# SUFFICIENT CONDITIONS FOR MATCHINGS

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## 1. Introduction

A graph $G$ is said to possess a *perfect matching* if there is a subgraph of $G$ consisting of disjoint edges which together cover all the vertices of $G$. Clearly $G$ must then have an even number of vertices. A necessary and sufficient condition for $G$ to possess a perfect matching was obtained by Tutte (**3**). If $S$ is any set of vertices of $G$, let $p(S)$ denote the number of components of the graph $G-S$ with an odd number of vertices. Then the condition

$$\text{for all } S,\ p(S) \leqq |S|$$

is both necessary and sufficient for the existence of a perfect matching. A simple proof of this result is given in (**1**).

We consider certain conditions which are sufficient although not necessary. Roughly speaking, $G$ will have a perfect matching if there are enough edges. For example, if $|V(G)| = n$, $n$ even, where $V(G)$ denotes the set of vertices of $G$, and if each vertex is of degree $\geqq \frac{1}{2}n$, i.e. if each vertex has at least $\frac{1}{2}n$ edges incident with it, then it is almost trivial (see § **3**) to show that $G$ has a perfect matching. Instead of looking at each vertex separately, we can put a condition on the vertices collectively. If $X$ denotes any subset of $V(G)$, let

$$\Gamma(X) = \{y \in V(G) \colon y \text{ is joined by an edge to at least one vertex in } X\}.$$

Following Woodall (**4**), we define

$$\text{melt}\,(G) = \max\,\{c \colon \forall X \subset V(G),\ |\Gamma(X)| \geqq \min\,(c\,|X|,\ |V(G)|)\}.$$

Thus melt $(G)$ is the largest number $c$ such that any $k$ vertices of $G$ are collectively adjacent to at least min $(ck, n)$ vertices. We have already (**1**) shown that, if $n$ is even,

$$\text{melt}\,(G) \geqq \tfrac{4}{3} \Rightarrow G \text{ has a perfect matching.} \tag{1}$$

We note that this condition implies that each vertex is of degree $\geqq \frac{1}{4}n$. Indeed, we have in general

**Lemma.** *If* melt $(G) \geqq c > 1$, *then each vertex of $G$ has degree* $\geqq \dfrac{c-1}{c}n$ *where* $n = |V(G)|$.

**Proof.** Suppose there is a vertex $v$ of degree $\leqq \dfrac{c-1}{c}n$. Then there are

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$\geqq \frac{n}{c}$ vertices none of which is joined by an edge to $v$. But these vertices must be joined to at least $c \cdot \frac{n}{c} = n$ vertices, a contradiction.

In the next section, we combine the two types of condition above to prove

**Theorem 1.** *Let $G$ have $n$ vertices, $n$ even. Let $c$ be any fixed number, $\frac{1}{4} \leqq c \leqq \frac{1}{2}$, and suppose that*

(i) *each vertex is of degree $\geqq cn$,*

(ii) melt $(G) \geqq \dfrac{3-4c}{2-2c}$.

*Then $G$ possesses a perfect matching.*

*Note* 1. $c = \frac{1}{2}$ gives the trivial result mentioned above, and $c = \frac{1}{4}$ gives the result (1).

*Note* 2. The theorem is also true for other values of $c$, but if $c > \frac{1}{2}$ condition (i) by itself is sufficient, whereas if $c < \frac{1}{4}$ then condition (ii) by itself suffices.

*Note* 3. Condition (ii) implies, by the lemma, that each vertex has degree $\geqq \dfrac{1-2c}{3-4c}\, n$, but this is less than $cn$ if $c > \frac{1}{4}$.

*Note* 4. The result is best possible. If $A$, $B$ are graphs let $A+B$ denote the graph obtained by joining every vertex of $A$ to every vertex of $B$. Take $A = aK_3 \cup bK_1$ and $B = (a+b-2)K_1$ where $K_n$ denotes the complete graph on $n$ vertices.

Following the suggestion of the referee, who is to be thanked for his careful consideration of the original version of this paper, we shall deduce Theorem 1 from the following stronger theorem which is proved along the same lines but more simply.

**Theorem 2.** *Let $G$ have $n$ vertices, $n$ even, and suppose that*

$$|\Gamma(X)| \geqq \min\left(2\,|X| - \frac{n}{2},\, n\right)$$

*for all sets $X$ of vertices of $G$. Then* either *$G$ has a perfect matching* or *there exist subsets $X$, $Y$ of $V(G)$, $X \nsubseteq Y$, such that*

$$|X| = \tfrac{1}{4}(3n-6),\ |Y| = \tfrac{1}{4}(3n-2),\ |\Gamma(X)| = 2\,|X| - \frac{n}{2},\ |\Gamma(Y)| = 2\,|Y| - \frac{n}{2}.$$

An example of a graph in which the second possibility occurs is $G = 3K_3 + K_1$. Theorem 2 is proved in the next section, but we now show that

*Theorem* 2 *implies Theorem* 1. We assume Theorem 2 and the hypotheses of Theorem 1. Let $W$ be any set of vertices of $G$. If $|W|>(1-c)n$, then, since the degree of each vertex of $G$ is $\geqq cn$, we cannot have a vertex of $G$ joined to no vertex of $W$. Thus $|\Gamma(W)| = V(G)$. So suppose $|W| \leqq (1-c)n$. Then

$$|\Gamma(W)| \geqq \frac{3-4c}{2-2c}|W| \geqq 2|W| - \frac{n}{2}. \tag{2}$$

It follows from Theorem 2 that $G$ possesses a perfect matching unless there exist two sets $X$, $Y$ as in Theorem 2. Then, by (2),

$$\frac{3-4c}{2-2c}|W| = 2|W| - \frac{n}{2}$$

for $W = X$ and for $W = Y$, giving $|X| = |Y|$, a contradiction.

Theorem 2 is proved in the next section. In the remainder of this paper we shall generalize in one theorem both Theorem 1 and a result of Woodall (**4**) concerned with the maximum number of disjoint edges in a graph with no perfect matching. Woodall's argument was based on that of (**1**), and now we in turn extend his result.

## 2. Proof of Theorem 2

We suppose there is no perfect matching of $G$. Then by Tutte's theorem there is a set $S$ of vertices of $G$ for which $p(S)>|S|$. Using the fact that $p(S) \equiv |S| \pmod 2$, we must then have

$$p(S) \geqq |S|+2.$$

*Case* 1. Suppose that $|S| \geqq \frac{1}{4}(n-6)$. Let $m$ denote the number of 1-components of $G-S$ (i.e. the number of components with just one vertex). Then

$$\begin{aligned} n &\geqq |S|+m+3(p(S)-m) \qquad (3)\\ &\geqq 4|S|+6-2m, \end{aligned}$$

whence

$$n-m \leqq \tfrac{3}{2}n-2|S|-3. \tag{4}$$

But, if $m>0$,

$$\begin{aligned} n-m &\geqq |\Gamma(G-S)| \geqq 2|G-S| - \frac{n}{2}\\ &= 2n-2|S|-\frac{n}{2}, \end{aligned}$$

whence

$$n-m \geqq \tfrac{3}{2}n-2|S|. \tag{5}$$

Since (4) and (5) contradict one another, we must have $m = 0$. Thus, from (3),

$$n \geqq 4\,|\,S\,|+6$$

i.e.

$$|\,S\,| \leqq \tfrac{1}{4}(n-6),$$

whence

$$|\,S\,| = \tfrac{1}{4}(n-6).$$

Equality here implies that each component of $G-S$ must have exactly 3 vertices. If we let $X$ denote the set of vertices in all but one of these components we then have $|\,X\,| = \frac{1}{4}(3n-6)$ and $|\,\Gamma(X)\,| \leqq |\,X\,|+|\,S\,| = n-3 = 2\,|\,X\,|-\frac{n}{2}$. Similarly, if $Y$ denotes the same set with one more vertex of $G-S$ added, then we also have $|\,Y\,| = \frac{1}{4}(3n-2)$ and $|\,\Gamma(Y)\,| \leqq |\,Y\,|+|\,S\,| = n-1 = 2\,|\,Y\,|-\frac{n}{2}$.

*Case* 2. Suppose now that $|\,S\,| < \frac{1}{4}(n-6)$. Let $h$ denote the number of vertices in all but the smallest component of $G-S$. Since there are $\geqq |\,S\,|+2$ components of $G-S$, containing between them $n-|\,S\,|$ vertices, we must have

$$h \geqq \frac{|\,S\,|+1}{|\,S\,|+2}(n-|\,S\,|). \tag{6}$$

These $h$ vertices can be adjacent to at most $h+|\,S\,|<n$ vertices; on the other hand, they are by hypothesis joined to at least $2h-\frac{n}{2}$ vertices. Thus

$$h \leqq |\,S\,|+\frac{n}{2}. \tag{7}$$

From (6) and (7), eliminating $h$, we obtain

$$|\,S\,| \geqq \tfrac{1}{4}(n-6),$$

a contradiction.

**3. Extension to imperfect matchings**

A related question is the following. Given a condition on a graph $G$ which does not imply that $G$ possesses a perfect matching, can we estimate how many disjoint edges can be found in $G$? Corresponding to the two types of condition already studied, we have the following results for a graph with $n$ vertices.

1. If each vertex is of degree $\geqq cn$, $0 \leqq c \leqq \frac{1}{2}$, then we can find at least $[cn]$ disjoint edges.

2. If melt $(G) \geqq c$, then there are at least

$$\frac{c}{c+1}\,n \text{ disjoint edges if } 0<c \leqq \tfrac{1}{2} \tag{8}$$

$$\left[\frac{3c-2}{3c}\,n\right] \text{disjoint edges if } 1<c \leqq \tfrac{4}{3}. \tag{9}$$

Result 2 is due to Woodall (4), with (1) as the special case $c = \frac{4}{3}$. Result 1 is almost trivial (although best possible—consider a bipartite graph). For suppose that each vertex is of degree $\geqq k$, and that $h<k$ disjoint edges have so far been found. If no two remaining vertices are joined by an edge, select any two of them, say $v_1$ and $v_2$. Then it is easy to see that there must be a pair $v_3$, $v_4$ of vertices, joined by one of the edges already chosen, such that $v_1$ is joined to $v_3$ and $v_2$ to $v_4$. With this new pairing we now have $h+1$ disjoint edges, and the process can be repeated if $h+1<k$. We now state

**Theorem 3.** *Let G be a graph with n vertices. Suppose that*

(i) *each vertex is of degree* $\geqq dn$,

(ii) $\text{melt}\,(G) \geqq \dfrac{3-4d-3f}{2-2d}$,

*where* $4d+3f \geqq 1$, $2d+3f \leqq 1$, $d \geqq 0$, $f \geqq 0$. *Then G possesses at least* $\left[\dfrac{n}{2}(1-f)\right]$ *disjoint edges.*

The special case $f = 0$ is Theorem 1, and the case $f = \frac{1}{3}(1-4d)$ is Woodall's result (9). The referee has suggested that it may be possible to deduce this result from an analogue to Theorem 2 in the same way as Theorem 1 was deduced from Theorem 2. However, we preserve here our original proof. Instead of Tutte's condition we use Berge's extension ((2); see also (4) for a simpler proof): for $G$ to possess at least $t$ disjoint edges, it is necessary and sufficient that $p(S)-|S| \leqq n-2t$ for all sets $S$ of vertices of $G$. We shall in fact prove that, for all $S$,

$$p(S) \leqq |S|+nf+\tfrac{5}{3}$$

since this will imply that there are at least $\dfrac{n}{2}(1-f)-\frac{5}{6}$ and hence at least $\left[\dfrac{n}{2}(1-f)\right]$ disjoint edges.

## 4. Proof of Theorem 3

In view of the above remarks, we may suppose that there exists a set $S$ of vertices of $G$ such that

$$p(S)>|S|+nf+\tfrac{5}{3} \tag{10}$$

and show that this leads to a contradiction.

*Case* 1. $|S| \geqq dn$. Let $m$ denote the number of 1-components in $G-S$. If $m = 0$,

$$n \geqq |S|+3p(S)>4|S|+3fn \geqq (4d+3f)n \geqq n,$$

so we must have $m>0$. Thus

$$n-m \geqq |\Gamma(G-S)| \geqq \frac{3-4d-3f}{2-2d}(n-|S|),$$

whence

$$m \leqq \frac{3-4d-3f}{2-2d} \mid S \mid - \frac{1-2d-3f}{2-2d} n. \tag{11}$$

But we also have, from (3), ignoring the term $\frac{5}{3}$ in (10),

$$n > 4 \mid S \mid - 2m + 3nf,$$

i.e.

$$m > 2 \mid S \mid - \tfrac{1}{2}(1-3f)n. \tag{12}$$

Eliminating $m$ from (11) and (12), we obtain $\mid S \mid < dn$, a contradiction.

*Case* 2. $\mid S \mid < dn$. Here there can be no 1-components, so that each odd component contains at least

$$\max(3, dn - \mid S \mid + 1) \tag{13}$$

vertices. From now on we can assume that $4d+3f>1$.

*Case* 2(*a*). Suppose there is at least one 3-component. Then (13) yields

$$dn - \mid S \mid = \beta, \quad 0 < \beta \leqq 2. \tag{14}$$

Then (3) and (10) give

$$n > 4 \mid S \mid + 3nf + 5 = (4d+3f)n - 4\beta + 5$$

so that

$$n(4d+3f-1) < 4\beta - 5. \tag{15}$$

Considering on the other hand all but one of the odd components we have, from the definition of melt ($G$),

$$n - 3 \geqq \frac{3-4d-3f}{2-2d}(n - \mid S \mid - 3).$$

Substituting for $\mid S \mid$ from (14), this gives

$$n(1-d)(4d+3f-1) \geqq (3-\beta)(4d+3f-1) - 6d + 2\beta > 2\beta - 6d.$$

Thus, by (15), we must have

$$2\beta - 6d < (1-d)(4\beta - 5)$$

whence

$$d > \tfrac{1}{3}.$$

It follows that

$$4d+3f-1 > \tfrac{1}{3}. \tag{16}$$

(15) and (16), with $\beta \leqq 2$, now yield $n<9$, and a contradiction easily follows.

*Case* 2 (*b*). Suppose now there is no 3-component. Here we shall show that $\mid S \mid$ is bounded. First of all, if $\mid S \mid < \frac{1}{2}dn$, then

$$n > \mid S \mid + (\mid S \mid + nf + \tfrac{5}{3})(\tfrac{1}{2}dn + 1)$$

so that

$$\tfrac{1}{2}dn \mid S \mid < \mid S \mid (2 + \tfrac{1}{2}dn) < n(1 - f - \tfrac{5}{6}d - \tfrac{1}{2}fdn). \tag{17}$$

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If $dn<4$ then $|S|<1$ whereas, if $dn \geqq 4$, then $\frac{1}{2}fdn \geqq 2f$ and (17) yields $|S|<6$. Secondly, if $\frac{1}{2}dn \leqq |S|<dn$, then

$$n>|S|+(|S|+nf)(dn-|S|+1)$$

whence

$$dn-|S|<\frac{n+nf}{|S|+nf}-2<\frac{1+f}{\frac{1}{2}d+f}-2<8.$$

Thus

$$8+\tfrac{1}{6}n(1-5f)>8+|S|>dn,$$

$$40>(6d+5f-1)n>\tfrac{1}{2}n,$$

so that $n<80$. But

$$n \geqq |S|+5(|S|+2)$$

so we must have $|S| \leqq 11$.

Thus in any case, $|S| \leqq 11$. It remains finally to consider each possible value of $|S|$ in turn. In each case we argue as follows.

Let $h$ denote the number of vertices in all but one of the odd components of $G-S$. Then

$$|S|+h \geqq \frac{3-4d-3f}{2-2d}h$$

whence

$$|S| \geqq \frac{1-2d-3f}{2-2d}h.$$

Thus

$$|S| \geqq \frac{1-2d-3f}{2-2d}.5.(|S|+1). \tag{18}$$

For any specific value of $|S|$, (18) gives a lower bound for $d$, and so for $dn$. For example, if $|S| = 5$, (18) yields

$$d \geqq \tfrac{2}{5}-\tfrac{9}{5}f$$

and

$$dn \geqq \tfrac{2}{5}n-\tfrac{9}{5}\theta$$

where $\theta = fn$. Having obtained this bound for $dn$, we obtain a contradiction by estimating $n$ in two different ways. For we have

$$n>|S|+5(|S|+\theta+1)$$

and also

$$n>(|S|+\theta+1)(dn-|S|+1).$$

With $|S| = 5$, these give

$$n>35+5\theta \tag{19}$$

and

$$n>(6+\theta)(\tfrac{2}{5}n-\tfrac{9}{5}\theta-4)$$

i.e.

$$n<\frac{9\theta^2+74\theta+120}{2\theta+7}. \tag{20}$$

(19) and (20) contradict one another. The theorem is now proved.

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