BULL. AUSTRAL. MATH. SOC. 60C40
VOL. 34 (1986) 427-431.

REPRESENTING A DISTRIBUTION BY STOPPING A BROWNIAN MOTION: ROOT'S CONSTRUCTION

SHEY SHIUNG SHEU

A closed subset C of $[0,\infty] \times [-\infty,\infty]$ is called a *barrier* if

(i) $(\infty,x) \in C$, $\forall x$,

(ii) $(t,\pm\infty) \in C$, $\forall t$,

(iii) $(t,x) \in C$ implies $(s,x) \in C$, $\forall s \geq t$.

Given a Brownian motion $(B(t))$ starting at the origin and a barrier C, let $\tau(C)$ be $\inf\{t : (t,B(t)) \in C\}$. A random variable X (or a distribution F) is called *achievable* if there exists a barrier C so that $B(\tau(C))$ is distributed as $X(F)$. In this paper we shall show that if X is bounded above or below with finite mean or if X has zero mean and $E(|X| \log^{+}|X|) < \infty$ then X is achievable. This result gives a partial answer to a problem raised by Loynes [7].

1. Introduction

In dealing with various limit theorems for sums of independent random variables, Skorohod (see [9], page 163) introduced a method to imbed a mean-zero random variable X into a Brownian motion $B(t)$, $t \geq 0$, starting at the origin; that is, he found a stopping time τ (relative to a filtration generally larger than the Brownian filtration) so that B_τ has the same distribution as X (denoted by $B_\tau \sim X$) and, furthermore, $E(X^2) = E(\tau)$. If one requires τ to be a stopping time relative to the Brownian filtration (τ depends only on Brownian paths), whether such τ can be still constructed has been a research problem for many authors

Received 28 January 1986.

https://doi.org/10.1017/S0004972700010315 Published online by Cambridge University Press

(see Root [8], Dubins [6], Chacon and Walsh [4], Azéma and Yor [1], Bass [2], Vallois [10], etc.). Among these constructions, Root's seems most intuitive. His stopping time is the hitting time of a certain set in the compactified time-state space $H \equiv [0,\infty] \times [-\infty,\infty]$. A closed set C in H will be called a *barrier* if

(i) $(\infty,x) \in C$ for all $x \in C$,

(ii) $(t,\pm\infty) \in C$ for all t ,

(iii) $(t,x) \in C$ implies $(s,x) \in C$ for all $s \geqslant t$.

The space of barriers will be compact under the Hausdorff metric. For a barrier C , let $\tau(C) = \inf\{t , B(T)) \in C\}$. Root proved that if X has zero mean and finite variance, then there exists a barrier C so that $B(\tau(C)) \sim X$, and $E(\tau(C)) = E(X^2)$. Loynes [7] defines a random variable X to be *achievable* if there exists a barrier C so that $B(\tau(C)) \sim X$. He posed the problem of finding conditions for X to be achievable. In this respect, any X with zero mean and finite variance is achievable; any degenerate random variable is achievable and, therefore, being zero-mean is not a necessary condition. In fact, Loynes [7] showed that if X is concentrated on a half line $(-\infty,b]$ and $E(X) \geqslant 0$, or on $[a,\infty)$ and $E(X) \leqslant 0$, then X is achievable. He also pointed out that if X is achievable, then $E(|X|^p) < \infty$ for all p , $0 < p < 1$. Unfortunately, Loynes' results do not cover important cases such as Poisson distributions (X concentrated on the positive half line but $E(X) > 0$) . In this paper, we shall improve his results.

2. Main results

Call a sequence of random variables $\{X_n\}$ *stochastically bounded* if $\forall\varepsilon > 0$, $\exists A > 0$ such that $P(|X_n| \geqslant A) \leqslant \varepsilon$ for all n .

LEMMA 2.1. *Let $\{C_n\}_{n=1}^{\infty}$ be a sequence of barriers such that $C_n \to C_\infty$. Then the corresponding hitting times $\tau(C_n) \to \tau(C_\infty)$ in probability. In particular, if C_∞ consists of points at ∞ only , then $\tau(C_n)$ is not stochastically bounded.*

Proof. This is just a rephrase of Lemma 1 in Loynes [7]. □

LEMMA 2.2. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of random variables such that* X_n *converges to* X_∞ *in distribution. Let* $\{C_n\}_{n=1}^{\infty}$ *be a sequence of barriers such that* $C_n \to C_\infty$ *and* C_∞ *consists of at least one finite point. If* $B_{\tau(C_n)} \sim X_n$, $1 \leqslant n < \infty$, *then* $B_{\tau(C_\infty)} \sim X_\infty$.

Proof. By Lemma 2.1, $\tau(C_n) \to \tau(C_\infty)$ in probability. By assumption, $P(\tau(C_\infty) < \infty) = 1$. Therefore, there exists a subsequence $\tau(C_{n'}) \to \tau(C_\infty)$ almost surely. By the continuity of Brownian paths, we conclude $B_{\tau(C_\infty)} \sim X_\infty$. □

THEOREM 2.3. *Any random variable* X *bounded below or above with finite mean is achievable. In particular, the Poisson random variable is achievable.*

Proof. Without loss of generality, we may assume that $X \geqslant b > -\infty$. By Loynes' results, we may also assume $M = E(X) > 0$. Let

$$Y_n = \begin{cases} kM & \text{with probability } \frac{1}{2}(1-\frac{1}{n}) \\ b & \text{with probability } \frac{1}{2}(1-\frac{1}{n}) \\ -\frac{1}{2}(n-1)(kM+b) & \text{with probability } \frac{1}{n}, \end{cases}$$

where k is chosen so that $k > 0$, $(k-2)M+b > 0$. Since Y_n has mean zero and finite variance, it is achievable and the barrier can be expressed as $\{(t,x) : t \geqslant 0 , x = kM \text{ or } -\frac{1}{2}(n-1)(kM+b)\} \cup \{(t,x) : t \geqslant t_n , x = b\}$ for some $t_n > 0$. Let $X_n = X$ if $X \leqslant n$; $X_n = n$ if $X > n$, and let $M_n = E(X_n)$. Let

$$Z_n = \begin{cases} X_n & \text{with probability } 1 - \frac{1}{n} \\ -(n-1)M_n & \text{with probability } \frac{1}{n} \end{cases}$$

Z_n has mean zero and finite variance and hence, is achievable. The corresponding barrier can be expressed as $C_n = \{(t,x) : b \leqslant x \leqslant n ,$ $t \geqslant t_n(x)\} \cup \{(t,x) : t \geqslant 0 , x = -(n-1)M_n\}$. For $n \geqslant kM$, let $t'_n = \inf\{t_n(x) : b \leqslant x \leqslant kM\}$. Since $n > kM$, $-(n-1)M_n > -\frac{1}{2}(n-1)(kM+b)$,

https://doi.org/10.1017/S0004972700010315 Published online by Cambridge University Press

we have $t'_n \leq t_n$. But Y_n converges in distribution to Y , where $P(Y=kM) = P(Y=b) = \frac{1}{2}$. Therefore, t_n will converge to a finite number and consequently, C_n will not diverge to infinity. Since Z_n converges to X in distribution, X is achievable by Lemma 2.1 and Lemma 2.2. □

THEOREM 2.4. *If X is a random variable satisfying $E(X) = 0$, $E(|X| \log^+|X|) < \infty$, then X is achievable.*

Proof. We may assume that X is neither bounded above nor bounded below. Then there exist sequences $a_n \to -\infty$, $b_n \to \infty$ such that if $X_n = X$, when $a_n \leq X \leq b_n$; $X_n = 0$, when $X < a_n$ or $X > b_n$, and $E(X_n) = 0$. Of course, X_n is achievable. Let $\tau_n = \tau(C_n)$ be the stopping time such that $B_{\tau_n} \sim X_n$ and $E(\tau_n) = E(X_n^2)$. By the famous Burkholder-Gundy's inequality (see Theorem 6.1 in Burkholder [3]), we have

$$E(\sqrt{\tau_n}) \leq c\, E\,(\sup_{0\leq t\leq\tau_n} |B(t)|)\ .$$

By Doob's inequality (see Doob [5], page 317) and the fact that $\sup_{0\leq t\leq\tau_n} |B(t)|$ is bounded, we have

$$\begin{aligned} E(\sup_{0\leq t\leq\tau_n} |B(t)|) &\leq \frac{e}{e-1} + \frac{e}{e-1} E(|X_n| \log^+|X_n|) \\ &\leq \frac{e}{e-1} + \frac{e}{e-1} E(|X| \log^+|X|) \\ &< \infty\ . \end{aligned}$$

Hence, $\{E(\sqrt{\tau_n})\}$ is bounded, which implies $\{\tau_n\}$ is stochastically bounded. Since X_n converges to X in distribution, X is achievable by Lemma 2.1 and Lemma 2.2. □

References

[1] J. Azéma and M. Yor, "Une solution simple au probléme de Skorokhod", *Sem. Prob. XIII, Lecture Notes in Math.* 721 (1979), Springer, 90-115.

[2] R.F. Bass, "Skorohod imbedding via stochastic integrals", *Sem. Prob. XVII, Lecture Notes in Math.*, 986 (1983), Springer, 221-224.

[3] D.L. Burkholder, "Distribution function inequalities for martingales", *Ann. Prob.*, (1) 1 (1973), 19-42.

[4] R.V. Chacon and J.B. Walsh, "One-dimensional potential embedding", *Sem. Prob. X, Lecture Notes in Math.*, 511 (1976), Springer, 19-23.

[5] J.L. Doob, *Stochastic Process* (Wiley, New York, 1953).

[6] L.E. Dubins, "On a theorem of Skorohod", *Ann. Math. Stat.*, 39 (1968), 2094-2097.

[7] R.M. Loynes, "Stopping times on Brownian motion: some properties of Root's construction", *Z. Wahr. verw Gebiete*, 16 (1970), 211-218.

[8] D.H. Root, "The existence of certain stopping times on Brownian motion", *Ann. Math. Stat.*, (2) 40 (1967), 715-718.

[9] A.V. Skorokhod, *Studies in the Theory of Random Processes* (Addison-Wesley, Reading, 1965).

[10] P. Vallois, "Le probleme de Skorokhod sur R: une approche avec le temps local", *Sem. Prob. XVII, Lecture Notes in Math.*, 986 (1983), Springer, 227-239.

Institute of Applied Mathematics,
National Tsing Hua University,
Hsinchu, Taiwan 30043,
Republic of China.