ON TERNARY QUADRATIC FORMS THAT REPRESENT ZERO

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Serre [**6**] has recently created a theory of some generality in response to a query from Manin about the size of the number $N(x)$ of (indefinite) ternary quadratic forms $AX^2+BY^2+CZ^2$ that represent zero and have coefficients of magnitudes not exceeding x. In particular, as one of several examples used to illustrate his methods, he shewed that

$$N(x)=O\left(\frac{x^3}{\log^{\frac{3}{2}} x}\right)$$

but then indirectly asked if the corresponding lower bound

$$N(x)>\frac{A_1x^3}{\log^{\frac{3}{2}} x}$$

were also true for sufficiently large values of x and some positive absolute constant A_1. It is this lower bound that it is the purpose of the present communication to establish so that in fact

$$N(x)\asymp\frac{x^3}{\log^{\frac{3}{2}} x}$$

in the sense that the true order of magnitude of $N(x)$ is $x^3/\log^{\frac{3}{2}} x$.

The main constituents in our method are Legendre's celebrated criterion for the non-trivial solubility of $AX^2+BY^2+CZ^2=0$ in integers [**5**], Burgess's estimate for character sums containing the Jacobi symbol [**1**], and a variant of the author's individual lower bound technique described in [**3**] and in Chapter 6 of his tract [**4**]. Since it is a lower bound that is sought, it suffices to obtain it for the contribution $N'(x)$ to $N(x)$ due to a set of quadratic forms to which Legendre's criterion is immediately applicable. A suitable set of such forms being those with *square-free* coefficients $A=a>0$, $B=b>0$, $C=-c<0$ that conform to simultaneous conditions $\Theta(a,b,c)$ defined by

$$(b,c)=(c,a)=(a,b)=1;\qquad (a,2)=1;\qquad b\equiv c\equiv 1, \bmod 4, \tag{1}$$

the necessary and sufficient condition that $aX^2+bY^2-cZ^2=0$ be a zero form is that bc, ca, $-ab$ be quadratic residues, moduli a, b, c, respectively. Hence an indefinite ternary form answering to (1) contributes to $N'(x)$ if and only if the function $\tau(a,b,c)$ defined through the Jacobi symbol as

$$\prod_{p\mid a}\left\{1+\left(\frac{bc}{p}\right)\right\}\prod_{p\mid b}\left\{1+\left(\frac{ca}{p}\right)\right\}\prod_{p\mid c}\left\{1+\left(\frac{-ab}{p}\right)\right\}=\sum_{d\mid a}\left(\frac{bc}{d}\right)\sum_{e\mid b}\left(\frac{ca}{e}\right)\sum_{f\mid c}\left(\frac{-ab}{f}\right) \tag{2}$$

be positive. However

$$\sum_{\substack{a,b,c\le x\\ \Theta(a,b,c)}}\tau(a,b,c) \tag{3}$$

is itself merely a sum in which each contributing element of $N'(x)$ is counted with a weight $2^{\omega(abc)}$ derived from the number $\omega(abc)$ of distinct prime factors of abc, whereas the corresponding exact expression

$$\sum_{\substack{a,b,c\le x\\ \Theta(a,b,c)}} \frac{\tau(a,b,c)}{2^{\omega(abc)}} \tag{4}$$

for $N'(x)$ is awkward to estimate asymptotically because of the presence of the reciprocal of $2^{\omega(abc)}$. We therefore have recourse to the already mentioned method of **[4]**, by which a satisfactory lower bound for (4) can be found provided that a sufficiently good asymptotic formula with remainder term can be elicited for the generalizations

$$N'(x,\Delta) = \sum_{\substack{a,b,c\le x\\ \Theta(a,b,c)\\ abc\equiv 0,\ \mathrm{mod}\ \Delta}} \tau(a,b,c) \tag{5}$$

of the unsatisfactory (3) when Δ is an odd square-free number not exceeding a small fixed power ξ of x.

Proceeding therefore to the treatment of $N'(x,\Delta)$, we first remark that each triplet a, b, c permitted by the conditions of summation therein satisfies congruences of the type

$$a\equiv 0,\ \mathrm{mod}\ \alpha,\qquad b\equiv 0,\ \mathrm{mod}\ \beta,\qquad c\equiv 0,\ \mathrm{mod}\ \gamma, \tag{6}$$

for one and only one representation $\alpha\beta\gamma$ of Δ as a product of three factors. Therefore

$$\begin{aligned} N'(x,\Delta) &= \sum_{\alpha\beta\gamma=\Delta} \sum_{\substack{a,b,c\le x\\ \Theta(a,b,c)\\ a\equiv 0,\ \mathrm{mod}\ \alpha;\ b\equiv 0,\ \mathrm{mod}\ \beta;\ c\equiv 0,\ \mathrm{mod}\ \gamma}} \tau(a,b,c)\\ &= \sum_{\alpha\beta\gamma=\Delta} N'(x;\alpha,\beta,\gamma),\ \text{say}, \end{aligned} \tag{7}$$

where in future it may be assumed that α, β, γ are odd square-free numbers satisfying

$$(\beta,\gamma)=(\gamma,\alpha)=(\alpha,\beta)=1,\qquad \alpha,\beta,\gamma\le x^{\frac{1}{15}} \tag{8}$$

because the inequality

$$\Delta\le\xi=x^{\frac{1}{15}} \tag{9}$$

constitutes a suitable constraint for Δ in the current situation.

Before we embark on the detailed analysis that (7) initiates, it is opportune to emphasize that we shall continue to let the letters a, b, c (with or without distinguishing marks) denote positive square-free numbers and that hence certain other symbols arising therefrom such as r, s, t, d, e, f (again with or without distinguishing marks) may also be regarded as being subject to the same restriction; we shall also reserve the letter l to denote a positive integer. Moreover, in the next phase of development immediately following, we shall find it notationally advantageous to submit all summations to (6) and the relevant conditions in (1) without explicitly saying so; in particular, such properties as $(\alpha,\beta)=1$ are automatically given. With this convention understood from this explanation

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and a perusal of how it is applied, we deduce from (7) and (2) that

$$\begin{aligned} N'(x;\alpha,\beta,\gamma) &= \sum_{b,c\le x}\ \sum_{e|b;f|c}\ \sum_{a\le x}\left(\frac{ac}{e}\right)\left(\frac{-ab}{f}\right)\sum_{d|a}\left(\frac{bc}{d}\right) \\ &= \sum_{b,c\le x}\ \sum_{e|b;f|c}\left(\frac{c}{e}\right)\left(\frac{-b}{f}\right)\sum_{\substack{a\le x\\ d|a}}\left(\frac{a}{ef}\right)\left(\frac{bc}{d}\right) \\ &= \sum_{b,c\le x}\ \sum_{e|b;f|c}\left(\frac{c}{e}\right)\left(\frac{-b}{f}\right)Q(b,c,e,f),\ \text{say}, \end{aligned} \tag{10}$$

where

$$Q(b,c,e,f) = \sum_{\substack{a\le x\\ d|a}}\left(\frac{a}{ef}\right)\left(\frac{d}{bc}\right) \tag{11}$$

by the law of quadratic reciprocity and the latent condition $b\equiv c\equiv 1$, mod 4. Hence, writing $a=dr$, $b=es$, $c=ft$ in view of the divisibility conditions, we end the first phase of the estimation of $N(x;\alpha,\beta,\gamma)$ by reformulating (11) as

$$Q(b,c,e,f) = \sum_{dr\le x}\left(\frac{dr}{ef}\right)\left(\frac{d}{efst}\right) = \sum_{dr\le x}\left(\frac{r}{ef}\right)\left(\frac{d}{st}\right), \tag{12}$$

where dr is to satisfy the conditions laid down for a.

The second phase of the estimation requires us to restore the usual procedure of explicitly stating conditions of summation and begins by our distinguishing between the case where ef and st are both greater than 1 and that where they are not. In the former, so long as we confirm our previous agreement that r, d, with or without a subscript, denote square-free numbers, the conditions of summation in (12) are tantamount to

$$dr\equiv 0,\ \text{mod}\ \alpha,\ (d,2rbc)=(r,2dbc)=1, \qquad dr\le x, \tag{13}$$

in which the first constituent implies that

$$d=d_1\alpha/(\alpha,r) \quad \text{and} \quad r=r_1\alpha/(\alpha,d) \tag{14}$$

for certain integers d_1, r_1. Hence, since the last constituent means that either $d\le\eta$ or $r\le x/\eta$ for any suitable value of η to be chosen later in terms of e,f,s,t, equation (12) can be transformed into

$$Q(b,c,e,f) = \sum_{d\le\eta}\ \sum_{r\le x/d} + \sum_{r\le x/\eta}\ \sum_{\eta<d\le x/r} = \sum\nolimits_1 + \sum\nolimits_2,\ \text{say}, \tag{15}$$

whose right-hand side is the sum of two similar terms that are estimated in a common manner. Indeed, because of (13), (14) and the implicit understanding that $(\alpha,bc)=1$, we have reduced the assessment to the treatment of the inner sums in the inequalities

$$\left|\sum\nolimits_1\right| \le \sum_{d\le\eta}\left|\sum_{\substack{r_1\le x(\alpha,d)/\alpha d\\ (r_1,2dbc\alpha)=1}}\left(\frac{r_1}{ef}\right)\right|, \tag{16}$$

$$\left|\sum\nolimits_2\right| \le \sum_{r\le x/\eta}\left|\sum_{\substack{\eta(\alpha,r)/\alpha r<d_1\le x(\alpha,r)/\alpha r\\ (d_1,2rbc\alpha)=1}}\left(\frac{d_1}{st}\right)\right| \tag{17}$$

by means of a corollary of Burgess's theorem that is stated in the following lemma.

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LEMMA 1. *Let k be a given odd square-free modulus exceeding* 1 *and P a given positive integer. Then, for any integer ρ exceeding* 2, *both of which are less than u^{A_2} for some positive absolute constant A_2,*

$$\sum_{\substack{l \leq y \\ (l,P)=1;\, \mu(l)\neq 0}} \left(\frac{l}{k}\right) = O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon}\right),$$

where the constant implied by the O-notation depends at most on ρ, A_2, and ϵ. The same estimate is still available when the summand in the given sum is affected by the non-principal character $\chi(l)$, mod 4, *as a multiplier.*

Burgess's results [**1**] certainly assert that the bound provided by the lemma is valid for the sum derived from the originally given sum by deleting the subsidiary conditions $(l, P)=1$ and $\mu(l)\neq 0$. Hence, first investigating the effect of including the former condition, we find

$$\begin{aligned}
\sum_{\substack{l \leq y \\ (l,P)=1}} \left(\frac{l}{k}\right) &= \sum_{l \leq y} \left(\frac{l}{k}\right) \sum_{m \mid l;\, m \mid P} \mu(m) = \sum_{m \mid P} \mu(m) \sum_{\substack{l \leq y \\ l \equiv 0, \bmod m}} \left(\frac{l}{k}\right) \\
&= \sum_{m \mid P} \mu(m) \left(\frac{m}{k}\right) \sum_{l' \leq y/m} \left(\frac{l'}{k}\right) \\
&= O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon} \sum_{m \mid P} \frac{1}{m^{1-\frac{1}{\rho}}}\right) \\
&= O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon} P^{\epsilon}\right) = O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon}\right).
\end{aligned}$$

From this, including the other condition $\mu(l)\neq 0$, we obtain

$$\begin{aligned}
\sum_{\substack{l \leq y \\ (l,P)=1;\, \mu(l)\neq 0}} \left(\frac{l}{k}\right) &= \sum_{\substack{l \leq y \\ (l,P)=1}} \left(\frac{l}{k}\right) \sum_{m^2 \mid l} \mu(m) = \sum_{\substack{m \leq y^{\frac{1}{2}} \\ (m,P)=1}} \mu(m) \sum_{\substack{l \leq y;\, (l,P)=1 \\ l \equiv 0, \bmod m^2}} \left(\frac{l}{k}\right) \\
&= \sum_{\substack{m \leq y^{\frac{1}{2}} \\ (m,P)=1}} \mu(m) \left(\frac{m^2}{k}\right) \sum_{\substack{l' \leq y/m^2 \\ (l',P)=1}} \left(\frac{l'}{k}\right) \\
&= O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon} \sum_{m \leq y^{\frac{1}{2}}} \frac{1}{m^{2(1-\frac{1}{\rho})}}\right) \\
&= O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon} \sum_{m=1}^{\infty} \frac{1}{m^{\frac{4}{3}}}\right) = O\left(y^{1-\frac{1}{\rho}} k^{\frac{\rho+1}{4\rho^2}} u^{\epsilon}\right),
\end{aligned}$$

as asserted†, the second part of the lemma being similarly established because $\chi(l)(l \mid k)$ is a non-principal character, mod $4k$.

Applying to (16) the result of the lemma for $\rho = 3$, we get

$$\sum\nolimits_1 = O\left(\frac{x^{\frac{2}{3}+\epsilon}(ef)^{\frac{1}{9}}}{\alpha^{\frac{2}{3}}} \sum_{d \leq \eta} \frac{(\alpha, d)^{\frac{2}{3}}}{d^{\frac{2}{3}}}\right) = O\left(\frac{x^{\frac{2}{3}+\epsilon}(ef)^{\frac{1}{9}} \eta^{\frac{1}{3}} \sigma_{-\frac{1}{3}}(\alpha)}{\alpha^{\frac{2}{3}}}\right) = O\left(\frac{x^{\frac{2}{3}+\epsilon}(ef)^{\frac{1}{9}} \eta^{\frac{1}{3}}}{\alpha^{\frac{2}{3}}}\right),$$

† We should note it is not so satisfactory to add the side conditions in the opposite order; also that a slight change in the proof would enable us to include the case $\rho = 2$, which we shall not find it convenient to use here.

while similarly

$$\sum_2 = O\left(\frac{x^{\frac{2}{3}+\epsilon}(st)^{\frac{1}{9}}}{\alpha^{\frac{2}{3}}} \sum_{r \le x/\eta} \frac{(\alpha, r)^{\frac{2}{3}}}{r^{\frac{2}{3}}}\right) = O\left(\frac{x^{1+\epsilon}(st)^{\frac{1}{9}}}{\alpha^{\frac{2}{3}}\eta^{\frac{1}{3}}}\right)$$

from (17).

The quantities within the O-symbols in these estimates are equal when

$$\eta = \left(\frac{st}{ef}\right)^{\frac{1}{6}} x^{\frac{1}{2}},$$

and is it therefore inferred from (15) that

$$Q(b, c, e, f) = O\left(\frac{x^{\frac{5}{6}+\epsilon}(efst)^{\frac{1}{18}}}{\alpha^{\frac{2}{3}}}\right) = O\left(\frac{x^{\frac{5}{6}+\epsilon}(bc)^{\frac{1}{18}}}{\alpha^{\frac{2}{3}}}\right)$$

when neither ef nor st equals 1. The contribution of the case in point to $N'(x; \alpha, \beta, \gamma)$ thus amounts in all to only

$$O\left(\frac{x^{\frac{5}{6}+\epsilon}}{\alpha^{\frac{2}{3}}} \sum_{\substack{b,c \le x \\ b \equiv 0, \bmod \beta;\ c \equiv 0, \bmod \gamma}} d(b)d(c)(bc)^{\frac{1}{18}}\right) = O\left(\frac{x^{\frac{17}{18}+\epsilon}}{\alpha^{\frac{2}{3}}} \sum_{\substack{b,c \le x \\ b \equiv 0, \bmod \beta;\ c \equiv 0, \bmod \gamma}} 1\right)$$

$$= O\left(\frac{x^{\frac{53}{18}+\epsilon}}{\alpha^{\frac{2}{3}}\beta\gamma}\right) = O\left(\frac{x^{\frac{107}{36}}}{\alpha\beta\gamma}\right) \tag{18}$$

by (10) and (8).

To appraise the effect of the respective cases $ef = 1$ and $st = 1$, we note the corresponding values of e, f, s, t are then $1, 1, b, c$ and $b, c, 1, 1$ so that in either instance

$$Q(b, c, e, f) = \sum_{rd \le x} \left(\frac{r}{bc}\right)$$

by (12) and symmetry provided that rd satisfy the apposite conditions laid down for a in (1) and (6). Also the coefficient of $Q(b, c, e, f)$ in (10) is 1 because it is obviously so in the former instance and is

$$\left(\frac{c}{b}\right)\left(\frac{-b}{c}\right) = \left(\frac{-1}{c}\right)\left(\frac{c}{b}\right)\left(\frac{b}{c}\right) = 1$$

in the latter by (1) and the law of quadratic reciprocity. Hence, since the two cases only concur when $b = c = 1$, the remaining part of $N'(x; \alpha, \beta, \gamma)$ is

$$2 \sum_{\substack{r \le x;\ d \le x/r \\ rd \equiv 0, \bmod \alpha}} \sum_{\substack{b \le x \\ b \equiv 0, \bmod \beta \\ b \equiv 1, \bmod 4}} \left(\frac{r}{b}\right) \sum_{\substack{c \le x \\ c \equiv 0, \bmod \gamma \\ c \equiv 1, \bmod 4}} \left(\frac{r}{c}\right) + O\left(\frac{x^{1+\epsilon}}{\alpha}\right), \tag{19}$$

wherein the variables of summation are understood to be subject to the additional restraint

$$(b, c) = (b, rd) = (c, rd) = (r, d) = (bcrd, 2) = 1. \tag{20}$$

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We then consider that part of the first term in (19) that arises from values of r other than 1. In these circumstances and when $(rdb, \gamma)=1$, the innermost sum in this term is

$$\sum_{\substack{c\le x;\,(c,2db)=1\\ c\equiv 0,\ \mathrm{mod}\ \gamma\\ c\equiv 1,\ \mathrm{mod}\ 4}}\left(\frac{c}{r}\right)=\left(\frac{\gamma}{r}\right)\sum_{\substack{c'\le x/\gamma;\,(c',2db\gamma)=1\\ c'\equiv(-1|\gamma),\ \mathrm{mod}\ 4}}\left(\frac{c'}{r}\right)$$

$$=\frac{1}{2}\left(\frac{\gamma}{r}\right)\sum_{\substack{c'\le x/\gamma\\ (c',2db\gamma)=1}}\left\{\left(\frac{c'}{r}\right)+\left(\frac{-1}{\gamma}\right)\left(\frac{c'}{r}\right)\left(\frac{-1}{c'}\right)\right\}$$

$$=O\left\{\left(\frac{x}{\gamma}\right)^{\frac{2}{3}+\epsilon}r^{\frac{1}{9}}\right\}=O\left(\frac{x^{\frac{7}{9}+\epsilon}}{\gamma^{\frac{2}{3}}}\right)$$

by both parts of Lemma 1, whence the quantity in question is

$$O\left(\frac{x^{\frac{7}{9}+\epsilon}}{\gamma^{\frac{2}{3}}}\sum_{\substack{a\le x\\ a\equiv 0,\ \mathrm{mod}\ \alpha}}d(a)\sum_{\substack{b\le x\\ b\equiv 0,\ \mathrm{mod}\ \beta}}1\right)=O\left(\frac{x^{\frac{25}{9}+\epsilon}}{\alpha\beta\gamma^{\frac{2}{3}}}\right)=O\left(\frac{x^{\frac{107}{36}}}{\alpha\beta\gamma}\right) \tag{21}$$

in virtue of (8).

The main element in the eventual formula for $N'(x;\alpha,\beta,\gamma)$ emerges from the remaining part of the first term in (19), the treatment of which begins with the following counterpart of Lemma 1 for principal characters.

LEMMA 2. *For $j=\pm 1$ and any odd integer M, we have*

$$\sum_{\substack{l\le y;\,(l,M)=1\\ l\equiv j,\ \mathrm{mod}\ 4}}\mu^2(l)=\frac{2}{\pi^2}\psi(M)y+O(y^{\frac{1}{2}}d(M)),$$

where

$$\psi(M)=\prod_{p|M}\left(1+\frac{1}{p}\right)^{-1}.$$

The proof is suppressed because it involves the same order of development as in Lemma 1.

From this, we derive the following result.

LEMMA 3. *With the notation of Lemma 2, we have*

$$\sum_{\substack{l\le y;\,(l,M)=1\\ l\equiv 0,\ \mathrm{mod}\ k\\ l\equiv j,\ \mathrm{mod}\ 4}}\mu^2(l)=\frac{2\psi(Mk)y}{\pi^2 k}+O\left(\frac{y^{\frac{1}{2}}d(Mk)}{k^{\frac{1}{2}}}\right)$$

and

$$\sum_{\substack{l\le y;\,(l,2M)=1\\ l\equiv 0,\ \mathrm{mod}\ k}}\mu^2(l)=\frac{4\psi(Mk)y}{\pi^2 k}+O\left(\frac{y^{\frac{1}{2}}d(Mk)}{k^{\frac{1}{2}}}\right)$$

if k be an odd square-free integer relatively prime to M.

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The first part is a consequence of the previous lemma because the left-side of the proposed equation equals

$$\sum_{\substack{l' \le y/k;\, (l', Mk)=1 \\ l' \equiv j(-1|k),\ \mathrm{mod}\ 4}} \mu^2(l'),$$

the second part then following by adding the formulae for $j=1$ and $j=-1$.

As another preliminary to the estimation of the principal part of (19) corresponding to the restriction of r to the value 1, it is appropriate to replace the symbol d by a in order to achieve notational balance and to reflect the genesis of r and d from the coefficient originally denoted by a. The sum S to be assessed thus assumes the form

$$2\sum_{a\le x}\sum_{b\le x}\sum_{c\le x} 1$$

in which $\Theta(a,b,c)$ and (6) are implicit in the conditions of summation, and this through the transformation $a=\alpha a'$, $b=\beta b'$, $c=\gamma c'$ becomes

$$2\sum_{\substack{a'\le x/\alpha \\ (a',2\alpha\beta\gamma)=1}} \sum_{\substack{b'\le x/\beta \\ (b',\alpha\beta\gamma)=1 \\ b'\equiv(-1|\beta),\ \mathrm{mod}\ 4}} \sum_{\substack{c'\le x/\gamma \\ (c',\alpha\beta\gamma)=1 \\ c'\equiv(-1|\gamma),\ \mathrm{mod}\ 4}} 1$$

so long as the new variables of summation be subject to the supplementary constraint $(b',c')=(c',a')=(a',b')=1$. Eliminating the need for this qualification about the summatory conditions by one of the Möbius formulae, we then obtain the formula

$$\begin{aligned} S &= 2\sum_{\substack{a'\le x/\alpha \\ (a',2\alpha\beta\gamma)=1}} \sum_{\substack{b'\le c/\beta \\ (b',\alpha\beta\gamma)=1 \\ b'\equiv(-1|\beta),\ \mathrm{mod}\ 4}} \sum_{\substack{c'\le x/\gamma \\ (c',\alpha\beta\gamma)=1 \\ c'\equiv(-1|\gamma),\ \mathrm{mod}\ 4}} \\ &\qquad \sum_{d'|b';\, d'|c'} \mu(d') \sum_{e'|c';\, e'|a'} \mu(e') \sum_{f'|a';\, f'|b'} \mu(f') \\ &= 2\sum_{\substack{d',e'f' \\ (d'e'f',2\alpha\beta\gamma)=1}} \mu(d')\mu(e')\mu(f') \\ &\qquad \sum_{\substack{a'\le x/\alpha;\,(a',2\alpha\beta\gamma)=1 \\ a'\equiv 0,\ \mathrm{mod}[e',f']}} 1 \sum_{\substack{b'\le x/\beta;\,(b',\alpha\beta\gamma)=1 \\ b'\equiv 0,\ \mathrm{mod}[f',d'] \\ b'\equiv(-1|\beta),\ \mathrm{mod}\ 4}} 1 \sum_{\substack{c'\le x/\gamma;\,(c',\alpha\beta\gamma)=1 \\ c'\equiv 0,\ \mathrm{mod}[d',e'] \\ c'\equiv(-1|\gamma),\ \mathrm{mod}\ 4}} 1 \end{aligned}$$

containing inner sums that can be evaluated by Lemma 3. Hence, since the product of these sums is certainly zero when at least one of the inequalities $[e',f']>x/\alpha$; $[f',d']>x/\beta$; $[d',e']>x/\gamma$ holds, we infer that

$$\begin{aligned} S &= \frac{32x^3\psi^3(\alpha\beta\gamma)}{\pi^6\alpha\beta\gamma} \sum_{\substack{\alpha[e',f'],\beta[f',d'],\gamma[d',e']\le x \\ (d'e'f',2\alpha\beta\gamma)=1}} \frac{\mu(d')\mu(e')\mu(f')\psi([e',f'])\psi([f',d'])\psi([d',e'])}{[e',f'][f',d'][d',e']} \\ &\quad + O\left(x^{\frac{5}{2}+\epsilon} \sum_{d',e'f'\le x} \frac{1}{[e',f']^{\frac{1}{2}}[f',d'][d',e']}\right) \qquad (22) \end{aligned}$$

by an argument that uses symmetry and a comparison between the sizes of the main terms and remainder terms in the formulae of Lemma 3. The sums in this are treated through the principles underlying the following extension of Euler's theorem to multiplicative functions of three variables.

LEMMA 4. *Let $f(\lambda, \mu, \nu)$ be a multiplicative function, namely, a function with the property that*

$$f(\lambda_1\lambda_2, \mu_1\mu_2, \nu_1\nu_2) = f(\lambda_1, \mu_1, \nu_1)f(\lambda_2, \mu_2, \nu_2)$$

whenever $(\lambda_1\mu_1\nu_1, \lambda_2\mu_2\nu_2) = 1$. Then

$$\sum_{\lambda,\mu,\nu=1}^{\infty} f(\lambda, \mu, \nu) = \prod_p \sum_{\rho,\sigma,\tau=0}^{\infty} f(p^\rho, p^\sigma, p^\tau)$$

if the infinite product

$$\prod_p \sum_{\rho,\sigma,\tau=0}^{\infty} |f(p^\rho, p^\sigma, p^\tau)|$$

be convergent.

The removal of the three inequalities from the conditions of summation in the first sum in (22) entails the introduction of a compensating term that is

$$O\Bigg(\sum_{[e'f']>x/\max(\alpha,\beta,\gamma)} \frac{\psi[e',f'])\psi([f',d'])\psi([d',e'])}{[e',f'][f',d'][d',e']}\Bigg)$$
$$= O\Bigg(\frac{\{\max(\alpha,\beta,\gamma)\}^{\frac{1}{2}}}{x^{\frac{1}{2}}} \sum_{d',e',f'} \frac{1}{[e',f']^{\frac{1}{2}}[f',d'][d',e']}\Bigg)$$

so that

$$S = \frac{32x^3\psi^3(\alpha\beta\gamma)}{\pi^6\alpha\beta\gamma} \sum_{\substack{d',e',f' \\ (d'e'f',2\alpha\beta\gamma)=1}} \frac{\mu(d')\mu(e')\mu(f')\psi([e',f'])\psi([f',d'])\psi([d',e'])}{[e',f'][f',d'][d',e']}$$
$$+ O\Bigg(x^{\frac{5}{2}+\epsilon} \sum_{d',e',f'} \frac{1}{[e',f']^{\frac{1}{2}}[f',d'][d',e']}\Bigg) = \frac{32x^3\psi^3(\alpha\beta\gamma)}{\pi^6\alpha\beta\gamma}\sum\nolimits_3 + O\big(x^{\frac{5}{2}+\epsilon}\sum\nolimits_4\big), \text{ say,} \tag{23}$$

provided that the series $\sum_4$ be convergent. But this condition is easily seen to be ensured by Lemma 4, which also gives

$$\sum\nolimits_3 = \prod_{p \nmid 2\alpha\beta\gamma} \left\{1 - \frac{3}{p^2}\left(1+\frac{1}{p}\right)^{-2} + \frac{2}{p^3}\left(1+\frac{1}{p}\right)^{-3}\right\}$$
$$= \prod_{p \nmid 2\alpha\beta\gamma} \left(1+\frac{1}{p}\right)^{-3}\left(1+\frac{3}{p}\right) = A_3 \prod_{p|\alpha\beta\gamma}\left(1+\frac{1}{p}\right)^{3}\left(1+\frac{3}{p}\right)^{-1},$$

where A_3 is the first of a series of further positive constants A_i to be now introduced. Hence, setting

$$\theta(M) = \prod_{p|M}\left(1+\frac{3}{p}\right)^{-1} \tag{24}$$

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and referring to (23), we first get

$$S=\frac{A_4\theta(\alpha\beta\gamma)x^3}{\alpha\beta\gamma}+O(x^{\frac{5}{2}+\epsilon})$$

and then, by (18), (19), and (21), the estimate

$$N'(x;\alpha,\beta,\gamma)=\frac{A_4\theta(\alpha\beta\gamma)x^3}{\alpha\beta\gamma}+O\left(\frac{x^{\frac{107}{36}}}{\alpha\beta\gamma}\right)$$

when (8) holds. From this and (7) then flows the formula

$$N'(x,\Delta)=\frac{A_4\theta(\Delta)3^{\omega(\Delta)}x^3}{\Delta}+O\left(\frac{3^{\omega(\Delta)}x^{\frac{107}{36}}}{\Delta}\right) \tag{25}$$

that was the first goal we were set.

The ground has been fully prepared for the application of the lower bound method, the principles of which are outlined with the aid of the description in Chapter 6 of our tract [**4**]. Wishing to formulate a suitable lower bound for the function $2^{-\omega(n)}$ when $n\le x^3$, let $\omega_v(n)$ be the number of distinct prime factors of n not exceeding a number $v=\xi^h$ that is a power to a suitably small fixed positive exponent h of the number ξ in (9). Then, since n cannot possess more than $3\log x/\log v$ distinct prime factors exceeding v so that

$$2^{-\omega(n)}>A_5 2^{-\omega_v(n)}, \tag{26}$$

it suffices to concentrate attention on $2^{-\omega_v(n)}$ by using the method of [**4**] to bound the function

$$f(n)=\prod_{p\mid n}f(p)\le 2^{-\omega_v(n)} \tag{27}$$

defined here as the first of the two multiplicative functions given by

$$f(p)=\begin{cases}1-\dfrac{1}{2}\left(1+\dfrac{3}{p}\right)<\dfrac{1}{2}, & \text{if}\quad p\le v,\\ 1, & \text{otherwise},\end{cases}$$

and

$$f_1(p)=1-f(p)=\frac{1}{2\theta(p)}>\frac{1}{2} \tag{28}$$

for $p\le v$. Therefore, letting d_1 now denote generally a square-free number (possibly 1) composed entirely of prime factors not exceeding v, we introduce a function $\rho(d_1)$ that induces a lower bound sieve in the sense that it meets Selberg's characteristic requirement that

$$\sum_{d_1\mid n}\rho(d_1)\le\sum_{d_1\mid n}\mu(d_1),$$

which implies that

$$f(n)\ge\sum_{d_1\mid n}\rho(d_1)f_1(d_1), \tag{29}$$

by the reasoning on p. 101 of **[4]**. Also, by the standard theory of sieves (see, for example, **[2]**), the function $\rho(d_1)$ can be assumed to have the properties

$$\rho(d_1)=\begin{cases} O(1), & \text{if } d_1 \le \xi, \\ 0, & \text{if } d_1 > \xi, \end{cases} \tag{30}$$

and

$$\sum_{(d_1,2)=1}\left(\frac{3}{2}\right)^{\omega(d_1)}\frac{\rho(d_1)}{d_1} > \frac{A_6}{\log^{\frac{3}{2}} x} \tag{31}$$

if, with a sufficiently small value of h, it be chosen appropriately to act as the agent of a sieve of dimension $\frac{3}{2}$ related to primes between 3 and v.

All the necessary elements are available for the estimation of $N(x)$. First, by (4), (26), (27), (29), (5), and (25) taken in turn,

$$\begin{aligned} N(x) \ge N'(x) &\ge A_5 \sum_{\substack{a,b,c\le x \\ \Theta(a,b,c)}} \tau(a,b,c) \sum_{d_1|abc} \rho(d_1) f_1(d_1) \\ &= A_5 \sum_{(d_1,2)=1} \rho(d_1) f_1(d_1) N'(x,d_1) \\ &= A_7 \sum_{(d_1,2)=1} \rho(d_1) f_1(d_1) \left\{ \frac{\theta(d_1) 3^{\omega(d_1)} x^3}{d_1} + O\left(\frac{3^{\omega(d_1)} x^{\frac{107}{36}}}{d_1}\right)\right\}. \end{aligned}$$

Hence, now using (28), (30), and (31), we conclude that

$$\begin{aligned} N(x) &\ge A_7 x^3 \sum_{(d_1,2)=1}\left(\frac{3}{2}\right)^{\omega(d_1)}\frac{\rho(d_1)}{d_1} + O\left(x^{\frac{107}{36}} \sum_{d_1\le\xi} \frac{3^{\omega(d_1)}}{d_1}\right) \\ &\ge \frac{A_8 x^3}{\log^{\frac{3}{2}} x} + O(x^{\frac{107}{36}} \log^3 x) > \frac{A_9 x^3}{\log^{\frac{3}{2}} x}. \end{aligned}$$

The result announced at the beginning of the article has therefore been substantiated in the form of the following theorem.

THEOREM. *Let $N(x)$ be the number of (indefinite) ternary quadratic forms $AX^2 + BY^2 + CZ^2$ representing zero whose coefficients are in magnitude not more than x. Then*

$$N(x) > \frac{Dx^3}{\sqrt{\log^3 x}}$$

for some positive absolute constant D and x sufficiently large.

REFERENCES

1. D. A. Burgess, On character sums and L-series. II, *Proc. London Math. Soc.* (3) **13** (1963), 524–536.

2. H. Halberstam and H. E. Richert, *Sieve methods* (Academic Press, 1974).

3. C. Hooley, On the intervals between numbers that are sums of two squares. III, *J. Reine Angew. Math.* **267** (1974), 207–218.

4. C. Hooley, *Applications of sieve methods to the theory of numbers* (Cambridge University Press, 1976).

5. A.-M. Legendre, *Théorie des Nombres*, 3rd ed. (Firmin–Didot, 1830), I, 33–39.

6. J.-P. Serre, Spécialisation des éléments de $\mathrm{Br}_2(\mathbf{Q}(T_1, \ldots, T_n))$, *C.R. Acad. Sci. Paris Sér. I Math.* **311** (1990), 397–402.

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