SOME CHARACTERIZATIONS OF THE ULTRASPHERICAL POLYNOMIALS

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(received January 2, 1968)

1. Introduction. Let $P_n^{(\alpha)}(x)$ be the n^{th} ultraspherical polynomial. Also let $p_n^{(\alpha)}(x) = P_n^{(\alpha)}(x)/P_n^{(\alpha)}(1)$. The following generating relation is well known (3, p.98).

$$2^{\alpha-\frac{1}{2}}\Gamma(\alpha+\tfrac{1}{2})e^{xt}\{t\sqrt{1-x^2}\}^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(t\sqrt{1-x^2}) = \sum_{n=o}^{\infty} p_n^{(\alpha)} \frac{t^n}{n!} .$$

It can also be written as

$$e^{xt}\,{}_0F_1(-;\alpha+\tfrac{1}{2};-\frac{t^2(1-x^2)}{4}) = \sum_{n=o}^{\infty} p_n^{(\alpha)}(x)\frac{t^n}{n!} . \tag{1.1}$$

This suggests the consideration of the class of polynomial sets $\{Q_n(x),\ n = 0, 1, 2, \ldots\}$, $Q_n(x)$ is of exact degree n and

$$e^{xt}\varphi(t\sqrt{1-x^2}) = \sum_{n=o}^{\infty} Q_n(x)\frac{t^n}{n!} , \tag{1.2}$$

holds where $\varphi(u)$ has a formal power series expansion, $\varphi(o) = 1$, $\varphi(-u) = \varphi(u)$. It is obvious that the set of normalized ultraspherical polynomials is only one of many possible sets in the above class. For example if $\varphi(u) = 1$ then $Q_n(x) = x^n$.

It is therefore of interest to see what else is required in order to characterize the ultraspherical polynomials by means of (1.2). We give below four such characterizations. We show that if we require that $\{Q_n(x)\}$ be orthogonal, hypergeometric of certain type, that $Q_n(x)$, $n = 0, 1, 2, \ldots$ satisfy a differential equation of the second order of Sturm-Liouville type, or that the set $\{Q_n(x)\}$ satisfy a certain functional equation, then $Q_n(x)$ is essentially the n^{th} ultraspherical polynomial.

Canad. Math. Bull. vol. 11, no. 3, 1968

We note that (1.2) imply that $\{Q_n(x)\}$ satisfy the equation (1)

$$(1.3) \qquad (1-x^2)Q_{n+1}(x) = (n+1)\{Q_n(x) - xQ_{n+1}(x)\} .$$

In fact one can show that (1.3) is equivalent to (1.2). We also note that (1.2) implies

$$(1.4) \qquad Q_n(1) = 1, \quad Q_n(-1) = (-1)^n \quad n = 0, 1, 2, \ldots$$

$$(1.5) \qquad Q_n(-x) = (-1)^n Q_n(x) .$$

If $Q_n(x) = \sum_{k=o}^{n} b(n, k)x^k$ then (1.3) (or equivalently (1.2)) yields

$$(1.6) \quad (n+1)b(n, k) - (n-k+2)b(n+1, k-1) - (k+1)b(n+1, k+1) = 0 .$$

We further remark that (1.5) implies

$$b(n, n-1) = b(n, n-3) = \ldots = 0.$$

For brevity we denote $b(n, n)$ by b_n. We shall also use the notation

$$(a)_n = \begin{cases} a(a+1) \ldots (a+n-1) & (n \geq 1) \\ 1 & (n = 0) \end{cases}$$

$${}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \,;\, x \\ \beta_1, \beta_2, \ldots, \beta_q \,;\end{matrix}\right] = \sum_{n=o}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{n!(\beta_1)_n \ldots (\beta_q)_n} x^n .$$

2. We first assume that the set $\{Q_n(x)\}$ is an orthogonal set and hence satisfies the recurrence relation (3, p. 42).

$$(2.1) \qquad Q_{n+1}(x) = (A_n x + B_n)Q_n(x) + C_n Q_{n-1}(x) \qquad (n \geq 1)$$

where $Q_o(x) = 1$, $Q_1(x) = A_o x + B_o$, $A_n = b_{n+1}/b_n$ and $A_n C_n \neq 0$.

Putting $x = 1$ and $x = -1$ in (2.1) and making use of (1.4) we get, respectively,

$$A_n + B_n + C_n = 1, \qquad A_n - B_n + C_n = 1$$

so that $B_n = 0$ for all n and hence

$$(2.2) \qquad 1 = A_n + C_n .$$

Putting $k = n$ in (1.6) we get

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$$(2.3)\qquad (n+1)\{b_{n+1}-b_n\} = -2b(n+1, n-1) .$$

Now equating coefficients of x^{n-1} in both sides of (2.1) we get, using (2.2) and (2.3),

$$\frac{1}{2}(n+1)(b_{n+1}-b_n) = \frac{1}{2} n \frac{b_{n+1}}{b_n} (b_n-b_{n-1}) - (1 - \frac{b_{n+1}}{b_n}) b_{n-1}$$

which may be simplified to

$$(2.4)\qquad b_n b_{n+1} - (n+1)b_n^2 + 2b_n b_{n-1} + (n-2)b_{n+1} b_{n-1} = 0 ,$$

$$b_o = 1, \quad b_1 = 1 .$$

To solve (2.4) make the substitution

$$b_n = \frac{(\lambda)_n 2^n}{(2\lambda)_n} k_n \qquad (k_o = k_1 = 1, \quad k_n \neq 0 \text{ for } n \geq 2) .$$

We get on putting $f_n = k_n/k_{n-1}$ $(f_1 = 1, \ f_n \neq 0$ for $n \geq 2)$

$$(2.5)\qquad 2 \frac{(\alpha+n-1)(\alpha+n)}{(2\alpha+n)(2\alpha+n-1)} f_n f_{n+1} - (n+1) \frac{\alpha+n-1}{2\alpha+n-1} f_n$$

$$+ (n-2) \frac{\alpha+n}{2\alpha+n} + 1 = 0 .$$

Putting $f_n = 1 + g_n$ $(g_1 = 0, \ g_n \neq -1 \ n \geq 2)$ formula (2.5) becomes

$$(2.6)\qquad n \frac{\alpha+n-1}{2\alpha+n} g_n - (n-1) \frac{\alpha+n}{2\alpha+n-1} g_{n+1}$$

$$- 2 \frac{(\alpha+n-1)(\alpha+n)}{(2\alpha+n-1)(2\alpha+n)} g_n g_{n+1} = 0 .$$

If $g_k = 0$ for some $k \geq 2$ then $g_n = 0$ for all $n \geq k$, conversely if $g_{n+1} = 0$ then $g_n = 0$ for all $n \geq 2$. Hence we have the two cases:

<u>Case 1</u>. If $g_2 = 0$ then $g_n = 0$ for all n and the result is that

$$b_n = \frac{(\lambda)_n 2^n}{(2\lambda)_n} .$$

Case 2. If $g_2 \neq 0$ then $g_n \neq 0$ for all n. Thus put $g_n = 1/G_n$ and formula (2.6) yields

$$\frac{n}{2(\lambda+n)(2\lambda+n)} G_{n+1} - \frac{n-1}{2(\lambda+n-1)(2\lambda+n-1)} G_n = \frac{1}{(2\lambda+n-1)(2\lambda+n)} .$$

Thus

$$\frac{(n-1)}{2(\lambda+n-1)(2\lambda+n-1)} G_n = \frac{-1}{2\lambda+n-1} + C$$

where C is an arbitrary constant. Hence

$$(2.7) \qquad G_n = -\frac{2(\lambda+n-1)}{n-1} + C\frac{(\lambda+n-1)(2\lambda+n-1)}{(n-1)}, \quad (n \geq 2) .$$

Case 2a. If $C = 0$ then $k_n = \dfrac{(2\lambda)_n}{2^n(\lambda)_n}$ so that $b_n = 1$ and this case was ruled out.

Case 2b. Now suppose that $C \neq 0$. Then

$$\frac{k_n}{k_{n-1}} = \frac{n^2+\alpha_1 n+\beta_1}{n^2+\alpha_2 n+\beta_2} \text{ where } \alpha_1 - \alpha_2 = \frac{1}{C} = \beta_2 - \beta_1 .$$

Thus

$$k_n = \frac{(2\lambda)_n}{(\lambda)_n} \frac{(\lambda - \frac{1}{C})_n}{(2\lambda - \frac{2}{C})_n} ,$$

so that

$$b_n = \frac{(\lambda - \frac{1}{C})_n 2^n}{(2\lambda - \frac{2}{C})_n} .$$

To summarize we have seen that in Cases 1 and 2b

$$b_n = \frac{(\lambda)_n 2^n}{(2\lambda)_n} .$$

so that $A_n = 2\dfrac{\lambda+n}{2\lambda+n}$ and $C_n = -\dfrac{n}{2\lambda+n}$, and hence from (2.1) we conclude that

$$Q_n(x) = P_n^{(\lambda)}(x) .$$

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Now the following theorem follows:

THEOREM 1. The only polynomial set $\{Q_n(x)\}$, where deg $Q_n = n$, which is orthogonal and satisfies (1.2) is the set of ultraspherical polynomials.

3. In this section we assume that the set $\{Q_n(x)\}$ satisfy (1.2) and

$$(3.1) \qquad Q_n(x) = {}_pF_q\left[\begin{matrix} -n,\ n+\gamma,\ \alpha_1, \alpha_2, \ldots, \alpha_{p-2} ; \\ \beta_1, \beta_2, \ldots, \beta_q ; \end{matrix}\ 1-x\right]$$

where the parameters $\gamma, \alpha_1, \ldots, \alpha_{p-2}, \beta_1, \ldots, \beta_q$ are arbitrary complex numbers with $\beta_k \neq -m$, a negative integer

Put

$$Q_n(x) = \sum_{k=o}^{n} C(n,k)(1-x)^k$$

in (1.2) and then equate coefficients of $(1-x)^k$. We get

$$(3.2) \quad (n-k+2)C(n+1,k-1) + (n+1)C(n,k) - (n-2k+1)C(n+1,k) = 0 .$$

Put

$$C(n,k) = \frac{(-n)_k(n+\gamma)_k(\alpha_1)_k \ldots (\alpha_{p-2})_k}{k!\ (\beta_1)_k \ldots (\beta_q)_k}$$

in (3.2) and simplify. It then follows that

$$\frac{k(\beta_1+k-1)(\beta_2+k-1) \ldots (\beta_q+k-1)}{(\alpha_1+k-1)(\alpha_2+k-1) \ldots (\alpha_{p-2}+k-1)} = k(2k+\gamma-1)$$

so that

$$b(n,k) = \frac{(-n)_k(n+\gamma)_k}{k!\ (\frac{r-1}{2})_k 2^k} .$$

Consequently we have proved

THEOREM 2. The only hypergeometric polynomial of the type (3.1) which has a generating function (1.2) is the set of ultraspherical polynomials.

4. We now assume that the function $\varphi(u)$ in (1.1) involves a parameter α (so that $Q_n(x) = Q_n(x,\alpha)$ in such a fashion so that

$$(4.1) \qquad \frac{d}{dx} Q_n(x,\alpha) = \frac{n(n+2\alpha)}{2\alpha+1} Q_{n-1}(x,\alpha+1) \qquad (n \geq 1) .$$

Multiply (4.1) by $t^n/n!$ and sum over $n \geq 1$. We get

$$\frac{d}{dx} \sum_{n=1}^{\infty} Q_n(x,\alpha) \frac{t^n}{n!} = \frac{t^{1-2\alpha}}{2\alpha+1} \frac{d}{dt} \left\{ t^{2\alpha+1} \sum_{n=0}^{\infty} Q_n(x,\alpha+1) \frac{t^n}{n!} \right\} .$$

Thus

$$(2\alpha+1) \left\{ t\,\varphi_\alpha(t\sqrt{1-x^2}) - \frac{tx}{\sqrt{1-x^2}} \varphi'_\alpha(t\sqrt{1-x^2}) \right\}$$

$$= t^2 \left\{ x\,\varphi_{\alpha+1}(t\sqrt{1-x^2}) + \sqrt{1-x^2}\,\varphi'_{\alpha+1}(t\sqrt{1-x^2}) \right\}$$

$$+ (2\alpha+1)\, t\, \varphi_{\alpha+1}(t\sqrt{1-x^2}) .$$

If

$$\varphi_\alpha(u) = \sum_{k=0}^{\infty} C(\alpha,k)\, u^{2k}$$

then (4.2) imply

$$(4.3) \quad (2\alpha+1) \Sigma\, C(\alpha,k) t^{2k+1} (1-x^2)^k - (2\alpha+1)x\ \Sigma\, 2k\, C(\alpha,k) t^{2k} (1-x^2)^{k-1}$$

$$= x\, \Sigma\, C(\alpha+1,k) t^{2k+2} (1-x^2)^k + \Sigma\, 2k\, C(\alpha+1,k) t^{2k+1} (1-x^2)^k$$

$$+ (2\alpha+1)\ \Sigma\ C(\alpha+1,k) t^{2k+1} (1-x^2)^k .$$

Equating the coefficient of t^{2n} we get

$$(4.4) \qquad C(\alpha,n) = -\frac{1}{(2n)(2\alpha+1)} C(\alpha+1, n-1)$$

$$= (-1)^n \frac{1}{2^{2n}\, n!(\alpha+\frac{1}{2})_n} C(\alpha+n, 0) .$$

On the other hand the coefficients of t^{2n+1} yield

$$C(\alpha,n) = \frac{\alpha+n+\frac{1}{2}}{\alpha+\frac{1}{2}} C(\alpha+1,n)$$

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which is also satisfied by (4.4) provided that

$$C(\alpha+n, 0) = C(\alpha+n+1, 0) \text{ for all } n .$$

Thus $C(\alpha+n, 0)$ is independent of n. Call it $C(\alpha)$, where $C(\alpha) = C(\alpha+1)$.

Thus it follows that

$$\varphi_\alpha(u) = C(\alpha)\ {}_0F_1(-; \alpha+\frac{1}{2}; -\frac{u^2}{4}) .$$

Consequently we have proved

THEOREM 3. *If the polynomial set* $\{Q_n(x, \alpha)\}$ *is generated by* (1.2) *and satisfies* (4.1) *then*

$$Q_n(x, \alpha) = C(\alpha)\ P_n^{(\alpha)}(x)$$

where $C(\alpha)$ *is an arbitrary constant such that* $C(\alpha) = C(\alpha+1)$.

5. Finally we prove in this section the following theorem

THEOREM 4. *If the polynomial set* $\{Q_n(x)\}$ *satisfies* (1.2) *and a differential equation of the second order*

$$(5.1) \qquad A(x)Q_n''(x) + B(x)A_n'(x) + \lambda_n Q_n(x) = 0$$

where $A(x)$ *and* $B(x)$ *are polynomials of the second and first degrees respectively, and* λ_n *is independent of* x *then* $Q_n(x)$ *is essentially the ultraspherical polynomial.*

Proof. Since the degree of $Q_n(x)$ is exactly n and since $Q_1(x) = C_{11}x$, $Q_2(x) = C_{21}x + C_{22}$ we see that (5.1) reduces to

$$(5.2) \qquad (\alpha x^2 + \gamma)\ Q_n''(x) + bx\ Q_n'(x) - n(\alpha n-\alpha+b)Q_n(x) = 0$$

where $\alpha^2 + b^2 \neq 0$.

If $G(x, t) = e^{xt}\ \varphi(t\sqrt{1-x^2})$ then (5.2) implies

$$(5.3) \qquad (\alpha x^2+\gamma)\frac{\partial^2 G}{\partial x^2} + bx - \alpha t^2\ \frac{\partial^2 G}{\partial t^2} - bt\ \frac{\partial G}{\partial t} = 0 .$$

Putting $x = 0$ in (5.3) we get

$$(5.4) \qquad \alpha t\varphi''(t) + (b+\gamma)\varphi'(t) - \gamma t\varphi(t) = 0 .$$

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It is easy to see that $\alpha = 0$ is not possible. For then $\varphi(u) = e^{cu^2}$. Furthermore (1.1) with $\varphi(u) = e^{cu^2}$ and (5.2) with $\varphi = 0$ are not compatible.

We may now assume that $\alpha \neq 0$. It then follows that

$$\varphi(t) = {}_0F_1\left(- ; \frac{b+\gamma+1}{2} ; -\frac{\alpha t^2}{4\gamma}\right) .$$

In view of (1.1) we see that the conclusion of the theorem is true.

6. Remark. Illief (2) has shown that the ultraspherical polynomials are the only orthogonal system of polynomials in the class of polynomial sets defined by means of

$$e^{xt} \varphi(t\sqrt{1-x^2}) = \sum_{n=o}^{\infty} Q_n(x)\, t^n/n!$$

where $\varphi(u)$ is an even function which is either a polynomial whose zeros are all real or an entire function which is the limit of such polynomials.

It is thus apparent that the hypothesis in Theorem 1 is weaker and the theorem may be considered as a slight improvement over that which was obtained by Illief.

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https://doi.org/10.4153/CMB-1968-054-1 Published online by Cambridge University Press