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BOUNDEDNESS OF SOME INTEGRAL OPERATORS

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ABSTRACT We apply the expression for the norm of a function in the weighted Lorentz space, with respect to the distribution function, to obtain as a simple consequence some weighted inequalities for integral operators

1. **Introduction.** Given a measure space $\mathcal{M}$ and a function $k\colon \mathcal{M} \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, we define the operator

$$T_k f(x) = \int_0^\infty k(x,t) f(t)\, dt.$$

The boundedness of this operator

$$T_k\colon L^{p_0}(w_0) \longrightarrow L^{p_1}(d\mu), \tag{1}$$

for nonincreasing functions, where w_0 is a nonnegative locally integrable function (that is a weight) in $\mathbb{R}^+$ and $d\mu$ is a measure on $\mathcal{M}$, has been widely studied for particular choices of the kernel k (see [1], [2], [6], [7], ...).

In particular, if $k(x,t) = x^{-1}a(tx^{-1})$, the weak boundedness of

$$T_k\colon L^{p_0}(w_0) \longrightarrow L^{p_1,\infty}(w_1),$$

with w_1 a weight in $\mathbb{R}^+$, has been completely solved by K. Andersen in [1]. If a satisfies some extra condition, he also gets the strong boundedness of the operator $T_k\colon L^p(w) \longrightarrow L^p(w)$. A related work can be found in [5] where the authors consider the boundedness of a particular case of the operator T_k, with $k(x,t) = \chi_{[0,x]}(t)\varphi(t/x)$ but with no monotone restriction on the functions f.

In [7], E. Sawyer solved question (1), for $1 < p_0, p_1$, via the study of T_k^* whenever this operator can be easily identified and its boundedness easily studied. His argument is based upon a duality type result for nonincreasing functions (see Theorem 3.1). Results about particular cases of operators T_k have many other proofs (see [2], [6], ...).

Our point of view consists mainly in studying this type of question as a consequence of the boundedness of an operator T associated to T_k in the weighted Lorentz spaces. To be precise, let $\Lambda^p_\sigma(w)$ be the space of all measurable functions f on a measure space $\mathcal{N}$ such that $\|f\|_{\Lambda^p_\sigma(w)} = \left(\int_0^\infty \left(f^*_\sigma(x)\right)^p w(x)\, dx\right)^{1/p} < +\infty$, where σ is a σ-finite measure on $\mathcal{N}$ and w is a locally integrable function (that is, a weight) and f^*_σ denotes the rearrangement

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function with respect to the measure $d\sigma$. Then, we try to characterize the measures σ_j and the weights w_j such that $T\colon \Lambda^{p_0}_{\sigma_0}(w_0) \longrightarrow \Lambda^{p_1}_{\sigma_1}(w_1)$, or $T\colon \Lambda^{p_0}_{\sigma_0}(w_0) \longrightarrow \Lambda^{p_1,\infty}_{\sigma_1}(w_1)$ are bounded, where the weak space $\Lambda^{p,\infty}_{\sigma}(w)$ is defined as in [4], namely

$$\|f\|_{\Lambda^{p,\infty}_{\sigma}(w)} = \sup_{y>0} y\Big(\int_0^{\lambda^\sigma_f(y)} w(t)\,dt\Big)^{1/p} < +\infty.$$

Given a σ-finite measure σ on $\mathcal{N}$, we shall denote by $\sigma(A) = \int_A d\sigma(x)$ and $\lambda^\sigma_f(y) = \sigma\big(\{x : |f(x)| > y\}\big)$. When $d\sigma(x) = u(x)\,dx$, we shall write $u(A)$, λ^u_f and f^*_u respectively. Finally, if we are working with the Lebesgue measure, u is omitted and we simply write $|A|$, λ_f or f^*. We shall write $L^p_{\mathrm{dec}}(w)$ to denote the set of all nonincreasing functions in $L^p(w)$. The expression $f \approx g$ will indicate the existence of two positive constants a and b such that $af \leq g \leq bf$, and constants such as C may change from one occurrence to the next.

The paper is organized as follows. In Section 2, the boundedness of the operator $T_k\colon L^{p_0}_{\mathrm{dec}}(w_0) \longrightarrow L^{p_1}(w_1)$ is completely solved in the range $0 < p_0 \leq 1$, $p_0 \leq p_1$ as a consequence of a more general result (see Theorem 2.4 and Proposition 2.5). In Section 3, we study the weak boundedness of $T_k\colon \Lambda^{p_0}_{u_0}(w_0) \longrightarrow \Lambda^{p_1,\infty}_{u_1}(w_1)$ whenever T_k satisfies a weak monotone property condition. Also, if T_kf is a nonincreasing function for f nonincreasing, then we get a characterization of the boundedness of the operator $T_k\colon L^{p_0}_{\mathrm{dec}}(w_0) \longrightarrow L^{p_1,\infty}(w_1)$ which gives another proof of the result of K. Andersen we mentioned above. In Section 4, we finish with a very simple proof of the boundedness of the generalized Hardy operator and its generalized conjugate operator, for the case $L^p(w)$. This proof is closely related to the (also very simple) proof of Neugebauer for the Hardy operator (see [6]).

2. Case $0 < p_0 \leq 1$.

In [4], the following formula using the distribution function was proved.

THEOREM 2.1. *Let $(\mathcal{N}, \sigma)$ be a measure space and w a weight in $\mathbb{R}^+$. Then, for $0 < p < \infty$, we get*

$$\int_0^\infty \big(f^*_\sigma(t)\big)^p w(t)\,dt = p\int_0^\infty y^{p-1}\Big(\int_0^{\lambda^\sigma_f(y)} w(t)\,dt\Big)\,dy.$$

To prove it, it suffices to check it for simple functions.

It is trivial to show that for particular choices of k we can obtain both the Hardy operator $Sf(x) = x^{-1}\int_0^x f(t)\,dt$ and its conjugate $\tilde{S}f(x) = \int_x^\infty f(t)t^{-1}\,dt$. Then, a first application of Theorem 2.1 is given by the following result.

COROLLARY 2.2. *(i) If f is a nonincreasing function, $\int_0^\infty k(x,t)f(t)\,dt = \int_0^\infty \int_0^{\lambda_f(y)} k(x,t)\,dt\,dy$.*

(ii) $S(f^*_\sigma)(x) = \int_0^\infty \min\big(1, \lambda^\sigma_f(y)/x\big)\,dx$.

(iii) $\tilde{S}(f^*_\sigma)(x) = \int_0^\infty \log^+\big(\lambda^\sigma_f(y)/x\big)\,dy$.

By standard arguments using a dyadic decomposition, one can easily obtain the following discretization formula, (see [4]).

COROLLARY 2.3. *For every measurable function f in $\Lambda^p_\sigma(w)$, and $0 < p \leq \infty$,*

$$\|f\|_{\Lambda^p_\sigma(w)} \approx \Big(\sum_{k=-\infty}^{+\infty} 2^{kp}\Big(\int_0^{\lambda^\sigma_f(2^k)} w(t)\,dt\Big)\Big)^{1/p}.$$

The main result of this section is the following:

THEOREM 2.4. *Let $(\mathcal{N}, d\sigma)$ and $(\mathcal{M}, d\mu)$ be two σ-finite measure spaces. Given a measurable function f in $\mathcal{N}$, we can define $Tf(x) = T_k(f^*_\sigma)(x)$ for every $x \in \mathcal{M}$. Let σ_0 be another σ-finite measure in $\mathcal{N}$ and w_0 a weight in $\mathbb{R}^+$. Then, if $0 < p_0 \leq 1$ and $p_0 \leq p_1$, the operator $T\colon \Lambda^{p_0}_{\sigma_0}(w_0) \longrightarrow L^{p_1}(d\mu)$ is bounded if and only if, there exists a constant $C > 0$ such that*

$$\Big(\int_{\mathcal{M}}\Big(\int_0^{\sigma(A)} k(x,t)\,dt\Big)^{p_1} d\mu(x)\Big)^{1/p_1} \leq C\Big(\int_0^{\sigma_0(A)} w_0(x)\,dx\Big)^{1/p_0}, \tag{2}$$

for all measurable sets A in $\mathcal{N}$.

PROOF. To prove the necessity condition, it is enough to apply the hypothesis to the characteristic function $f = \chi_A$.

Conversely, condition (2) implies that

$$\Big(\int_{\mathcal{M}}\Big(\int_0^{\lambda^\sigma_f(y)} k(x,t)\,dt\Big)^{p_1} d\mu(x)\Big)^{1/p_1} \leq C\Big(\int_0^{\lambda^{\sigma_0}_f(y)} w_0(x)\,dx\Big)^{1/p_0}.$$

Then, if $p_1 \leq 1$, we get using Theorem 2.1 and Corollary 2.3, that

$$\begin{aligned}\Big(\int_{\mathcal{M}}\big(Tf(x)\big)^{p_1} d\mu(x)\Big)^{1/p_1} &= \Big(\int_{\mathcal{M}}\Big(\int_0^\infty \int_0^{\lambda^\sigma_f(y)} k(x,t)\,dt\,dy\Big)^{p_1} d\mu(x)\Big)^{1/p_1}\\ &\leq \Big(\int_{\mathcal{M}}\Big(\int_0^\infty y^{p_1-1}\Big(\int_0^{\lambda^\sigma_f(y)} k(x,t)\,dt\Big)^{p_1} dy\Big)\,d\mu(x)\Big)^{1/p_1}\\ &\leq C\Big(\int_0^\infty y^{p_1-1}\Big(\int_0^{\lambda^{\sigma_0}_f(y)} w_0(x)\,dx\Big)^{p_1/p_0} dy\Big)^{1/p_1}.\end{aligned}$$

Finally, since $p_0/p_1 \leq 1$, using again Corollary 2.3, we get

$$\Big(\int_{\mathcal{M}}\big(Tf(x)\big)^{p_1} d\mu(x)\Big)^{1/p_1} \leq C\Big(\int_0^\infty y^{p_0-1}\int_0^{\lambda^{\sigma_0}_f(y)} w_0(x)\,dx\,dy\Big)^{1/p_0} \approx \|f\|_{\Lambda^{p_0}_{\sigma_0}(w_0)}.$$

Now, for $p_1 > 1$,

$$Tf(x) = T_k(f^*_\sigma)(x) = \int_0^\infty \int_0^{\lambda^\sigma_f(y)} k(x,t)\,dt\,dy.$$

Hence, by Minkowski integral inequality and the hypothesis,

$$\begin{aligned}\|Tf\|_{L^{p_1}(d\mu)} &\leq \int_0^\infty \Big\|\int_0^{\lambda^\sigma_f(y)} k(\cdot,t)\,dt\Big\|_{L^{p_1}(d\mu)} dy\\ &\leq C\int_0^\infty \Big(\int_0^{\lambda^{\sigma_0}_f(y)} w_0(x)\,dx\Big)^{1/p_0} dy.\end{aligned}$$

Finally, since $p_0 \leq 1$, we get, by Corollary 2.3,

$$\|Tf\|^{p_0}_{L^{p_1}(d\mu)} \leq C\int_0^\infty y^{p_0-1}\Big(\int_0^{\lambda_f^{\sigma_0}(y)} w_0(x)\,dx\Big)\,dy = C\|f\|^{p_0}_{\Lambda^{p_0}_{\sigma_0}(w_0)}. \qquad ■$$

PROPOSITION 2.5. *Let w_0 and w_1 be two weights in $\mathbb{R}^+$, $0 < p_0 \leq 1$ and $p_0 \leq p_1$. Then, the operator $T_k\colon L^{p_0}_{\mathrm{dec}}(w_0) \longrightarrow L^{p_1}(w_1)$ is bounded, if and only if, for every $r > 0$,*

$$\Big(\int_0^\infty\Big(\int_0^r k(x,t)\,dt\Big)^{p_1} w_1(x)\,dx\Big)^{1/p_1} \leq C\Big(\int_0^r w_0(x)\,dx\Big)^{1/p_0}.$$

PROOF. To prove the necessity condition, it is enough to apply the hypothesis to the characteristic function $f = \chi_{(0,r)}$. Conversely, by Theorem 2.4, with both σ and σ_0 equals the Lebesgue measure, and $d\mu(x) = w_1(x)\,dx$, we obtain that $Tf = T_k f$ for every nonincreasing function and $T\colon \Lambda^{p_0}(w_0) \longrightarrow L^{p_1}(w_1)$. It now remains to observe that $L^{p_0}_{\mathrm{dec}}(w_0)$ is a subspace of $\Lambda^{p_0}(w_0)$. ■

PROPOSITION 2.6. *Let u_0, u_1 be two weights in $\mathbb{R}^n$ and w_0, w_1 two weights in $\mathbb{R}^+$. Then, if $0 < p_0 \leq 1$ and $p_0 \leq p_1$,*

(a)

$$\Big(\int_0^\infty\Big(\frac{1}{x}\int_0^x f^*_{u_1}(s)\,ds\Big)^{p_1} w_1(x)\,dx\Big)^{1/p_1} \leq C\Big(\int_0^\infty f^*_{u_0}(x)^{p_0} w_0(x)\,dx\Big)^{1/p_0},$$

if and only if,

$$\Big(\int_0^{u_1(A)} w_1(x)\,dx + u_1(A)^{p_1}\int_{u_1(A)}^\infty \frac{w_1(x)}{x^{p_1}}\,dx\Big)^{1/p_1} \leq C\Big(\int_0^{u_0(A)} w_0(x)\,dx\Big)^{1/p_0},$$

for every measurable set $A \subset \mathbb{R}^n$.

(b) The Hardy operator S is bounded from $L^{p_0}_{\mathrm{dec}}(w_0)$ into $L^{p_1}(w_1)$ if and only if,

$$\Big(\int_0^r w_1(x)\,dx + r^{p_1}\int_r^\infty \frac{w_1(x)}{x^{p_1}}\,dx\Big)^{1/p_1} \leq C\Big(\int_0^r w_0(x)\,dx\Big)^{1/p_0},$$

for every $r > 0$.

PROOF. It suffices to consider $\mathcal{M} = \mathbb{R}^+$, $\mathcal{N} = \mathbb{R}^n$, $d\mu(x) = w_1(x)\,dx$, $d\sigma(x) = u_1(x)\,dx$, $d\sigma_0(x) = u_0(x)\,dx$ and $k(x,t) = x^{-1}\chi_{[0,x]}(t)$ in Theorem 2.4. ■

REMARK 2.7. If Mf is the Hardy-Littlewood maximal function of f and using the fact that $(Mf)^*(x) \approx S(f^*)(x)$, the above proposition gives a characterization of the weights u_0, w_0 and w_1 for which M is bounded from $\Lambda^{p_0}_{u_0}(w_0)$ into $\Lambda^{p_1}(w_1)$, for $0 < p_0 \leq 1$ and $p_0 \leq p_1$. If $p \geq 1$, the characterization of the boundedness of M in $\Lambda^p(w)$ was first given by Ariño and Muckenhoupt in [2]. In the case $1 < p_0, p_1$ and $u_0 = 1$, the boundedness of M from $\Lambda^{p_0}(w_0)$ into $\Lambda^{p_1}(w_1)$ was proved by E. Sawyer in [7].

PROPOSITION 2.8. *Let u_0, u_1 be two weights in $\mathbb{R}^n$ and w_0, w_1 two weights in $\mathbb{R}^+$. Then, if $0 < p_0 \leq 1$ and $p_0 \leq p_1$,*

(a)

$$\left(\int_0^\infty \left(\int_x^\infty f_{u_1}^*(s)\,\frac{ds}{s}\right)^{p_1} w_1(x)\,dx\right)^{1/p_1} \leq C\left(\int_0^\infty f_{u_0}^*(x)^{p_0} w_0(x)\,dx\right)^{1/p_0},$$

if and only if,

$$\left(\int_0^\infty \left(\log^+\left(\frac{u_1(A)}{x}\right)\right)^{p_1} w_1(x)\,dx\right)^{1/p_1} \leq C\left(\int_0^{u_0(A)} w_0(x)\,dx\right)^{1/p_0},$$

for every measurable set $A \subset \mathbb{R}^n$.

(b) The conjugate Hardy operator $\tilde{S}$ is bounded from $L^{p_0}_{\mathrm{dec}}(w_0)$ into $L^{p_1}(w_1)$ if and only if,

$$\left(\int_0^\infty \left(\log^+\left(\frac{r}{x}\right)\right)^{p_1} w_1(x)\,dx\right)^{1/p_1} \leq C\left(\int_0^r w_0(x)\,dx\right)^{1/p_0},$$

for every $r > 0$.

PROOF. It suffices to consider $\mathcal{M} = \mathbb{R}^+$, $\mathcal{N} = \mathbb{R}^n$, $d\mu(x) = w_1(x)\,dx$, $d\sigma(x) = u_1(x)\,dx$ and $k(x,t) = t^{-1}\chi_{[x,\infty]}(t)$ in Theorem 2.4. ∎

Another easy application of Theorem 2.4 is the boundedness of the Calderón operator. Recall that for $1 \leq r_0 < r_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$, $q_0 \neq q_1$ and $m = (1/q_0 - 1/q_1)/(1/r_0 - 1/r_1)$ the Calderón operator is defined by

$$Sf(t) = t^{-1/q_0} \int_0^{t^m} s^{1/r_0} f(s)\,\frac{ds}{s} + t^{-1/q_1} \int_{t^m}^\infty s^{1/r_1} f(s)\,\frac{ds}{s}.$$

This operator plays an important role in the theory of rearrangement invariant spaces (see [3]). Let us write S^1 for the first integral term and S^2 for the second one, so that $S = S^1 + S^2$.

PROPOSITION 2.9. *Let $(\mathcal{N}_0, \sigma_0)$ and $(\mathcal{N}_1, \sigma_1)$ be two σ-finite measure spaces. Then, if $0 < p_0 \leq 1$ and $p_0 \leq p_1$, we get that*

(a)

$$\|S^1(f_{\sigma_1}^*)\|_{L^{p_1}(w_1)} \leq C\|f\|_{\Lambda^{p_0}_{\sigma_0}(w_0)},$$

if and only if,

$$\left(\int_0^{\sigma_1(A)^{1/m}} x^{(m/r_0 - 1/q_0)p_1} w_1(x)\,dx + \sigma_1(A)^{p_1/r_0} \int_{\sigma_1(A)^{1/m}}^\infty x^{-p_1/q_0} w_1(x)\,dx\right)^{1/p_1}$$
$$\leq C\left(\int_0^{\sigma_0(A)} w_0(x)\,dx\right)^{1/p_0},$$

for every measurable set $A \subset \mathcal{N}_1$.

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(b)

$$\|S^2(f^*_{\sigma_1})\|_{L^{p_1}(w_1)} \le C\|f\|_{\Lambda^{p_0}_{\sigma_0}(w_0)},$$

if and only if,

$$\Big(\int_0^{\sigma_1(A)^{1/m}} x^{-p_1/q_1}(\sigma_1(A)-x^m)^{p_1/r_1} w_1(x)\,dx\Big)^{1/p_1} \le C\Big(\int_0^{\sigma_0(A)} w_0(x)\,dx\Big)^{1/p_0},$$

for every measurable set $A \subset \mathcal{N}_1$.

PROOF. (a) It suffices to consider $\mathcal{M} = \mathbb{R}^+$, $\mathcal{N} = N_1$, $d\mu(x) = w_1(x)\,dx$, $d\sigma(x) = d\sigma_1(x)$ and $k(x,t) = x^{-1/q_0}t^{1/r_0-1}\chi_{[0,x^m]}(t)$ in Theorem 2.4.

(b) It suffices to consider $k(x,t) = x^{-1/q_1}t^{1/r_1-1}\chi_{[x^m,\infty)}(t)$ and $\mathcal{M}$, $\mathcal{N}$, $d\mu$, $d\sigma$, $d\sigma_0$ as in (a), in Theorem 2.4. ∎

3. **Some results in the case $p_0 > 0$.** The following result is due to E. Sawyer (see [7]) and it will be used very often in what follows.

THEOREM 3.1. *Suppose $1 < p < +\infty$ and that $v(x)$ and $g(x)$ are nonnegative measurable functions on $\mathbb{R}^+$, with v locally integrable. Then*

(3)
$$\sup \frac{\int_0^\infty f(x)g(x)\,dx}{\big(\int_0^\infty f(x)^p v(x)\,dx\big)^{1/p}} \approx \Big(\int_0^\infty \Big(\int_0^x g\Big)^{p'} v(x)\Big(\int_0^x v(t)\,dt\Big)^{-p'}dx\Big)^{1/p'} + \frac{\int_0^\infty g(t)\,dt}{\big(\int_0^\infty v(t)\,dt\big)^{1/p}},$$

where the supremum is taken over all nonnegative and nonincreasing functions f.

Moreover, the right side of (3) can be replaced with the integral

$$\Big(\int_0^\infty \Big(\int_0^x g(t)\,dt\Big)^{p'-1}\Big(\int_0^x v(t)\,dt\Big)^{1-p'} g(x)\,dx\Big)^{1/p'}.$$

Using the ideas developed in [1], we can give an easy proof for the $\le$ inequality.

PROOF. For the $\ge$ inequality we have to consider the function

$$f(x) = \Big(\int_x^{+\infty} \frac{g(t)}{\int_0^t v(s)\,ds}\,dt\Big)^{p'-1},$$

(see [7]). Conversely, set

$$h(t) = \left(\int_t^\infty \Big(\int_0^x g(s)\,ds\Big)^{p'-1}\Big(\int_0^x v(s)\,ds\Big)^{-p'} v(x)\,dx + \left(\frac{\int_0^\infty g(s)\,ds}{\int_0^\infty v(s)\,ds}\right)^{p'-1}\right)^{1/p'}.$$

Then,

$$\begin{aligned}\int_0^\infty f(x)g(x)\,dx &= \int_0^\infty f(x)g(x)h(x)h(x)^{-1}\,dx \\ &\le \Big(\int_0^\infty f^p(x)h^{-p}(x)g(x)\,dx\Big)^{1/p}\Big(\int_0^\infty h^{p'}(x)g(x)\,dx\Big)^{1/p'}.\end{aligned}$$

Applying Fubini to the second factor in the previous inequality we obtain the right hand side of (3). For the first factor, we observe that

$$h(x)^{-p} \leq \Big(\int_0^x g(t)\,dt\Big)^{-1}\Big(\int_0^x v(t)\,dt\Big)$$

and thus, by Theorem 2.1,

$$\int_0^\infty f^p(x)h^{-p}(x)g(x)\,dx = p\int_0^\infty y^{p-1}\int_0^{\lambda_f(y)} h^{-p}(x)g(x)\,dx\,dy.$$

Integrating by parts the inner integral and erasing the negative terms one has that the previous expression can be bounded, up to multiplicative constants, by

$$\int_0^\infty y^{p-1}\Big(\int_0^{\lambda_f(y)} g(x)\,dx\Big)h^{-p}\big(\lambda_f(y)\big)\,dy$$
$$\leq \int_0^\infty y^{p-1}\Big(\int_0^{\lambda_f(y)} v(x)\,dx\Big)\,dy \approx \int_0^\infty f^p(x)v(x)\,dx. \quad \blacksquare$$

Using this result, E. Sawyer proves that if $\int_0^\infty w(x)\,dx = +\infty$, then the dual space of $\Lambda_u^p(w)$ can be identified with the space $\Gamma_u^{p'}(\tilde{w})$, defined by the norm

$$\|f\|_{\Gamma_u^{p'}(\tilde{w})} = \Big(\int_0^\infty\Big(\frac{1}{x}\int_0^x f_u^*(s)\,ds\Big)^{p'}\tilde{w}(x)\,dx\Big)^{1/p'} < +\infty,$$

where $\tilde{w}(x) = \big(x^{-1}\int_0^x w(t)\,dt\big)^{-p'}w(x)$.

For $p \leq 1$, we also have (see [4]) the following result.

THEOREM 3.2. *Suppose $p \leq 1$ and that $v(x)$ and $g(x)$ are nonnegative measurable functions on $\mathbb{R}^+$ with v locally integrable. Then*

$$\sup \frac{\int_0^\infty f(x)g(x)\,dx}{\big(\int_0^\infty f(x)^p v(x)\,dx\big)^{1/p}} \approx \sup_{r>0}\Big(\int_0^r g(x)\,dx\Big(\int_0^r v(x)\,dx\Big)^{-1/p}\Big),$$

where the supremum is taken over all nonnegative and nonincreasing functions f.

The first inmediate consequence is the following.

THEOREM 3.3. *Let $T_k f(x) = \int_0^\infty k(x,t)f(t)\,dt$ and let us assume that $T_k f$ is a nonincreasing function whenever f is a nonincreasing function. Then, the operator $T_k\colon L^{p_0}_{\mathrm{dec}}(w_0) \longrightarrow \Lambda^{p_1,\infty}(w_1)$ is bounded if and only if,*

(a) if $p_0 > 1$,

$$\sup_{z>0}\Bigg(\Big(\int_0^\infty\Big(\int_0^y k(z,t)\,dt\Big)^{p_0'}\Big(\int_0^y w_0(t)\,dt\Big)^{-p_0'}w_0(y)\,dy\Big)^{1/p_0'}$$
$$+\int_0^\infty k(z,t)\,dt\Big(\int_0^\infty w_0(s)\,ds\Big)^{-1/p_0}\Bigg)\Big(\int_0^z w_1(s)\,ds\Big)^{1/p_1} < +\infty,$$

(b) if $p_0 \leq 1$,

$$\sup_{z>0}\left(\sup_{r>0}\left(\int_0^r k(z,x)\,dx\right)\left(\int_0^r w_0(x)\,dx\right)^{-1/p_0}\right)\left(\int_0^z w_1(s)\,ds\right)^{1/p_1} < +\infty$$

PROOF Observe that to show that $\sup_{y>0} y\left(\int_0^{\lambda_{T_kf}(y)} w_1(x)\,dx\right)^{1/p_1} \leq \|f\|_{L^{p_0}(w_0)}$ it is enough to consider values of y equals $T_kf(z)$ for all z, and thus, we have to see that

$$\sup_{z>0} T_kf(z)\left(\int_0^z w_1(x)\,dx\right)^{1/p_1} \leq C\|f\|_{L^{p_0}(w_0)},$$

for all nonincreasing f This is equivalent to showing that

$$\sup_{z>0}\left(\left(\sup_{f\in L^{p_0}_{\text{dec}}(w_0)} \frac{T_kf(z)}{\|f\|_{L^{p_0}(w_0)}}\right)\left(\int_0^z w_1(x)\,dx\right)^{1/p_1}\right) \leq C < +\infty,$$

and Theorems 3 1 and 3 2, lead us to the conclusion ∎

In particular, if $k(x,t) = x^{-1}a(tx^{-1})$ this was proved by K Andersen (see [1]) The following two results give us the weak boundedness of an integral operator when it satisfies a monotone type condition

THEOREM 3 4 *Let* $T_kf(x) = \int_0^\infty k(x,t)f(t)\,dt$ *and let us assume that, for every* x, *there exists a measurable set* I_x *of positive measure such that* $T_kf(x) \leq T_kf(t)$, *for every* $t \in I_x$ *and every* f, *and if* $T_kf(x) < T_kf(t)$ *for some* f, *then* $t \in I_x$ *Then,* $T_k\ \Lambda^{p_0}_{u_0}(w_0) \longrightarrow \Lambda^{p_1\,\infty}_{u_1}(w_1)$ *is bounded if and only if,*

(a) if $p_0 > 1$,

$$\sup_{z>0}\left(\left(\|u_0^{-1}k(z,\ )\|_{\Gamma^{p_0'}_{u_0}(w_0)} + \int_0^\infty k(z,t)\,dt\left(\int_0^\infty w_0(s)\,ds\right)^{1/p_0}\right)\left(\int_0^{u_1(I_z)} w_1(t)\,dt\right)^{1/p_1}\right) < +\infty,$$

(b) if $p_0 \leq 1$,

$$\sup_{z>0}\left(\sup_{r>0}\left(\int_0^r k(z,t)\,dt\right)\left(\int_0^r w_0(t)\,dt\right)^{-1/p_0}\left(\int_0^{u_1(I_z)} w_1(t)\,dt\right)^{1/p_1}\right) < +\infty$$

PROOF Let $f \in \Lambda^{p_0}_{u_0}(w_0)$ and assume that $f \geq 0$ Then, for every $t \in I_x$,

$$\int_0^\infty k(x,s)f(s)\,ds \leq T_kf(t)$$

and, hence, if we write $\xi < \int_0^\infty k(x,s)f(s)\,ds$, we get

$$u_1(I_x) \leq \int_{\{t\ T_kf(t)>\xi\}} u_1(s)\,ds = \lambda^{u_1}_{T_kf}(\xi)$$

Then,

$$\xi\Big(\int_0^{u_1(I_x)} w_1(s)\,ds\Big)^{1/p_1} \leq \xi\Big(\int_0^{\lambda^{u_1}_{T_kf}(\xi)} w_1(s)\,ds\Big)^{1/p_1}$$
$$\leq \sup_{y>0} y\Big(\int_0^{\lambda^{u_1}_{T_kf}(y)} w_1(s)\,ds\Big)^{1/p_1} = \|T_kf\|_{\Lambda^{p_1\infty}_{u_1}(w_1)} \leq C\|f\|_{\Lambda^{p_0}_{u_0}(w_0)}.$$

Therefore, taking the supremum over all $\xi < \big(\int_0^\infty k(x,s)f(s)\,ds\big)$, we get

$$\sup_{x>0}\Bigg(\sup_f \frac{\int_0^\infty k(x,s)f(s)\,ds}{\|f\|_{\Lambda^{p_0}_{u_0}(w_0)}}\Bigg)\Big(\int_0^{u_1(I_x)} w_1(s)\,ds\Big)^{1/p_1} < \infty,$$

and we get the conclusion by Theorems 3.1 and 3.2.

Conversely, we shall only prove (a) (the proof of (b) is entirely analogous). Let $f \geq 0$ in $\Lambda^{p_0}_{u_0}(w_0)$ and set x_j such that if $\sup_{x>0}\int_0^\infty k(x,s)f(s)\,ds > 2^k$, then $T_kf(x_j) = 2^j$. Then, since,

$$\|T_kf\|^{p_1}_{\Lambda^{p_1\infty}_{u_1}(w_1)} \leq C\sup_{j\in\mathbb{Z}} 2^{jp_1}\int_0^{\lambda_{T_kf}(2^j)} w_1(s)\,ds,$$

and

$$\lambda_{T_kf}(2^j) = \int_{\{x\,(T_kf)(x)>2^j\}} u_1(x)\,dx \leq u_1(I_{x_j}),$$

we get,

$$\|T_kf\|^{p_1}_{\Lambda^{p_1\infty}_{u_1}(w_1)} \leq C\sup_{j\in\mathbb{Z}}\Big(\int_0^\infty k(x_j,t)f(t)\,dt\Big)^{p_1}\Big(\int_0^{u_1(I_{x_j})} w_1(s)\,ds\Big)$$
$$\leq C\sup_{j\in\mathbb{Z}}\|f\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}\Bigg(\|k(x_j,\cdot)u_0^{-1}\|_{\Gamma^{p_0'}_{u_0}(\tilde{w}_0)}$$
$$+\int_0^\infty k(x_j,t)\,dt\Big(\int_0^\infty w_0(s)\,ds\Big)^{-1/p_0}\Bigg)\Big(\int_0^{u_1(I_{x_k})} w_1(s)\,ds\Big)$$
$$\leq C\sup_{j\in\mathbb{Z}}\|f\|^{p_1}_{\Lambda^{p_0}_{u_0}(w_0)}. \qquad \blacksquare$$

The following result will give us the strong boundedness of the operator T_k for a particular choice of k. As a consequence, we partially obtain a result of E. Sawyer (see [7]).

THEOREM 3.5. *Let $1 < p_0 \leq p_1$ and let $k(x,t) = \chi_{[0,x]}(t)\phi(t)$, where ϕ is a nonincreasing locally integrable function in $\mathbb{R}^+$. Then, if w_0 is a nondecreasing weight in $\mathbb{R}^+$, the operator T_k: $L^{p_0}_{\mathrm{dec}}(w_0) \longrightarrow L^{p_1}(w_1)$ is bounded if and only if,*

$$\sup_{z>0}\Bigg(\Bigg(\int_0^\infty\Big(\int_0^{\min(x,z)}\phi(t)\,dt\Big)^{p_0'}\frac{w_0(x)}{\big(\int_0^x w_0(t)\,dt\big)^{p_0'}}\,dx\Bigg)^{1/p_0'}$$
$$+\int_0^z\phi(t)\,dt\Big(\int_0^\infty w_0\Big)^{-1/p_0}\Bigg)\Big(\int_z^\infty w_1(t)\,dt\Big)^{1/p_1} < +\infty.$$

PROOF. To prove the necessary condition, we observe that $k(x,t)$ satisfies the hypothesis of the previous theorem and since $L^{p_1,\infty}(w_1) = \Lambda_{w_1}^{p_1,\infty}(1)$ we get the result as in Theorem 3.4. Conversely, we proceed as in Theorem 3.4, but in this case we observe that $2^j \approx \int_{x_{j-1}}^{x_j} f(s)\phi(s)\,ds$ and hence if we call $f_j = f\chi_{(0,x_j-x_{j-1})}$, we get

$$\begin{aligned}
\|Tf\|_{L^{p_1}(w_1)}^{p_1} &\le C\sum_{j\in\mathbb{Z}}\Big(\int_{x_{j-1}}^{x_j} f(s)\phi(s)\,ds\Big)^{p_1}\Big(\int_{x_j}^{\infty} w_1(s)\,ds\Big)\\
&\le C\sum_{j\in\mathbb{Z}}\Big(\int_0^{x_j-x_{j-1}} f(s+x_{j-1})\phi(s)\,ds\Big)^{p_1}\Big(\int_{x_j}^{\infty} w_1(s)\,ds\Big)\\
&\le C\sum_{j\in\mathbb{Z}}\|f_j\|_{L^{p_0}(w_0)}^{p_1}\left(\left(\int_0^\infty\Big(\int_0^{\min(x,x_j)}\phi(t)\,dt\Big)^{p_0'}\frac{w_0(x)}{\big(\int_0^x w_0(t)\,dt\big)^{p_0'}}\,dx\right)^{1/p_0'}\right.\\
&\qquad\left.+\int_0^{x_j}\phi(t)\,dt\Big(\int_0^\infty w_0\Big)^{-1/p_0}\right)\Big(\int_{x_j}^\infty w_1(s)\,ds\Big)\\
&\le C\sum_{j\in\mathbb{Z}}\|f_j\|_{L^{p_0}(w_0)}^{p_1} \le C\sum_{j\in\mathbb{Z}}\Big(\int_{x_{j-1}}^{x_j} f(s)w_0(s-x_{j-1})\,ds\Big)^{p_1/p_0}\\
&\le C\sum_{j\in\mathbb{Z}}\Big(\int_{x_{j-1}}^{x_j} f(s)w_0(s)\,ds\Big)^{p_1/p_0} \le C\|f\|_{L^{p_0}(w_0)}^{p_1}. \qquad \blacksquare
\end{aligned}$$

The same proof works for $k(x,t) = \chi_{[x,\infty)}\phi(t)$, $k(x,t) = \chi_{[0,x^m]}\phi(t)$, $k(x,t) = \chi_{[x^m,\infty)}\phi(t)$ or in general for $k(x,t) = \chi_{[\varphi(x),\infty)}(t)\phi(t)$ and $k(x,t) = \chi_{[0,\varphi(x)]}(t)\phi(t)$ for every monotone function φ. Therefore, this can be applied to the Calderón operator. The corresponding results for $p_0 \le 1$ follow from Proposition 2.5.

4. **Generalized Hardy operator.** The following results are well known and they have been proved by several authors in many different ways (see [2], [6], [1]). We give, however, a quite simple proof using Theorem 2.1.

THEOREM 4.1. *Let $p > 1$ and $\Phi(x) = \int_0^x \phi(t)\,dt$. Then, the generalized Hardy operator*

$$S_\phi f(x) = \frac{1}{\Phi(x)}\int_0^x f(t)\phi(t)\,dt,$$

satisfies that

$$\|S_\phi f\|_{L^p(w)} \le C\|f\|_{L^p(w)}$$

for all f nonincreasing if and only if,

$$\Phi(r)^p\int_r^\infty \frac{w(x)}{\Phi(x)^p}\,dx \le C\int_0^r w(x)\,dx.$$

PROOF. To prove the necessary condition one just has to apply the hypothesis to $f = \chi_{(0,r)}$.

Conversely, let us observe that

$$\Big(\int_0^x f(t)\phi(t)\,dt\Big)^p = p\int_0^x \Big(\int_0^t f(s)\phi(s)\,ds\Big)^{p-1} f(t)\phi(t)\,dt$$
$$= p\int_0^x \Big(\frac{1}{\Phi(t)}\int_0^t f(s)\phi(s)\,ds\Big)^{p-1} f(t)\Phi(t)^{p-1}\phi(t)\,dt.$$

Let us write $g(t) = \big(\frac{1}{\Phi(t)}\int_0^t f(s)\phi(s)\,ds\big)^{p-1} f(t)$. Hence,

$$\|S_\phi f\|_{L^p(w)} = p^{1/p}\Big(\int_0^\infty \Big(\int_0^x g(t)\Phi(t)^{p-1}\phi(t)\,dt\Big)\frac{w(x)}{\Phi(x)^p}\,dx\Big)^{1/p}.$$

Now, since g is a nonincreasing function we get by Theorem 2.1,

$$\int_0^x g(t)\Phi(t)^{p-1}\phi(t)\,dt = \int_0^\infty \int_0^{\lambda_g(y)} \chi_{(0,x)}(t)\Phi(t)^{p-1}\phi(t)\,dt\,dy$$
$$= \frac{1}{p}\int_0^\infty \Phi\Big(\min\big(\lambda_g(y),x\big)\Big)^p\,dy.$$

Therefore,

$$\|S_\phi f\|^p_{L^p(w)} = \int_0^\infty \int_0^\infty \Phi\Big(\min\big(\lambda_g(y),x\big)\Big)^p\,dy\frac{w(x)}{\Phi(x)^p}\,dx$$
$$= \int_0^\infty \int_0^{\lambda_g(y)} w(x)\,dx + \Phi^p\big(\lambda_g(y)\big)\int_{\lambda_g(y)}^\infty \frac{w(x)}{\Phi(x)^p}\,dx\,dy$$
$$\leq C\int_0^\infty \int_0^{\lambda_g(y)} w(x)\,dx\,dy = C\int_0^\infty g(x)w(x)\,dx$$
$$= \int_0^\infty \Big(\frac{1}{\Phi(x)}\int_0^x f(s)\phi(s)\,ds\Big)^{p-1} f(x)w(x)\,dx$$
$$\leq C\|f\|_{L^p(w)}\|S_\phi f\|_{L^{p'}(w)} = C\|f\|_{L^p(w)}\|S_\phi f\|^{p-1}_{L^p(w)},$$

where the last inequality is obtained by using Hölder's inequality. ∎

THEOREM 4.2. *Let $p > 1$ and $\Phi(x) = \int_0^x \phi(t)\,dt$. Then, the generalized conjugate Hardy operator*

$$\tilde{S}_\phi f(x) = \int_x^\infty f(t)\phi(t)\frac{dt}{\Phi(t)},$$

satisfies that $\|\tilde{S}_\phi f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$ for all f nonincreasing, if and only if

$$\int_0^r \Big(\log\frac{\Phi(r)}{\Phi(x)}\Big)^p w(x)\,dx \leq C\int_0^r w(x)\,dx.$$

PROOF. One has to follow the same steps as in the previous proof but in this case we use the identity

$$\Big(\int_x^\infty f(t)\phi(t)\frac{dt}{\Phi(t)}\Big)^p = p\int_x^\infty \Big(\int_t^\infty f(s)\phi(s)\frac{ds}{\Phi(s)}\Big)^{p-1} f(t)\phi(t)\frac{dt}{\Phi(t)},$$

and wrıte $g(t) = \left(\int_t^\infty f(s)\phi(s)\frac{ds}{\Phi(s)}\right)^{p-1} f(t)$. ∎

REMARK 4.3. We observe that in Theorems 4.1 and 4.2 we can also prove, as a consequence of Proposıtion 2.5, the boundedness of these generalized Hardy operators in the case of $p \leq 1$.

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