Proceedings of the Edinburgh Mathematical Society: page 1 of 16
doi:10.1017/S0013091524000774

DEFECT RELATION FOR HOLOMORPHIC MAPS FROM COMPLEX DISCS INTO PROJECTIVE VARIETIES AND HYPERSURFACES

SI DUC QUANG

Department of Mathematics, Hanoi National University of Education, 136-Xuan Thuy, Cau Giay, Hanoi, Vietnam
Email: quangsd@hnue.edu.vn

(Received 28 November 2022)

Abstract In this paper, we establish a second main theorem for holomorphic maps with finite growth index on complex discs intersecting arbitrary families of hypersurfaces (fixed and moving) in projective varieties, which gives an above bound of the sum of truncated defects. Our result also generalizes and improves many previous second main theorems for holomorphic maps from $\mathbb{C}$ intersecting hypersurfaces (moving and fixed) in projective varieties.

Keywords: second main theorem; holomorphic map; hypersurface; finite growth index

2010 Mathematics Subject Classification: Primary 32H30;
Secondary 30D35; 32A22

1. Introduction

Let $\Delta(R) = \{z \in \mathbb{C} : |z| < R\}$ be a complex disc and r_0 be a fixed positive number so that $0 < r_0 < R$. Let ν be a divisor on $\Delta(R)$, which is regarded as a function on $\Delta(R)$ with values in $\mathbb{Z}$ such that $\operatorname{Supp}\nu := \{z; \nu(z) \neq 0\}$ is a discrete subset of $\Delta(R)$. Let k be a positive integer or $+\infty$. The truncated counting function of ν is defined by:

$$n^{[k]}(t) = \sum_{|z|\le t} \min\{k, \nu(z)\} \ (0 \le t \le R),$$

$$\text{and} \quad N^{[k]}(r,\nu) = \int_{r_0}^{r} \frac{n^{[k]}(t) - n^{[k]}(0)}{t} dt.$$

We will omit the character $^{[k]}$ if $k = +\infty$.

Let $\varphi : \Delta(R) \to \mathbb{C} \cup \{\infty\}$ be a non-constant meromorphic function. We denote by ν^0_φ (resp. ν^∞_φ) the divisor of zeros (resp. divisor of poles) of φ and set $\nu_\varphi = \nu^0_\varphi - \nu^\infty_\varphi$. As usual, we will write $N^{[k]}_\varphi(r)$ and $N^{[k]}_{1/\varphi}(r)$ for $N^{[k]}(r,\nu^0_\varphi)$ and $N^{[k]}(r,\nu^\infty_\varphi)$ respectively.

Let $f : \Delta(R) \to \mathbb{P}^N(\mathbb{C})$ be a holomorphic map and Ω be the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$. The characteristic function of f is defined by

$$T_f(r) := \int_0^r \frac{dt}{t} \int_{|z|<t} f^*\Omega.$$

In [8], Ru and Sibony defined the growth index of f by

$$c_f = \inf \left\{ c > 0 \,\Big|\, \int_0^R \exp(cT_f(r))dr = +\infty \right\}.$$

For the convenient, we will set $c_f = +\infty$ if

$$\left\{ c > 0 \,|\, \int_0^R \exp(cT_f(r))dr = +\infty \right\} = \emptyset.$$

A meromorphic function a on $\Delta(R)$ (which is regarded as a holomorphic map into $\mathbb{P}^1(\mathbb{C})$) is said to be small with respect to f if $\|T_a(r) = o(T_f(r))$ as $r \to R$. Here (and throughout this paper), the notation '$\|P$' means the assertion P holds for all $r \in (0, R)$ outside a subset S of $(0, R)$ with $\int_S \exp((c_f + \epsilon)T_f(r))dr < +\infty$ for some $\epsilon > 0$.

Denote by $\mathcal{H}$ the ring of all holomorphic functions on $\Delta(R)$. Let Q be a homogeneous polynomial in $\mathcal{H}[x_0, \dots, x_n]$ of degree $d \geq 1$ given by

$$Q(z) = \sum_{I \in \mathcal{T}_d} a_I(z)\omega^I,$$

where $\mathcal{T}_d = \{(i_0, \dots, i_N) \in \mathbf{N}_0^{N+1}\ ;\ i_0 + \cdots + i_N = d\}$, $\omega^I = \omega_0^{i_0} \cdots \omega_n^{i_N}$ for $I = (i_0, \dots, i_N)$ and all $a_I \in \mathcal{H}$ has no common zero. The homogeneous polynomial Q is called a moving hypersurface of $\mathbb{P}^N(\mathbf{C})$. Throughout this paper, by changing the homogeneous coordinates of $\mathbb{P}^N(\mathbb{C})$ if necessary, we may assume that $a_{I_0} \not\equiv 0$ each such given moving hypersurface Q, where $I_0 = (d, 0, \dots, 0)$. We put $\tilde{Q} = \sum_{I \in \mathcal{T}_d} \frac{a_I}{a_{I_0}} \omega^I$.

The moving hypersurface Q is said to be slow with respect to f if all $\frac{a_I}{a_{I_0}}$ $(I \in \mathcal{T}_d)$ are small with respect to f. Let $\mathbf{f} = (f_0, \dots, f_N)$ be a reduced representation of f. We define

$$Q(\mathbf{f})(z) = \sum_{I=(i_0,\dots,i_N) \in \mathcal{T}_d} a_I(z) f_0^{i_0}(z) \cdots f_N^{i_N}(z).$$

Then the truncated divisor $\nu^{[k]}_{Q(\mathbf{f})}$ does not depend on the choice of the reduced representation $\mathbf{f}$ and hence is written by $\nu_{Q(f)}$. Its truncated counting function is denoted simply by $N^{[k]}_{Q(f)}(r)$. The proximity function of f with respect to Q is define by

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

$$m_f(r,Q)=\int_0^{2\pi}\log\frac{\|\mathbf{f}\|^q\cdot\|Q\|}{|Q(\mathbf{f})|}(re^{i\theta})\frac{d\theta}{2\pi}.$$

If Q is slow with respect to f, then the first main theorem states that

$$\|\ dT_f(r)=m_f(r,Q)+N_{Q(f)}(r)+o(T_f(r)).$$

The truncated defect of f with respect to Q is defined by

$$\delta^{[k]}_{f,Q}=1-\limsup_{r\longrightarrow R}\frac{N^{[k]}_{Q(f)}(r)}{dT_f(r)}.$$

We omit the character $^{[k]}$ if $k=+\infty$. If all coefficients of Q are constant then we call Q a (fixed) hypersurface of $\mathbb{P}^N(\mathbb{C})$ and set $Q^*=\{(\omega_0:\cdots:\omega_N)\mid\sum_{I\in\mathcal{T}_d}a_I\omega^I=0\}$.

Let V be a smooth projective subvariety of $\mathbb{P}^N(\mathbb{C})$ of dimension n. Let $\mathcal{Q}=\{Q_1,\ldots,Q_q\}$ be a family of moving hypersurfaces in $\mathbb{P}^N(\mathbf{C})$, where $Q_i=\sum_{I\in\mathcal{T}_{d_i}}a_{iI}x^I$. Denote by $\mathcal{K}_{\mathcal{Q}}$ the smallest field which contains $\mathbb{C}$ and all functions $\frac{a_{iI}}{a_{iI_0}}$ $(I\in\mathcal{T}_{d_i})$. As usual, the family $\{Q_1,\ldots,Q_q\}$ is said to be in weakly ℓ-subgeneral position if $\bigcap_{s=1}^{\ell+1}Q_{j_s}(z)^*\cap V=\emptyset$ for every $1\le j_1<\cdots<j_{\ell+1}\le q$ and for generic points $z\in\Delta(R)$ (i.e. for all $z\in\Delta(R)$ outside a discrete subset). Here, we note that $\dim\emptyset=-\infty$. In [6], we define the distributive constant of the family $\mathcal{Q}$ with respect to V by

$$\Delta_V:=\max_{\emptyset\ne\Gamma\subset\{1,\ldots,q\}}\frac{\sharp\Gamma}{n-\dim\left(\bigcap_{j\in\Gamma}Q_j(z)^*\right)},$$

for generic points $z\in\Delta(R)$. From [6, Remark 3.7], we know that if $\mathcal{Q}$ is in weakly ℓ-subgeneral position with respect to V then $\Delta_V\le\ell-n+1$.

For the case of holomorphic curves from $\mathbb{C}$ into V and families of fixed hypersurfaces in general position (i.e. in n-subgeneral position), Ru [7] proved the following.

Theorem A. (see [7]) *Let f be an algebraically non-degenerate holomorphic map of $\mathbb{C}$ into a smooth subvariety $V\subset\mathbb{P}^N(\mathbf{C})$ of dimension n. Let $\{Q_i\}_{i=1}^q$ be a family of q hypersurfaces in general position with respective to V. Then for any $\epsilon>0$, we have*

$$\|\ (q-n-1-\epsilon)T_f(r)\le\sum_{i=1}^q\frac{1}{\deg Q_i}N_{Q_i(f)}(r).$$

From the above theorem, the number $n+1$ is an above bound of the sum of defects (without truncated multiplicity) for hypersufaces in this case. Later on, many mathematicians generalized Theorem A to the case of slowly moving hypersurfaces in general position with respect to V. Recently, in [4] Quang considered the case of holomorphic maps from $\Delta(R)$ into $\mathbb{P}^N(\mathbb{C})$ and proved the following result.

Theorem B. (reformulation of [4, Theorem 1.3]) *Let $V\subset\mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n\ge1$. Let $\{Q_1,\ldots,Q_q\}$ be a family of hypersurfaces in*

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

$\mathbb{P}^N(\mathbb{C})$ with the distributive constant Δ with respect to V, $\deg Q_i = d_i$ $(1 \le i \le q)$, and let d be the least common multiple of $d_1, \dots, d_q$. Let $f : \Delta(R) \to V$ be an algebraically non-degenerate holomorphic curve with $c_f < +\infty$. Then, for every $\epsilon > 0$,

$$\begin{aligned} \| \ & (q - \Delta(n+1) - \epsilon) T_f(r) \\ & \le \sum_{i=1}^{q} \frac{1}{d_i} N^{[L]}_{Q_i(f)}(r) + \frac{L c_f T_f(r)}{2d^{n+1}(2n+1)(n+1)(q!) \deg V}. \end{aligned}$$

where $L = \left[d^{n^2+n} \deg(V)^{n+1} e^n \Delta^n (2n+4)^n (n+1)^n (q!)^n \epsilon^{-n} \right]$.

Here, by $[x]$ we denote the largest integer not exceeding the real number x. However, the coefficient of $T_f(r)$ in the right hand side of the above inequality has a factor $c_f(q!\epsilon^{-1})^{n-1}$. Then if c_f or q is large enough, this coefficient always exceeds q and the theorem is meaningless. Hence, it may not imply the defect relation. Our purpose in this paper is to improve Theorem B by reducing the truncation level L and that coefficients so that they do not depend on q. In order to do so, we will apply the new below bound of Chow weight in [5] (see Lemma 2.2) and also give some new technique to control the error term occuring when the theorem on the estimate of Hilbert weights (Theorem 2.1) is applied. We also consider the case of moving hypersurfaces. Our main result is stated as follows.

Theorem 1.1. *Let f be a non-constant holomorphic map of $\Delta(R)$ into an n-dimensional smooth projective subvariety $V \subset \mathbb{P}^N(\mathbf{C})$ with finite growth index c_f. Let $\{Q_i\}_{i=1}^q$ be a family of slow (with respect to f) moving hypersurfaces with the distributive constant Δ_V with respect to V. Assume that f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$.*

a) *For any $\epsilon' > 0$ and $(n+1)\Delta_V > \epsilon > 0$, we have*

$$\begin{aligned} \| \ & (q - \Delta_V(n+1) - \epsilon) T_f(r) \\ & \le \sum_{j=1}^{q} \frac{1}{d} N^{[L-1]}_{Q_j(f)}(r) + \frac{(\Delta_V(n+1)+\epsilon)(c_f+\epsilon')(L-1)}{2du} T_f(r), \end{aligned}$$

where

$$L = d^n \deg V (u+1)^n \left[\left(1 + \frac{\epsilon}{2(n+1)\Delta_V}\right)^{\left[\frac{d^n \deg V (u+1)^{n+q}}{\log^2(1+\frac{\epsilon}{2(n+1)\Delta_V})}\right]+1} \right]$$

with $u = \lceil 2\Delta_V(2n+1)(n+1)d^n \deg V(\Delta_V(n+1)+\epsilon)\epsilon^{-1} \rceil$.

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

b) *Assume further that all Q_i $(1 \leq i \leq q)$ are assumed to be fixed hypersurfaces. For any $\epsilon' > 0$ and $\epsilon > 0$ we have*

$$\begin{aligned} \| \ & (q - \Delta_V(n+1) - \epsilon)T_f(r) \\ & \leq \sum_{j=1}^{q} \frac{1}{d} N^{[L'-1]}_{Q_j(f)}(r) + \frac{(\Delta_V(n+1)+\epsilon)(c_f+\epsilon')(L'-1)}{2du'} T_f(r), \end{aligned}$$

where $L' = \left[d^{n^2+n}(\deg V)^{n+1}e^n(2n+5)^n(\Delta_V^2(n+1)\epsilon^{-1}+\Delta_V)^n\right]$ *with* $u' = \lceil \Delta_V(2n+1)(n+1)d^n \deg V(\Delta_V(n+1)+\epsilon)\epsilon^{-1}\rceil$.

Here, $\lceil x \rceil$ stands for the smallest integer bigger than or equal to the real number x. By this theorem, we get the following truncated defect relation for fixed hypersurfaces.

Corollary 1.2. *With the assumption and notation as in Theorem 1.1 and suppose that all Q_i are fixed hypersurfaces. Then for any $\epsilon > 0$, we have*

$$\sum_{i=1}^{q} \delta^{[L-1]}_{f,Q_i} \leq \Delta_V(n+1) + \epsilon + \frac{(\Delta_V(n+1)+\epsilon)c_f(L-1)}{2du},$$

where $L = [d^{n^2+n}(\deg V)^{n+1}e^n(2n+5)^n(\Delta_V^2(n+1)\epsilon^{-1}+\Delta_V)^n]$ *with* $u = \lceil 2\Delta_V(2n+1)(n+1)d^n \deg V(\Delta_V(n+1)+\epsilon)\epsilon^{-1}\rceil$.

2. Some auxiliary results

Let $X \subset \mathbb{P}^n(\mathbb{C})$ be a projective variety of dimension k and degree δ. For $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}_{\geq 0}$ we write $\mathbf{x}^{\mathbf{a}}$ for the monomial $x_0^{a_0} \cdots x_n^{a_n}$. Let I_X be the prime ideal in $\mathbb{C}[x_0, \ldots, x_n]$ defining X. Let $\mathbb{C}[x_0, \ldots, x_n]_u$ be the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degree u (including 0). For $u = 1, 2, \ldots$, put $(I_X)_u := \mathbb{C}[x_0, \ldots, x_n]_u \cap I_X$ and define the Hilbert function H_X of X by

$$H_X(u) := \dim \mathbb{C}[x_0, \ldots, x_n]_u/(I_X)_u.$$

Let $\mathbf{c} = (c_0, \ldots, c_n)$ be a tuple in $\mathbb{R}^{n+1}_{\geq 0}$ and let $e_X(\mathbf{c})$ be the Chow weight of X with respect to $\mathbf{c}$. The u-th Hilbert weight $S_X(u, \mathbf{c})$ of X with respect to $\mathbf{c}$ is defined by

$$S_X(u, \mathbf{c}) := \max \sum_{i=1}^{H_X(u)} \mathbf{a}_i \cdot \mathbf{c},$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_X(u)}}$ whose residue classes modulo I_X form a basis of $\mathbb{C}[x_0, \ldots, x_n]_u/(I_X)_u$.

The following theorem is due to Evertse and Ferretti [2].

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

Theorem 2.1. (see [2, Theorem 4.1]) *Let $X \subset \mathbb{P}^n(\mathbb{C})$ be an algebraic variety of dimension k and degree δ. Let $u > \delta$ be an integer and let $\boldsymbol{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}_{\geqslant 0}$. Then*

$$\frac{1}{uH_X(u)} S_X(u, \boldsymbol{c}) \geq \frac{1}{(k+1)\delta} e_X(\boldsymbol{c}) - \frac{(2k+1)\delta}{u} \cdot \left(\max_{i=0,\ldots,n} c_i \right).$$

The following lemma is due to the Quang [5].

Lemma 2.2. (see [5, Lemma 3.2]) *Let Y be a projective subvariety of $\mathbb{P}^R(\mathbb{C})$ of dimension $k \geq 1$ and degree δ_Y. Let ℓ $(\ell \geq k+1)$ be an integer and let $\boldsymbol{c} = (c_0, \ldots, c_R)$ be a tuple of non-negative reals. Let $\mathcal{H} = \{H_0, \ldots, H_R\}$ be a set of hyperplanes in $\mathbb{P}^R(\mathbb{C})$ defined by $H_i = \{y_i = 0\}$ $(0 \leq i \leq R)$. Let $\{i_1, \ldots, i_\ell\}$ be a subset of $\{0, \ldots, R\}$ such that:*

(1) $c_{i_\ell} = \min\{c_{i_1}, \ldots, c_{i_\ell}\}$,
(2) $Y \cap \bigcap_{j=1}^{\ell-1} H_{i_j} \neq \emptyset$,
(3) *and* $Y \not\subset H_{i_j}$ *for all* $j = 1, \ldots, \ell$.

Let $\Delta_{\mathcal{H},Y}$ be the distributive constant of the family $\mathcal{H} = \{H_{i_j}\}_{j=1}^{\ell}$ with respect to Y. Then

$$e_Y(\boldsymbol{c}) \geq \frac{\delta_Y}{\Delta_{\mathcal{H},Y}} (c_{i_1} + \cdots + c_{i_\ell}).$$

The following theorem is due to Ru and Sibony [8].

Theorem 2.3. (reformulation of [8, Theorem 4.8]) *Let f be a linearly non-degenerate holomorphic map from $\Delta(R)$ $(0 < R \leq +\infty)$ into $\mathbb{P}^n(\mathbb{C})$. Let $H_1, \ldots, H_q$ be q arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Then, for every $\varepsilon > 0$, we have*

$$\begin{aligned} \Big\| \int_0^{2\pi} \max_K \log \sum_{j \in K} \frac{\|\mathbf{f}\|}{|H_j(\mathbf{f})|} \frac{d\theta}{2\pi} + N_W(r) \\ \leq (n+1) T_f(r) + \frac{n(n+1)}{2} (c_f + \epsilon) T_f(r), \end{aligned}$$

where $W = \det(f_i^{(k)}; 0 \leq i, k \leq n)$ for a reduced representation $\mathbf{f} = (f_0, \ldots, f_n)$ of f.

Note that, in the original theorem [8, Theorem 4.8], the last term of the right hand side of the above inequality is more complex and the inequality holds for all r outside an exceptional set $S \subset (0, R)$ such that $\int_S \exp((c_f + \epsilon)T_f(r))dr < +\infty$. In this reformulation, we just simplify that term but the exceptional set is a subset $S' \subset (0, R)$ such that $\int_{S'} \exp((c_f + \epsilon')T_f(r))dr < +\infty$ for some $\epsilon' > 0$.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be a family moving hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ given by

$$Q_i(z)(\mathbf{x}) = \sum_{I \in \mathcal{T}_{d_i}} a_{iI}(z)\mathbf{x}^I,$$

where $\mathbf{x} = (x_0, \ldots, x_N)$, $\mathbf{x}^I = x_0^{i_0} \cdots x_N^{i_N}$ for $I = (i_0, \ldots, i_N)$. Denote by $\mathcal{C}_{\mathcal{Q}}$ the set of all non-negative functions $h : \mathbf{C}^m \setminus A \longrightarrow [0, +\infty]$, which are of the form:

$$h = \frac{|g_1| + \cdots + |g_l|}{|g_{l+1}| + \cdots + |g_{l+k}|},$$

where $k, l \in \mathbf{N},\ g_1, ..., g_{l+k} \in \mathcal{K}_{\mathcal{Q}} \setminus \{0\}$ and A is a discrete subset of $\Delta(R)$, which may depend on $g_1, ..., g_{l+k}$. Then, for $h \in \mathcal{C}_{\mathcal{Q}}$ we have

$$\int_0^{2\pi} \log^+(h)\frac{d\theta}{2\pi} = O(\max T_{a_{iI}/a_{iJ}}(r)).$$

Also, for every moving hypersurface Q in $\mathcal{K}_{\mathcal{Q}}[x_0, \ldots, x_N]$ of degree d, we have

$$Q(z)(\mathbf{x}) \leq c(z)\|\mathbf{x}\|^d,$$

for some $c \in \mathcal{C}_{\mathcal{Q}}$.

Lemma 2.4. (see [6, Lemma 3.2]) *Let V be a projective variety of $\mathbb{P}^N(\mathbb{C})$. With the above notation, let $1 \leq j_1 \leq \cdots \leq j_k \leq q$. Suppose that there exists $z_0 \in \Delta(R)$ such that $V \cap \bigcap_{s=1}^k Q_{j_s}(z_0)^* = \emptyset$. Then we have $V \cap \bigcap_{s=1}^k Q_{j_s}(z)^* = \emptyset$ for every $z \in \Delta(R)$ outside a discrete subset, and there exists a function $c \in \mathcal{C}_{\mathcal{K}}$ such that*

$$\|\boldsymbol{f}(z)\| \leq c(z) \max_{1 \leq s \leq k} \{Q_{j_s}(\boldsymbol{f})(z)\}.$$

3. Proof of theorem 1.1

Replacing Q_j by $Q^{\frac{d}{d_j}}$ if necessary, we may assume that $Q_1, \ldots, Q_q$ have the same degree d and $Q_i = \sum_{I \in \mathcal{T}_d} a_{iI}x^I$ $(i = 1, \ldots, q)$. Take a point z_0 such that $a_{iI_0}(z_0) \neq 0$ for all i, and

$$\Delta_V = \max_{\emptyset \neq \Gamma \subset \{1,\ldots,q\}} \frac{\sharp\Gamma}{\dim V - \dim V \cap \bigcap_{j \in \Gamma} \widetilde{Q}_i(z_0)^*}.$$

It is suffice for us to consider the case where $q > \Delta_V(n+1)$. Denote by $\sigma_1, \ldots, \sigma_{n_0}$ all bijections from $\{0, \ldots, q-1\}$ into $\{1, \ldots, q\}$, where $n_0 = q!$. For each σ_i, it is easy to see that $\bigcap_{j=0}^{q-2} \tilde{Q}_{\sigma_i(j)}(z_0)^* \cap V = \emptyset$. Then there exists a smallest index $\ell_i \leq q-2$ such that $\bigcap_{j=0}^{\ell_i} \tilde{Q}_{\sigma_i(j)}(z_0)^* \cap V = \emptyset$. Hence $\bigcap_{j=0}^{\ell_i} \tilde{Q}_{\sigma_i(j)}(z)^* \cap V = \emptyset$ for generic points z and for all

$i = 1, \ldots, n_0$. Denote by $\mathcal{S}$ the set of all points $z \in \Delta(R)$ such that $\bigcap_{j=0}^{\ell_i} \tilde{Q}_{\sigma_i(j)}(z)^* \cap V \neq \emptyset$ for some i. Then $\mathcal{S}$ is a discrete subset of $\Delta(R)$.

By Lemma 2.4, there is a function $A \in \mathcal{C}_{\mathcal{Q}}$, chosen common for all σ_i, such that

$$\|\mathbf{f}(z)\|^d \leq A(z) \max_{0 \leq j \leq \ell_i} \frac{|\tilde{Q}_{\sigma_i(j)}(\mathbf{f})(z)|}{\|\tilde{Q}_{\sigma_i(j)}(z)\|} \ \forall i = 1, \ldots, n_0.$$

Denote by $S(i)$ the set of all z not in $\mathcal{S}$ such that $\tilde{Q}_j(\mathbf{f})(z) \neq 0$ for all $j = 1, \ldots, q$ and

$$\frac{|\tilde{Q}_{\sigma_i(0)}(\mathbf{f})(z)|}{\|\tilde{Q}_{\sigma_i(0)}(z)\|} \leq \frac{|\tilde{Q}_{\sigma_i(1)}(\mathbf{f})(z)|}{\|\tilde{Q}_{\sigma_i(1)}(z)\|} \leq \cdots \leq \frac{|\tilde{Q}_{\sigma_i(q-1)}(\mathbf{f})(z)|}{\|\tilde{Q}_{\sigma_i(q-1)}(z)\|}.$$

Therefore, for every generic point $z \in S(i)$, we have

$$\prod_{j=1}^{q} \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_j(z)\|}{|\tilde{Q}_j(\mathbf{f})(z)|} \leq C(z) \prod_{j=0}^{\ell_j} \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_{\sigma_i(j)}(z)\|}{|\tilde{Q}_{\sigma_i(j)}(\mathbf{f})(z)|},$$

where $C(z) = \sum_{i=1}^{n_0} A(z)^{q-\ell_i-1} \in \mathcal{C}_{\mathcal{Q}}$.

For $z \notin \mathcal{S}$, consider the mapping Φ_z from V into $\mathbb{P}^{q-1}(\mathbb{C})$ defined by

$$\Phi_z(\mathbf{x}) = (\tilde{Q}_1(z)(x) : \cdots : \tilde{Q}_q(z)(x))$$

for every $\mathbf{x} = (x_0 : \cdots : x_N) \in V$, where $x = (x_0, \ldots, x_N)$. We set

$$\tilde{\Phi}_z(x) = (\tilde{Q}_1(z)(x), \ldots, \tilde{Q}_q(z)(x)).$$

Let $Y_z = \Phi_z(V)$. Since $V \cap \bigcap_{j=1}^{q} \tilde{Q}_j(z)^* = \emptyset$, Φ_z is a finite morphism on V and Y_z is a projective subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ with $\dim Y_z = n$ and of degree

$$\delta_z := \deg Y_z \leq d^n . \deg V = \delta.$$

For every $\mathbf{a} = (a_1, \ldots, a_q) \in \mathbb{Z}_{\geq 0}^q$ and $\mathbf{y} = (y_1, \ldots, y_q)$ we denote $\mathbf{y}^{\mathbf{a}} = y_1^{a_1} \ldots y_q^{a_q}$. Let u be a positive integer. We set $\xi_u := \binom{q+u}{u}$ and define the $\mathbb{C}$-vector space

$$Y_{z,u} := \mathbb{C}[y_1, \ldots, y_q]_u / (I_{Y_z})_u.$$

Denote by $(I_Y)_u$ the subspace of the $\mathcal{K}_{\mathcal{Q}}$-vector space $\mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]_u$ consisting of all homogeneous polynomials $P \in \mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]_u$ (including the zero polynomial) such that

$$P(z)(\Phi_z(\mathbf{f}(z))) \equiv 0.$$

Let $(\tilde{R}_1, \ldots, \tilde{R}_p)$ be an $\mathcal{K}_{\mathcal{Q}}$-basis of $(I_Y)_u$. By enlarging $\mathcal{S}$ if necessary, we may assume that all zeros and poles of all non-zero coefficients of $\tilde{R}_i$ $(1 \leq i \leq p)$ are contained in

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

$\mathcal{S}$, also all above assertions for generic points z still hold for all $z \notin \mathcal{S}$. Choose $\xi_u - p$ non-zero monic monomial $v_1, \ldots, v_{\xi_u - p}$ of degree of u in variables $y_1, \ldots, y_q$ such that $\{\tilde{R}_1, \ldots, \tilde{R}_p, v_1, \ldots, v_{\xi_u - p}\}$ is a $\mathcal{K}_{\mathcal{Q}}$-basis of $\mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]_u$.

Denote by $\mathcal{T} = \{T_1, \ldots, T_{\xi_u}\}$ the set of all non-zero monic monomials of degree of u in variables $y_1, \ldots, y_q$. Then $\{T_1, \ldots, T_{\xi_u}\}$ is a $\mathcal{K}_{\mathcal{Q}}$-basis of $\mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]_u$, and also is an $\mathbb{C}$-basis of $\mathbb{C}[y_1, \ldots, y_q]_u$.

From [6, Claim 4.3], we have the following claim.

Claim 3.1. *There is a discrete subset $\mathcal{S}'$ of $\Delta(R)$ such that for all $z \notin \mathcal{S}'$, we have:*

(i) *the family of equivalent classes of $v_1, \ldots, v_{\xi_u - p}$ is a basis of $Y_{z,u}$ and the family $\{\tilde{R}_1(z), \ldots, \tilde{R}_p(z)\}$ is a $\mathbb{C}$-basis of $(I_{Y_z})_u$;*

(ii) *for a subset $\{v'_1, \ldots, v'_{\xi_u - p}\}$ of $\mathcal{T}$, if $\{\tilde{R}_1, \ldots, \tilde{R}_p, v'_1, \ldots, v'_{\xi_u - p}\}$ is a $\mathcal{K}_{\mathcal{Q}}$-basis of $\mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]_u$ then the set of equivalent classes of $v'_1, \ldots, v'_{\xi_u - p}$ modulo $(I_{Y_z})_u$ is a $\mathbb{C}$-basis of $Y_{z,u}$ for everey $z \notin \mathcal{S}$;*

(iii) *otherwise if $\{\tilde{R}_1, \ldots, \tilde{R}_p, v'_1, \ldots, v'_{\xi_u - p}\}$ is linearly dependent over $\mathcal{K}_{\mathcal{Q}}$ then the set of equivalent classes of $v'_1, \ldots, v'_{\xi_u - p}$ modulo $(I_{Y_z})_u$ is not a $\mathbb{C}$-basis of $Y_{z,u}$.*

Then, we have $\xi_u - p = H_{Y_z}(u)$ for all z outside $\mathcal{S} \cup \mathcal{S}'$. Now, consider the holomorphic map F from $\Delta(R)$ into $\mathbb{P}^{\xi_u - p - 1}(\mathbb{C})$ with the representation

$$\mathbf{F} = (v_1(\tilde{\Phi} \circ \mathbf{f}), \ldots, v_{\xi_u - p}(\tilde{\Phi} \circ \mathbf{f})).$$

Since f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$, F is linearly non-degenerate over $\mathcal{K}_{\mathcal{Q}}$.

Now, for $z \notin \mathcal{S} \cup \mathcal{S}'$, we set $\mathbf{c}_z = (c_{1,z}, \ldots, c_{q,z}) \in \mathbb{Z}^q$, where

$$c_{i,z} := \log \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_i(z)\|}{|\tilde{Q}_i(\mathbf{f})(z)|} \geq 0, \quad \text{for } i = 1, \ldots, q.$$

By the definition of the Hilbert weight, there are $\mathbf{a}_{1,z}, \ldots, \mathbf{a}_{\xi_u - p, z} \in \mathbb{N}^q$ with

$$\mathbf{a}_{i,z} = (a_{i,1,z}, \ldots, a_{i,q,z}),$$

where $a_{i,j,z} \in \{1, \ldots, \xi_u\}$, such that the residue classes modulo $(I_Y)_u$ of $\mathbf{y}^{\mathbf{a}_{1,z}}, \ldots,$ $\mathbf{y}^{\mathbf{a}_{\xi_u - p, z}}$ form a basic of $\mathbb{C}[y_1, \ldots, y_q]_u / (I_{Y_z})_u$ and

$$S_Y(u, \mathbf{c}_z) = \sum_{i=1}^{\xi_u - p} \mathbf{a}_{i,z} \cdot \mathbf{c}_z.$$

Note that $\mathbf{y}^{\mathbf{a}_{i,z}} \in \mathcal{T}$ and the set $\{\tilde{R}_1, \ldots, \tilde{R}_p, \mathbf{y}^{\mathbf{a}_{1,z}}, \ldots, \mathbf{y}^{\mathbf{a}_{\xi_u - p, z}}\}$ is a basis of $\mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]$ (by Claim 3.1(iii)). Therefore, the set of equivalent classes of $\{\mathbf{y}^{\mathbf{a}_{1,z}}, \ldots,$ $\mathbf{y}^{\mathbf{a}_{\xi_u - p, z}}\}$ is a basis of $\dfrac{\mathcal{K}_{\mathcal{Q}}[y_1, \ldots, y_q]_u}{I(Y)_u}$. Then $\mathbf{y}^{\mathbf{a}_{i,z}} = L_{i,z}\left(v_1, \ldots, v_{H_Y(u)}\right)$ modulo

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

$I(Y)_u$, where $L_{i,z}$ $(1 \leq i \leq \xi_u - p)$ are $\mathcal{K}_{\mathcal{Q}}$-independent linear forms with coefficients in $\mathcal{K}_{\mathcal{Q}}$. We have

$$\begin{aligned}\log \prod_{i=1}^{\xi_u - p} |L_{i,z}(\mathbf{F}(z))| &= \log \prod_{i=1}^{\xi_u - p} \prod_{j=1}^{q} |\tilde{Q}_j(\mathbf{f})(z)|^{a_{i,j,z}} \\ &= -S_Y(u, \mathbf{c}_z) + du(\xi_u - p) \log \|\mathbf{f}(z)\| + \log C_1(z),\end{aligned}$$

where $C_1 \in \mathcal{C}_{\mathcal{Q}}$. Note that the number of these linear forms $L_{i,z}$ is finite, at most ξ_u. Denote by $\mathcal{L}$ the set of all $L_{i,z}$ occurring in the above inequalities. The above inequality follows that

$$\begin{aligned}\log \prod_{i=1}^{\xi_u - p} \frac{\|\mathbf{F}(z)\| \cdot \|L_{i,z}\|}{|L_{i,z}(\mathbf{F}(z))|} =& S_Y(u, \mathbf{c}_z) - du(\xi_u - p) \log \|\mathbf{f}(z)\| \\ &+ (\xi_u - p) \log \|\mathbf{F}(z)\| + \log C_2,\end{aligned}$$

where $C_2(z) \in \mathcal{C}_{\mathcal{Q}}$. Then, we have

$$\begin{aligned}S_Y(u, \mathbf{c}_z) \leq& \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \frac{\|\mathbf{F}(z)\| \cdot \|L\|}{|L(\mathbf{f}(z))|} + du(\xi_u - p) \log \|\mathbf{f}(z)\| \\ &- (\xi_u - p) \log \|\mathbf{F}(z)\| + \log C_2(z),\end{aligned} \tag{3.2}$$

where the maximum is taken over all subsets $\mathcal{J} \subset \mathcal{L}$ with $\sharp\mathcal{J} = \xi_u - p$ and $\{L | L \in \mathcal{J}\}$ is linearly independent over $\mathcal{K}$. From Theorem 2.1 we have

$$\frac{1}{u(\xi_u - p)} S_{Y_z}(u, \mathbf{c}_z) \geq \frac{1}{(n+1)\delta_z} e_{Y_z}(\mathbf{c}_z) - \frac{(2n+1)\delta_z}{u} \max_{1 \leq i \leq q} c_{i,z}. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\begin{aligned}&\frac{1}{(n+1)\delta_z} e_{Y_z}(\mathbf{c}_z) \\ &\leq \frac{1}{u(\xi_u - p)} \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \frac{\|\mathbf{F}(z)\| \cdot \|L\|}{|L(\mathbf{F}(z))|} \\ &\quad + \frac{(2n+1)\delta_z}{u} \max_{1 \leq i \leq q} c_{i,z} + \log^+ C_3(z) \\ &\leq \frac{1}{u(\xi_u - p)} \max_{\mathcal{J} \subset \mathcal{L}} \prod_{L \in \mathcal{J}} \frac{\|\mathbf{F}(z)\| \cdot \|L\|}{|L(\mathbf{F}(z))|} \\ &\quad + \frac{(2n+1)\delta_z}{u} \sum_{1 \leq i \leq q} \log \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_i(z)\|}{|\tilde{Q}_i(\mathbf{f})(z)|} + \log^+ C_3(z),\end{aligned} \tag{3.4}$$

where $C_3(z) \in \mathcal{C}_{\mathcal{Q}}$, for every $z \in \Delta(R)$ outside a discrete subset.

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

Fix a point $z \in \Delta(R) \setminus (\mathcal{S} \cup \mathcal{S}')$. Choose $i \in \{1, \ldots, n_0\}$ such that

$$e_{\sigma_i(0),z} \leq e_{\sigma_i(1),z} \leq \cdots \leq e_{\sigma_i(q-1),z}.$$

Since $\bigcap_{j=0}^{\ell_i-1} \tilde{Q}_{\sigma_i(j)}(z)^* \cap V \neq \emptyset$, by Lemma 2.2, we have

$$\begin{aligned}\Delta_V e_{Y_z}(\mathbf{c}_z) &\geq (c_{\sigma_i(0),z} + \cdots + c_{\sigma_i(\ell_i),z}) \cdot \delta_z \\ &= \delta_z \Bigg(\sum_{j=0}^{\ell_i} \log \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_{\sigma_i(j)}(z)\|}{|\tilde{Q}_{\sigma_i(j)}(\mathbf{f})(z)|}\Bigg).\end{aligned} \tag{3.5}$$

Then, from (3.2), (3.4) and (3.5) we have

$$\begin{aligned}&\frac{1}{\Delta_V} \log \prod_{i=1}^{q} \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_i(z)\|}{|\tilde{Q}_i(\mathbf{f})(z)|} \\ &\quad \leq \frac{n+1}{u(\xi_u - p)} \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \frac{\|\mathbf{F}(z)\| \cdot \|L\|}{|L(\mathbf{F}(z))|} \\ &\quad + \frac{(2n+1)(n+1)\delta_z}{u} \sum_{1 \leq i \leq q} \log \frac{\|\mathbf{f}(z)\|^d \|\tilde{Q}_i(z)\|}{|\tilde{Q}_i(\mathbf{f})(z)|} \\ &\quad + \frac{1}{\Delta_V} \log C(z) + (n+1) \log^+ C_3(z),\end{aligned} \tag{3.6}$$

for every $z \in \Delta(R)$ outside a discrete subset.

Denote by Ψ the set of all the coefficients of all linear forms $L_{i,z}$ and suppose that $\Psi = \{a_1, \ldots, a_{q_0}\}$. Then, we see that $\sharp\mathcal{L} \leq \xi_u, \sharp\Psi = q_0 \leq \xi_u(\xi_u - p)$. For each positive integer m, denote by $\mathcal{L}(\Psi(m))$ the $C-$vector space generated by the set $\{a_1^{i_1} \ldots a_{q_0}^{i_{q_0}} | i_j \geq 0$ and $\sum_{j=1}^{q_0} i_j \leq m\}$. By Remark 3.4 in [10], there exists the smallest integer p' such that

$$\frac{\dim \mathcal{L}(\Psi(p'+1))}{\dim \mathcal{L}(\Psi(p'))} \leq 1 + \frac{\epsilon}{2\Delta_V(n+1)}.$$

Put $s = \dim \mathcal{L}(\Psi(p')), t = \dim \mathcal{L}(\Psi(p'+1))$. Again, by [10, Remark 3.4], we have

$$\begin{aligned}t &\leq \left[\left(1 + \frac{\epsilon}{2(n+1)\Delta_V}\right)^{\left[\frac{\sharp\Psi}{\log^2(1+\frac{\epsilon}{2(n+1)\Delta_V})}\right]+1} \right] \\ &\leq \left[\left(1 + \frac{\epsilon}{2(n+1)\Delta_V}\right)^{\left[\frac{d^n \deg V (u+1)^{n+q}}{\log^2(1+\frac{\epsilon}{2(n+1)\Delta_V})}\right]+1} \right].\end{aligned}$$

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

Here, the last inequality comes from the fact that $\xi_u \leq (u+1)^q$ and

$$\xi_u - p \leq \delta\binom{n+u}{n} \leq d^n \deg V\binom{n+u}{n} \leq d^n \deg V(u+1)^n.$$

Choose $\{b_1, \ldots, b_s\}$ an $\mathbb{C}$-basis of $\mathcal{L}(\Psi(p'))$ and $\{b_1, \ldots, b_t\}$ an $\mathbb{C}$-basis of $\mathcal{L}(\Psi(p'+1))$. Consider the holomorphic map $\tilde{F} : \Delta(R) \to \mathbb{P}^{t(\xi_u - p)-1}(\mathbb{C})$ with a presentation

$$\tilde{\mathbf{F}} = (b_1 v_1(\tilde{\Phi}\circ\mathbf{f}), \ldots, b_1 v_{\xi_u - p}(\tilde{\Phi}\circ\mathbf{f}), \ldots, b_t v_1(\tilde{\Phi}\circ\mathbf{f}), \ldots, b_t v_{\xi_u - p}(\tilde{\Phi}\circ\mathbf{f})).$$

Note that $\| \; T_{\tilde{F}}(r) = duT_f(r) + o(T_f(r))$ and $c_{\tilde{F}} = \frac{1}{du}c_f$. By Theorem 2.3, we have

$$\begin{aligned}
\Big\| \; & s\int_0^{2\pi} \max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}} \frac{\|\mathbf{F}\|}{|L(\mathbf{F})|}\frac{d\theta}{2\pi} - N_{W(\tilde{F})}(r) \\
& \leq \int_0^{2\pi} \max_{\mathcal{J}\subset\mathcal{L}} \log \prod_{L\in\mathcal{J}}\prod_{i=1}^{s} \frac{\|\tilde{\mathbf{F}}\|}{|b_i L(\mathbf{F})|} - N_{W(\tilde{F})}(r) + o(T_f(r)) \\
& \leq t(\xi_u - p)udT_f(r) + \frac{(t(\xi_u-p)-1)t(\xi_u-p)}{2}(c_{\tilde{F}} + \frac{\epsilon'}{2du})T_{\tilde{F}}(r),
\end{aligned} \tag{3.7}$$

where $\max_{\mathcal{J}\subset\mathcal{L}}$ is taken over all subsets $\mathcal{J}$ of the system $\mathcal{L}$ of linear forms such that $\mathcal{J}$ is linearly independent over $\mathcal{K}_{\mathcal{Q}}$, ϵ' is an arbitrary positive number. Then, by integrating (3.6) and using the above inequality, we obtain

$$\begin{aligned}
\Big\| \; & \left(\frac{1}{\Delta_V} - \frac{(2n+1)(n+1)\delta}{u}\right)\sum_{i=1}^{q} m_f(r, Q_i) \\
& \leq \frac{d(n+1)t}{s}T_f(r) - \frac{(n+1)}{u(\xi_u - p)s}N_{W(\tilde{F})}(r) \\
& + \frac{(t(\xi_u-p)-1)t(n+1)}{2su}\left(c_{\tilde{F}} + \frac{\epsilon'}{2du}\right)T_{\tilde{F}}(r).
\end{aligned}$$

We now estimate the quantity $N_{W(\tilde{F})}(r)$. Let $z \in \Delta(R)$ which is neither zero nor pole of any coefficients of $\tilde{Q}_i$ $(1 \leq i \leq q)$ and b_i $(1 \leq i \leq t)$. We set

$$c_i = \max\{0, \nu^0_{\tilde{Q}_i(\mathbf{f})}(z) - \xi_u + p + 1\} \; (1 \leq i \leq q) \text{ and } \mathbf{c} = (c_1, \ldots, c_q) \in \mathbb{Z}^q_{\geq 0}.$$

Then there are

$$\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,q}), a_{i,s} \in \{1, ..., u\}$$

such that $\mathbf{y}^{\mathbf{a}_1}, ..., \mathbf{y}^{\mathbf{a}_{H_Y(u)}}$ is a basic of $\mathbb{C}[y_1, \ldots, y_q]_u/(I_{Y_z})_u$ and

$$S_{Y_z}(u, \mathbf{c}) = \sum_{i=1}^{H_Y(u)} \mathbf{a}_i \cdot \mathbf{c}.$$

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

Similarly as above, we write $\mathbf{y}^{\mathbf{a}_i} = L_i(v_1, ..., v_{\xi_u - p})$, where $L_1, ..., L_{\xi_u - p}$ are linearly independent linear forms. We see that

$$\nu^0_{W(\tilde{F})}(z) \geq t \sum_{i=1}^{H_Y(u)} \max\{0, \nu^0_{L_i(\mathbf{F})}(z) - n_u\},$$

where $n_u = (\xi_u - p)t - 1$. It is easy to see that

$$\nu^0_{L_i(\mathbf{F})}(z) = \sum_{j=1}^{q} a_{i,j} \nu^0_{\tilde{Q}_j(\mathbf{f})}(z),$$

and hence

$$\max\{0, \nu^0_{L_i(\mathbf{F})}(z) - n_u\} \geq \sum_{j=1}^{q} a_{i,j} c_j = \mathbf{a}_i \cdot \mathbf{c}.$$

Thus, we have

$$\nu^0_{W(\tilde{F})}(z) \geq t \sum_{i=1}^{H_Y(u)} \mathbf{a}_i \cdot \mathbf{c} = t S_Y(u, \mathbf{c}). \tag{3.8}$$

Choose an index σ_{i_0} such that $\nu^0_{\tilde{Q}_{\sigma_{i_0}(0)}(\mathbf{f})}(z) \geq \nu^0_{\tilde{Q}_{\sigma_{i_0}(1)}(\mathbf{f})}(z) \geq \cdots \geq \nu^0_{\tilde{Q}_{\sigma_{i_0}(q-1)}(\mathbf{f})}(z)$. By Lemma 2.2 we have

$$\begin{aligned}
\Delta_V e_{Y_z}(\mathbf{c}) &\geq (c_{\sigma_{i_0}(0),z} + \cdots + c_{\sigma_{i_0}(\ell_{i_0}),z}) \cdot \delta_z \\
&= \delta_z \cdot \sum_{j=1}^{l_{i_0}} \max\{0, \nu^0_{\tilde{Q}_{\sigma_{i_0}(j)}(\mathbf{f})}(z) - n_u\} \\
&= \delta_z \cdot \sum_{j=1}^{q} \max\{0, \nu^0_{\tilde{Q}_j(\mathbf{f})}(z) - n_u\} + O(\nu_{R^{i_0}}(z)),
\end{aligned}$$

where R^i is the resultant of the family $\{Q_{\sigma_i(j)}\}_{j=0}^{\ell_i}$ for $i = 1, \ldots, q$.

On the other hand, by Theorem 2.1 we have that

$$\begin{aligned}
S_{Y_z}(u, \mathbf{c}) &\geq \frac{u(\xi_u - p)}{(n+1)\delta_z} e_Y(\mathbf{c}) - (2n+1)\delta_z(\xi_u - p) \max_{1 \leq i \leq q} c_i + O(\nu_{R^{i_0}}(z)) \\
&\geq \left(\frac{u(\xi_u - p)}{\Delta_V(n+1)} - (2n+1)\delta_z(\xi_u - p) \right) \\
&\quad \times \sum_{j=1}^{q} \max\{0, \nu^0_{\tilde{Q}_j(\mathbf{f})}(z) - n_u\} + O(\nu_{R^{i_0}}(z)).
\end{aligned}$$

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

Combining this inequality and (3.8), we have

$$\frac{(n+1)}{u(\xi_u-p)s}\nu^0_{W(\tilde F)}(z)\geq\frac{t}{us}\left(\frac{u}{\Delta_V}-(2n+1)(n+1)\delta_z\right)$$
$$\times\sum_{j=1}^q\max\{0,\nu^0_{\tilde Q_j(\mathbf{f})}(z)-n_u\}+O(\sum_{i=1}^{n_0}\nu_{R^i}(z)).$$

Integrating both sides of this inequality, we obtain

$$\Big\|\ \frac{(n+1)}{u(\xi_u-p)s}N_{W(\tilde F)}(r)\geq\frac{t}{s}\left(\frac{1}{\Delta_V}-\frac{(2n+1)(n+1)\delta}{u}\right)$$
$$\times\sum_{j=1}^q\left(N_{Q_j(f)}(r)-N^{[nu]}_{Q_j(f)}(r)\right)+o(T_f(r)).$$

Seting $m_0=\frac{1}{\Delta_V}-\frac{(2n+1)(n+1)\delta}{u}$ and combining inequalities (3.7) with the above inequality, we get

$$\Big\|\ \sum_{i=1}^q m_f(r,Q_i)\leq\frac{d(n+1)t}{sm_0}T_f(r)-\frac{t}{s}\sum_{j=1}^q\left(N_{Q_i(f)}(r)-N^{[nu]}_{Q_j(f)}(r)\right)$$
$$+\frac{(t(\xi_u-p)-1)t(n+1)}{2sm_0u}(c_{\tilde F}+\frac{\epsilon'}{2du})T_{\tilde F}(r)+o(T_f(r)).$$

This inequality implies that

$$\begin{aligned}&\|\ (q-\frac{(n+1)t}{sm_0})T_f(r)\\&\leq\sum_{j=1}^q\frac{1}{d}N^{[nu]}_{Q_j(f)}(r)+\frac{(t(\xi_u-p)-1)t(n+1)}{2sdm_0u}(c_f+\epsilon')T_f(r).\end{aligned}\tag{3.9}$$

a) We choose $u=\lceil 2\Delta_V(2n+1)(n+1)d^n\deg V(\Delta_V(n+1)+\epsilon)\epsilon^{-1}\rceil$. Then $u\geq\left\lceil\dfrac{\Delta_V(2n+1)(n+1)\delta(\Delta_V(n+1)+\epsilon)}{\Delta_V(n+1)+\epsilon-\frac{\Delta_V(n+1)t}{s}}\right\rceil$, and we have:

- $q-\dfrac{(n+1)t}{sm_0}\geq q-\dfrac{\Delta_V(n+1)t}{(1-\Delta_V(2n+1)(n+1)\delta/u)s}$
$\geq q-\Delta_V(n+1)-\epsilon$;
- $n_u+1=(\xi_u-p)t$
$\leq d^n\deg V(u+1)^n\left[\left(1+\frac{\epsilon}{2(n+1)\Delta_V}\right)^{\left[\frac{d^n\deg V(u+1)^{n+q}}{\log^2(1+\frac{\epsilon}{2(n+1)\Delta_V})}\right]+1}\right]=L;$

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

- $\dfrac{(t(\xi_u - p) - 1)t(n+1)}{2sdm_0u} < \dfrac{t(n+1)}{sm_0} \cdot \dfrac{(L-1)}{2du}$
$\leq \dfrac{(\Delta_V(n+1)+\epsilon)(L-1)}{2du}$.

Then, from (3.9) we have

$$\| \ (q - \Delta_V(n+1) - \epsilon)T_f(r)$$
$$\leq \sum_{j=1}^{q} \frac{1}{d} N^{[L-1]}_{Q_j(f)}(r) + \frac{(\Delta_V(n+1)+\epsilon)(c_f+\epsilon')(L-1)}{2du} T_f(r).$$

The assertion a) is proved.

b) If all Q_i are fixed hypersurfaces then $t = s = 1$. Choosing $u' = \lceil \Delta_V(2n+1)(n+1)d^n \deg V(\Delta_V(n+1)+\epsilon)\epsilon^{-1} \rceil$ and replacing u in the above by $u^{'}$, we have

- $q - \dfrac{(n+1)}{m_0} \geq q - \Delta_V(n+1) - \epsilon$,
- $n_{u'} + 1 = \xi_{u'} - p < \delta \begin{pmatrix} n+u' \\ n \end{pmatrix} \leq d^n \deg V \begin{pmatrix} n+u' \\ n \end{pmatrix}$
$\leq \left[d^n \deg V e^n \left(1 + \dfrac{u'}{n}\right)^n \right]$
$\leq \left[d^{n^2+n} (\deg V)^{n+1} e^n (2n+5)^n (\Delta_V^2 (n+1)\epsilon^{-1} + \Delta_V)^n \right] = L'$.

Similarly as above, from (3.9) we have the desired inequality of the assertion b).

Hence, the proof of the theorem is completed.□

Remark. For the case $\Delta(R) = \mathbb{C}$, we have $c_f = 0$ if f is non-constant. In this case, (3.7) can be re-written as follows:

$$\begin{aligned} \Big\| \ s \int_0^{2\pi} \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \frac{\|\mathbf{F}\|}{|L(\mathbf{F})|} \frac{d\theta}{2\pi} - N_{W(\tilde{F})}(r) \\ \leq \int_0^{2\pi} \max_{\mathcal{J} \subset \mathcal{L}} \log \prod_{L \in \mathcal{J}} \prod_{i=1}^{s} \frac{\|\tilde{\mathbf{F}}\|}{|b_i L(\mathbf{F})|} - N_{W(\tilde{F})}(r) + o(T_f(r)) \\ \leq t(\xi_u - p)udT_f(r) + o(T_f(r)). \end{aligned} \tag{3.10}$$

Here, the notation "$\|$" means that the inequality hold for all $r \in [1, +\infty)$ outside a set of finite measure.

Using (3.10) instead of (3.7), from the above proof, we obtain the following second main theorem for holomorphic maps from $\mathbb{C}$.

Theorem 3.11. *Let V be a smooth subvriety of dimension n of $\mathbb{P}^N(\mathbb{C})$. Let $f : \mathbb{C} \to V$ be a holomorphic mapping. Let $\mathcal{Q} = \{Q_1, \ldots, Q_q\}$ be a set of slowly (with respect to f) moving hypersurfaces with the distributive constant Δ_V with respect to V. Assume that*

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press

f is algebraically non-degenerate over $\mathcal{K}_{\mathcal{Q}}$. Let $d = lcm(\deg Q_1, \dots, \deg Q_q)$. Then for every $(n+1)\Delta_V > \epsilon > 0$,

$$\| \ (q - \Delta_V(n+1) - \epsilon)T_f(r) \leq \sum_{j=1}^{q} \frac{1}{\deg Q_j} N_f^{[L]}(r, Q_j), \tag{3.12}$$

where

$$L = d^n \deg V(u+1)^n \left[\left(1 + \frac{\epsilon}{2(n+1)\Delta_V}\right)^{\left[\frac{d^n \deg V(u+1)^{n+q}}{\log^2(1+\frac{\epsilon}{2(n+1)\Delta_V})}\right]+1} \right],$$

with $u = \lceil 2\Delta_V(2n+1)(n+1)d^n \deg V(\Delta_V(n+1)+\epsilon)\epsilon^{-1}\rceil$.

Moreover, if all Q_i $(1 \leq i \leq q)$ are assumed to be fixed hypersurfaces, then for every $\epsilon > 0$ we have the inequality (3.12) with $L = \left[d^{n^2+n}(\deg V)^{n+1}e^n(2n+5)^n(\Delta_V^2(n+1)\epsilon^{-1} + \Delta_V)^n\right]$.

We also note that our proof is valid for the case of holomorphic maps from higher dimension complex spaces $\mathbb{C}^m$ into V. This theorem is an improvement of many previous second main theorem for hypersurface targets, such as [1, 3, 4, 6, 7, 9].

Competing interests. No potential conflict of interest was reported by the author.

References

(1) G. Dethloff and T. V. Tan, Holomorphic curves into algebraic varities intersecting moving hypersurfaces targets, *Acta Math. Vietnamica*, **45**(1) (2020), 291–308.

(2) J. Evertse and R. Ferretti, Diophantine inequalities on projective varieties, *Int. Math. Res. Not.*, **2002**(25) (2002), 1295–1330.

(3) L. Giang, An explicit estimate on multiplicity truncation in the degenerated second main theorem, *Houston J. Math.*, **42**(2) (2016), 447–462.

(4) S. D. Quang, Generalizations of degeneracy second main theorem and Schmidt's subspace theorem, *Pacific J. Math.*, **318**(1) (2022), 153–188.

(5) S. D. Quang, An effective function field version of Schmidt's subspace theorem for projective varieties, with arbitrary families of homogenous polynomials, *J. Number Theory*, **241** (2022), 563–580.

(6) S. D. Quang, Meromorphic mappings into projective varieties with arbitrary families of moving hypersurfaces, *J. Geom. Anal.*, **32** (2022), 52.

(7) M. Ru, Holomorphic curves into algebraic varieties, *Ann. Math.*, **169**(1) (2009), 255–267.

(8) M. Ru and N. Sibony, The second main theorem in the hyperbolic case, *Math. Ann.*, **377**(1–2) (2020), 759–795.

(9) L. Shi, Degenerated second main theorem for holomorphic curves into algebraic varieties, *Int. J. Math.* **31**(06) (2020), 2050042.

(10) D. D. Thai and S. D. Quang, Cartan-Nochka theorem with truncated counting functions for moving targets, *Acta Math. Vietnamica*, **35**(1) (2010), 173–197.

https://doi.org/10.1017/S0013091524000774 Published online by Cambridge University Press