,j}$  and  $\hat{f}_{g,\delta} = \sum_{j \in \mathbb{Z}} \hat{f}_{g,\delta,j}$ . Then, we see that

$$\int (\hat{f}_{g,\delta} \overset{(k-1)\text{-times}}{*} \hat{f}_{g,\delta})(-\omega) \hat{v}_g(\omega) d\omega = \sum_{j_1, j_2, \dots, j_k} \int (\hat{f}_{g,\delta,j_1} \overset{(k-1)\text{-times}}{*} \hat{f}_{g,\delta,j_{k-1}})(-\omega) \hat{v}_{g,j_k}(\omega) d\omega.$$

Denote  $j^* = \max\{j_1, \dots, j_{k-1}\}$ . We see that  $\hat{f}_{g,\delta,j_1} \ast \dots \ast \hat{f}_{g,\delta,j_{k-1}}$  is supported on an annulus. We can estimate the location of this annulus. First, each term of form  $\hat{f}_{g,\delta,j}$  is supported on an annulus with inner radius  $2^{j-1}$  and outer radius  $2^{j+2}$ . Thus,  $\hat{f}_{g,\delta,j_1} \ast \dots \ast \hat{f}_{g,\delta,j_{k-1}}$  is supported on an annulus with inner radius at least  $2^{j^*-1}$  and outer radius at most  $(k-1)2^{j^*+2}$ . Thus, if either  $2^{j_1+2} < 2^{j^*-1}$  or  $2^{j_1-1} > (k-1)2^{j^*+2}$  we see that

$$\int \hat{f}_{g,\delta,j_1}(z) \dots \hat{f}_{g,\delta,j_{k-1}}(z) v_{g,j_k}(z) dz = \int (\hat{f}_{g,\delta,j_1} \ast \dots \ast \hat{f}_{g,\delta,j_{k-1}})(-\omega) \hat{v}_{g,j_k}(\omega) d\omega = 0.$$

For this reason, we only need to sum the terms indexed by  $j_1, \dots, j_k$  with  $|j^* - j_k| \leq C(k)$  for a constant  $C(k)$  depending only on  $k$ . Let  $j$  be any integer and we sum all the terms  $j_1, \dots, j_k$  with  $|j_1 - j| \leq C(k)/2$  and  $|j^* - j| \leq C(k)/2$ . The resulting sum is bounded from above by a constant (depending on  $k$ ) times the following expression

$$2^{j(n-s)(k-2)} \int \left| \sum_{|q| \leq C(k)/2} \hat{f}_{g,\delta,j+q}(z) \right| \left| \sum_{|q| \leq C(k)/2} v_{g,j+q}(z) \right| dz. \tag{6.3}$$

We need to sum the above expression for  $j \in \mathbb{Z}$ . If  $2^{j-C(k)/2} \leq 2c''\delta^{-1}$ , then (6.3) is bounded from above by

$$c'''' 2^{j(n-s)(k-2)} \int \left| \sum_{|q| \leq C(k)/2} v_{g,j+q}(z) \right|^2 dz.$$

Here we used Cauchy–Schwarz inequality, Plancherel’s theorem as well as the fact that  $\|\hat{\phi}^D\|_\infty = c''''$ . If  $2^{j-C(k)/2} > 2c''\delta^{-1}$ , then (6.3) is equal to 0. In all, the sum for  $j \in \mathbb{Z}$  of (6.3) can be bounded from above by

$$c'''' \sum_{2^j \leq 2c''\delta^{-1}} 2^{j(n-s)(k-2)} \int \left| \sum_{|q| \leq C(k)/2} v_{g,j+q}(z) \right|^2 dz.$$

It is easy to check that the sum with  $j \leq 0$  gives another constant  $C(k, s, v)$  depending on  $k, s$  and  $v$ . Then, we can summarise our results so far in the following inequality,

$$\int f_{g,\delta}^{k-1}(z) dv_g(z) \lesssim \sum_{1 \leq 2^j \leq 2c''\delta^{-1}} 2^{j(n-s)(k-2)} \int |\tilde{v}_{g,j}(z)|^2 dz + C(k, s, v),$$

where we have written  $\tilde{v}_{g,j} = \sum_q v_{g,j+q}$  for simplicity. The functions  $v_{g,j}$  are real-valued for all  $j \in \mathbb{Z}$  because  $v_g$  is a real-valued measure and  $\psi$  is a radial function. Then, we see that

$$\int |\tilde{v}_{g,j}(z)|^2 dz = \int |\hat{v}_{g,j}(\omega)|^2 d\omega.$$

Recall that  $v_g = \mu \ast g\mu$ , we see that

$$\int |\hat{v}_{g,j}(\omega)|^2 d\omega \leq \int_{|\omega| \in [2^{j-C(k)/2-1}, 2^{j+C(k)/2+2}]} |\hat{\mu}(\omega)|^2 |\hat{\mu}(g\omega)|^2 d\omega.$$

The integral against  $dg$  of the RHS above is a constant multiple of

$$\int_{2^{j-C(k)/2-1}}^{2^{j+C(k)/2+2}} \left( \int |\hat{\mu}(t\sigma)|^2 d\sigma \right)^2 t^{n-1} dt \lesssim \int_{2^{j-C(k)/2-1}}^{2^{j+C(k)/2+2}} \left( \int |\hat{\mu}(t\sigma)|^2 d\sigma \right) t^{n-1} t^{-\gamma_s+\epsilon} dt,$$

where the above inequality holds for all  $\epsilon > 0$ . As in the case when  $k = 2$ , we see that,

$$\int \int f_{g,\delta}^{k-1}(z)dv_g(z)dg \lesssim C(k, s, \nu) + \sum_{j:1 \leq 2^j \leq 2c''\delta^{-1}} 2^{-j(\gamma_s - \epsilon)} 2^{j(n-s)} 2^{j(n-s)(k-2)}.$$

If  $(n - s)(k - 1) - \gamma_s + \epsilon < 0$ , then  $\nu$  would be an  $L^2$  function, otherwise we see that

$$\int v_\delta^2(z)dz \lesssim \delta^{-((n-s)(k-1) - \gamma_s + \epsilon)}.$$

This concludes the proof for the case when  $k \geq 3$ . □

**7. Asymmetric distance sets.** Let  $n \geq 2$  be an integer. Let  $F_1, F_2$  are compact sets in  $\mathbb{R}^n$  with  $\dim_H F_1 = s_1, \dim_H F_2 = s_2$ . Let  $\mu_1, \mu_2$  be probability measures supported on  $F_1, F_2$ , respectively. For  $g \in \mathbb{O}(n)$ , the orthogonal group on  $\mathbb{R}^n$ , we construct a measure  $\nu_g$  as follows:

$$\int_{\mathbb{R}^n} f(z)dv_g(z) = \int_{F_1} \int_{F_2} f(u - gv)d\mu_1(u)d\mu_2(v), f \in C_0(\mathbb{R}^n).$$

In other words,  $\nu_g = \mu_1 * g\mu_2$ . We also construct a measure  $\nu$  by

$$\int f(t)d\nu(t) = \int f(|x_1 - x_2|)d\mu_1(x_1)d\mu_2(x_2).$$

It can be seen that  $\nu$  is supported on

$$D(F_1, F_2) = \{|x_1 - x_2| : x_1 \in F_1, x_2 \in F_2\}.$$

Most of the argument in previous sections can be used here. In particular, one can show that for each  $\epsilon > 0$

$$\|\nu_\delta\|_2^2 \lesssim \delta^{-(n-\gamma_{s_1} - s_2 - \epsilon)}$$

and

$$\|\nu_\delta\|_2^2 \lesssim \delta^{-(n-\gamma_{s_2} - s_1 - \epsilon)}.$$

Therefore, we see that if  $\max\{\gamma_{s_1} + s_2, \gamma_{s_2} + s_1\} > n$ , then  $D(F_1, F_2)$  has positive Lebesgue measure. If  $s_2 \geq s_1$ , then this is equivalent to  $s_2 + 0.5s_1 > 0.75n + 0.5$ . Now we turn to Corollary 1.5. With the same arguments as above, we see that if  $s_2 + 0.5s_1 > 0.75n + 0.5$ , then for almost all  $g \in \mathbb{O}(n)$ ,  $\nu_g$  is absolutely continuous with respect to the Lebesgue measure. In general, we can consider  $k \geq 3$  and obtain conditions for  $\nu_g$  to be  $L^k$  for almost all  $g \in \mathbb{O}(n)$ .

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REFERENCES

1. X. Du, L. Guth, Y. Ou, H. Wang, B. Wilson and R. Zhang, Weighted restriction estimates and application to Falconer distance set problem, preprint, [arXiv:1802.10186](https://arxiv.org/abs/1802.10186), to appear in Amer. J. Math. (2018).

2. X. Du and R. Zhang, Sharp  $L^2$  estimates of the Schrödinger maximal function in higher dimensions, *Ann. Math.(2)* **189** (2019), 837–861.
3. G. Elekes and M. Sharir, Incidences in three dimensions and distinct distances in the plane, in *Proceedings 26th ACM Symposium on Computational Geometry* (2010), 413–422.
4. B. Erdogan, A bilinear Fourier extension problem and applications to the distance set problem, *Int. Math. Res. Not.* **23** (2005), 1411–1425.
5. K. Falconer, *Fractal geometry: Mathematical foundations and applications* (Wiley, Hoboken, New Jersey, 2004).
6. J. Fraser, D. Howroyd and H. Yu, Dimension growth for iterated sumsets, *Math. Z.* **293** (2019), 1015–1042.
7. A. Greenleaf, A. Iosevich, B. Liu and E. Palsson, A group-theoretic viewpoint on Erdős-Falconer problems and the Mattila integral, *Revista Mat. Iberoamer* **31**(3) (2015), 799–810.
8. L. Guth, *Polynomial methods in combinatorics* (American Mathematical Society, Providence, Rhode Island, 2016).
9. L. Guth, A. Iosevich, Y-M. Ou and H. Wang, On Falconer’s distance set problem in the plane, *Invent. Math.* **219** (2020), 779–830.
10. L. Guth and N. Katz, On the Erdős distances problem in the plane, *Ann. Math.(2)* **181** (2015), 155–190.
11. T. Keleti and P. Shmerkin, New bounds on the dimensions of planar distance sets, *Geom. Funct. Anal.* **29** (2020), 1886–1948.
12. P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, UK, 1999).
13. P. Mattila, *Fourier analysis and Hausdorff dimension*, Cambridge Studies in Advanced Mathematics (Cambridge University Press, Cambridge, UK, 2015).
14. T. Orponen, On the distance sets of Ahlfors-David regular sets, *Adv. Math.* **307** (2017), 1029–1045.
15. P. Shmerkin, On the Hausdorff dimension of pinned distance sets, *Israel J. Math.* **230** (2019), 949–972.
16. T. Wolff, *Recent work connected with the Kakeya problem*, Prospects in Mathematics (American Mathematical Society, Providence, 1996).
17. T. Wolff, Decay of circular means of Fourier transforms of measures, *Int. Math. Res. Not.* **10** (1996), 547–567.
18. H. Yu, Dimensions of triangle sets, *Mathematika* **65**(2) (2019), 311–332.