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Rigidity for the perimeter inequality under Schwarz symmetrization

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In this paper, we give necessary and sufficient conditions for the rigidity of the perimeter inequality under Schwarz symmetrization. The term *rigidity* refers to the situation in which the equality cases are only obtained by translations of the symmetric set. In particular, we prove that the sufficient conditions for rigidity provided in M. Barchiesi, F. Cagnetti and N. Fusco [Stability of the Steiner symmetrization of convex sets. J. Eur. Math. Soc. 15 (2013), 1245-1278.] are also necessary.

Keywords: perimeter inequality; geometric variational problems; Schwarz symmetrization; functions of bounded variation; Sobolev functions

2020 *Mathematics Subject Classification:* 49Q20; 49K21; 49Q10; 52A40

1. Introduction

Symmetrization procedures have a wide range of applications in modern analysis, geometric variational problems and optimization. Understanding the behaviour of functional and perimeter inequalities under symmetrization allows to prove the existence of symmetric minimisers of geometric variational problems, and to provide comparison principles for solutions of PDEs (see, for instance [9, 14, 15, 16, 19, 21, 22, 24, 25] and the references therein).

Examples of set symmetrizations under which the volume is preserved and the perimeter does not increase include Steiner symmetrization, Ehrhard symmetrization, circular and spherical symmetrization. We say that rigidity holds for a perimeter inequality if the set of extremals is trivial. Showing rigidity can lead to proving the uniqueness of minimisers of variational problems. For example, proving the rigidity of Steiner's inequality for convex sets was substantial in the celebrated proof of the Euclidean isoperimetric inequality by De Giorgi (see, [10, 11]).

Later on, the study of rigidity was revived in the seminal paper of Chlebík, Cianchi and Fusco (see [8]), where the authors gave the sufficient conditions for

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rigidity of Steiner's inequality, also for sets that are not convex. Henceforth, necessary and sufficient conditions for rigidity for Steiner's inequality have been obtained in [5] in the case where the distribution function is a special function of bounded variation with locally finite jump set. In the Gauss space, necessary and sufficient conditions for rigidity of Ehrhard's inequality are given in [6]. In the last two papers, the results are stated in terms of *essential connectedness*. For an expository article of the aforementioned rigidity results, we refer to [4]. In [7], the authors provided the necessary and sufficient conditions for rigidity for perimeter inequality under spherical symmetrization, while in [20], sufficient conditions for rigidity have been given for the anisotropic Steiner's perimeter inequality. We further point out that, regarding the smooth case, the authors in [18] proved sufficient conditions for rigidity of perimeter inequality in *warped products*, for a wide class of symmetrizations, including Steiner, Schwarz and spherical symmetrization.

The literature about Steiner's perimeter inequality of a higher codimension is less explored. Particularly, sufficient conditions for rigidity for any codimension have been provided in [2], through a comprehensive analysis of the barycenter function. The problem of a complete characterization (that is, necessary *and* sufficient conditions) for the rigidity of generic higher codimensions, however, remains open.

A special case of interest is where the codimension is equal to $(n-1)$. In this case, Steiner's symmetrization of codimension $(n-1)$ is usually referred to as *Schwarz symmetrization*.

The purpose of this paper is to provide necessary and sufficient conditions for rigidity of equality cases for the perimeter inequality under Schwarz symmetrization. In particular, we prove that the sufficient conditions for rigidity shown in [2] are also necessary. Our results are established by following techniques developed in [7].

In the remainder of this introductory section, we recall the setting of the problem, and we state our main results.

1.1. Schwarz symmetrization

For $n \geqslant 2$ with $n \in \mathbb{N}$, we label each point $x \in \mathbb{R}^n$ as $x = (z, w)$, where $z \in \mathbb{R}$ and $w \in \mathbb{R}^{n-1}$.

Given a measurable set $E \subset \mathbb{R}^n$ and $z \in \mathbb{R}$, we define the $(n-1)$-*dimensional slice* of E at z as

$$E_z := \{w \in \mathbb{R}^{n-1} : (z, w) \in E\}. \tag{1.1}$$

For a Lebesgue measurable function $\ell : \mathbb{R} \to [0, \infty)$, we say that the set E is ℓ-*distributed* if

$$\ell(z) = \mathcal{H}^{n-1}(E_z) \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \mathbb{R}, \tag{1.2}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$.

We can associate to ℓ the function $r_\ell : \mathbb{R} \to [0, \infty)$, which is such that

$$\ell(z) = \mathcal{H}^{n-1}\left(B^{n-1}(0, r_\ell(z)\right), \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \mathbb{R},$$

where $B^{n-1}(w, \rho)$ denotes the open ball in $\mathbb{R}^{n-1}$ with radius ρ and centred at $w \in \mathbb{R}^{n-1}$.

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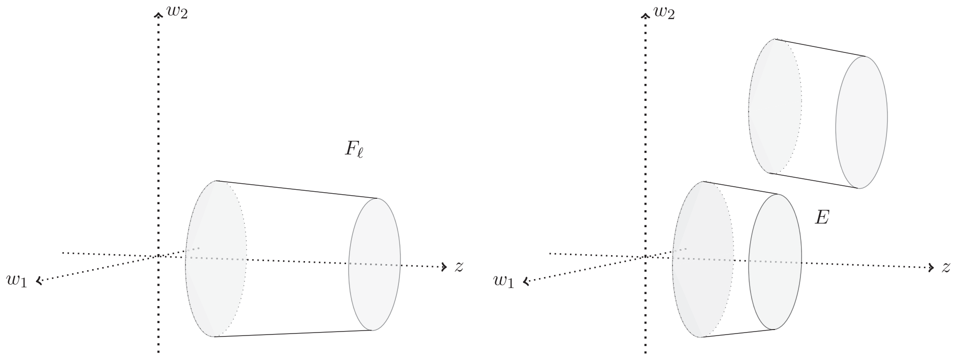

Figure 1. The symmetric set F_ℓ (left) of an ℓ-distrubuted set E (right) in case of $n = 3$. Note that, in general the slices of the set E do not need to be disks.

Note that $r_\ell(z)$ is the radius of an $(n-1)$-dimensional ball whose measure is $\ell(z)$, and can be explicitly written as

$$r_\ell(z) = \left(\frac{\ell(z)}{\omega_{n-1}}\right)^{1/n-1} \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \mathbb{R}, \tag{1.3}$$

where we set $\omega_{n-1} := \mathcal{H}^{n-1}(B^{n-1}(0, 1))$.

If $E \subset \mathbb{R}^n$ is ℓ-distributed, then the *Schwarz symmetric* set F_ℓ of E with respect to the axis $\{w = 0\}$ is defined as

$$F_\ell := \left\{x = (z, w) \in \mathbb{R} \times \mathbb{R}^{n-1} : |w| < r_\ell(z)\right\}; \tag{1.4}$$

see Figure 1.

This is the ℓ-distributed set whose cross sections are $(n-1)$-dimensional open balls centred at the z axis. We notice that the Schwarz symmetric set F_ℓ of an ℓ-distributed set E depends only on the function ℓ, and not on the particular ℓ-distributed set E under consideration.

Due to Fubini's theorem, Schwarz symmetrization preserves the volume, i.e. if E is ℓ-distributed and $\mathcal{H}^n(E) < \infty$, it turns out that $\mathcal{H}^n(E) = \mathcal{H}^n(F_\ell)$. Moreover, the perimeter inequality under Schwarz symmetrization holds, that is

$$P(F_\ell) \leqslant P(E) \text{for every } \ell\text{-distributed set } E \subset \mathbb{R}^n, \tag{1.5}$$

see, for instance, [**3**]. Here, $P(E)$ stands for the perimeter of E in $\mathbb{R}^n$ (see § 2.4). Additionally, a localized version of (1.5) holds, that is, if E is a set of finite perimeter, then

$$P(F_\ell; B \times \mathbb{R}^{n-1}) \leqslant P(E; B \times \mathbb{R}^{n-1}), \tag{1.6}$$

for every Borel set $B \subset \mathbb{R}$, see [**2**, Theorem 1.1].

The inequality (1.6) is well-known in the literature (see, for instance, [**3**], where this is proved through a careful approximation by polarizations). In [**2**], one can find an alternative and direct proof, which allowed the authors to give sufficient conditions for rigidity of Steiner's inequality of a general higher codimension k, where $1 < k \leqslant n - 1$.

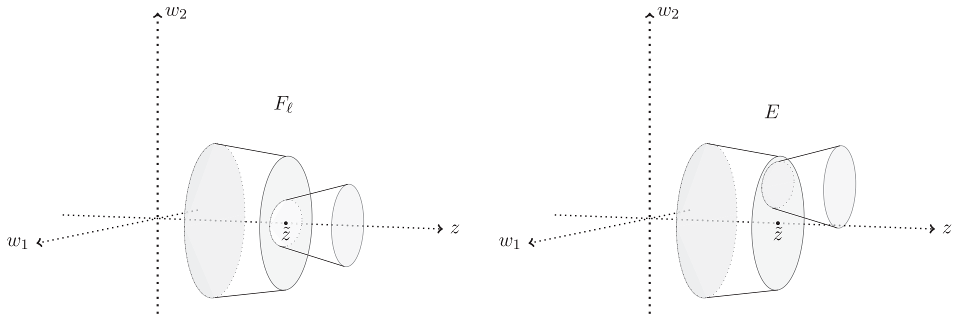

Figure 2. Rigidity ($\mathcal{RS}$) fails, since the (reduced) boundary $\partial^* F_\ell$ of F_ℓ has a non-negligible flat vertical part, thus violating ($\mathcal{RS}$). Note that the function ℓ is discontinuous at $\tilde{z}$, so that also (1.10) is violated.

1.2. Rigidity for perimeter inequality under Schwarz symmetrization

We shall now describe the main objective of the present paper. Given a Lebesgue measurable function $\ell : \mathbb{R} \to [0, \infty)$, such that F_ℓ is a set of finite perimeter and finite volume, we define the class of equality cases of (1.6) as

$$\mathcal{K}(\ell) = \{E \subset \mathbb{R}^n : E \text{ is } \ell\text{-distributed and } P(F_\ell) = P(E)\}. \tag{1.7}$$

Due to the invariance of the perimeter under translations along a direction $\tau \in \mathbb{R}^{n-1}$, as well as the definition of the symmetric set F_ℓ, the following inclusion is always true:

$$\mathcal{K}(\ell) \supset \{E \subset \mathbb{R}^n : \mathcal{H}^n(E \triangle (F_\ell + (0, \tau))) = 0 \text{ for some } \tau \in \mathbb{R}^{n-1}\}, \tag{1.8}$$

where $\triangle$ denotes the symmetric difference of sets. We say that *rigidity* holds for (1.6) if the opposite inclusion is also satisfied, i.e.

$$\mathcal{K}(\ell) = \{E \subset \mathbb{R}^n : \mathcal{H}^n(E \triangle (F_\ell + (0, \tau)) = 0 \text{ for some } \tau \in \mathbb{R}^{n-1}\}. \tag{$\mathcal{RS}$}$$

1.3. State of the art

Let us now give an account of the available results in the literature for the rigidity of (1.6). In general, not all equality cases of (1.6) can be written as a translation of the symmetric set F_ℓ. This can happen, for instance, if the (reduced) boundary $\partial^* F_\ell$ of F_ℓ contains flat vertical parts. In such a case, we can find an ℓ-distributed set E which preserves perimeter under symmetrization, and it is not equivalent to (a translation of) the symmetric set F_ℓ; see Figure 2.

In order to rule out this issue, the authors in [**2**] localized the problem, by considering an open set $\Omega \subset \mathbb{R}$, and imposing the following condition:

$$\mathcal{H}^{n-1}(\{(z, w) \in \partial^* F_\ell : \nu_w^{F_\ell}(z, w) = 0\} \cap (\Omega \times \mathbb{R}^{n-1})) = 0, \tag{1.9}$$

where $\nu_w^{F_\ell}(z, w)$ denotes the w-component of the measure-theoretic outer unit normal to the symmetric set F_ℓ. It turns out that (1.9) is related to the regularity of the function ℓ. Note that, in general, if E is a set of finite perimeter in $\mathbb{R}^n$, then

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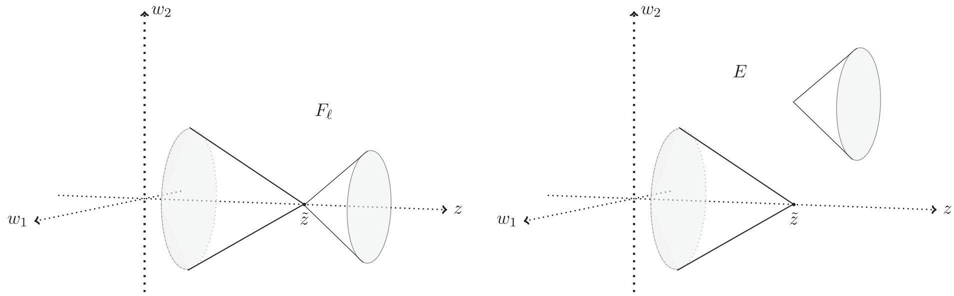

Figure 3. Rigidity ($\mathcal{RS}$) fails, since the set $\{\ell^\wedge > 0\}$ is disconnected by a point $\tilde{z} \in \mathbb{R}$, where $\ell(\tilde{z}) = 0$, thus, violating (1.11).

either F_ℓ is equivalent to $\mathbb{R}^n$, or ℓ is a function of Bounded Variation in $\mathbb{R}$ (see proposition 2.2).

In [**2**, Proposition 3.5], the authors showed that (1.9) is equivalent to asking that ℓ is a Sobolev function in Ω, as explained below.

PROPOSITION 1.1. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume in $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}$ be an open set. Then*

$$\mathcal{H}^{n-1}(\{(z, w) \in \partial^* F_\ell : \nu_w^{F_\ell}(z, w) = 0\} \cap (\Omega \times \mathbb{R}^{n-1})) = 0$$

if and only if

$$\ell \in W^{1,1}(\Omega). \tag{1.10}$$

Even if condition (1.9) (or, equivalently, (1.10)) is satisfied, rigidity can still be violated. In particular, this can happen when the symmetric set F_ℓ *is not connected in a suitable measure-theoretic way*, despite the fact that it can be connected from a topological point of view; see Figure 3.

Note that, once condition (1.9) [or, equivalently, (1.10) is imposed, we have that $\ell \in W^{1,1}(\Omega)$, and since Ω is a one-dimensional set, ℓ is absolutely continuous in Ω. Therefore the condition imposed in [**2**] to rule out situations as in Figure 3 can be written as

$$\ell^*(z) > 0 \text{ for all } z \in \Omega, \tag{1.11}$$

where ℓ^* denotes the Lebesgue representative of ℓ, see [**2**, condition (1.4)].

It turns out that (1.9) and (1.11) are sufficient for rigidity (see [**2**, Theorem 1.2]), as explained below.

THEOREM 1.2. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume. Let $\Omega \subset \mathbb{R}$ be a connected open set, and suppose that (1.9) and (1.11) are satisfied. If*

$$P(F_\ell; \Omega \times \mathbb{R}) = P(E; \Omega \times \mathbb{R}),$$

then $E \cap (\Omega \times \mathbb{R})$ is equivalent to (a translation along $\mathbb{R}^{n-1}$) of $F_\ell \cap (\Omega \times \mathbb{R})$. Here, $P(E; \Omega \times \mathbb{R})$ denotes the relative perimeter of E in $\Omega \times \mathbb{R}$.

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1.4. The main result

Our contribution is to show that conditions (1.9) and (1.11) are also *necessary* for rigidity. As we have already observed, the proof of the theorem 1.2 requires the localization of the problem in an open and connected set $\Omega \subset \mathbb{R}$, to impose the condition (1.10). We will show that this can be avoided. We also notice that, if F_ℓ is a set of finite perimeter and finite volume, in general, we only have that $\ell \in BV(\mathbb{R})$ and this means that ℓ may be discontinuous. Therefore, we need to rephrase condition (1.11) in terms of the approximate lim inf $\ell^\wedge$ of ℓ at every point $z \in \mathbb{R}$, see § 2. We are now able to state our main result. Below, $\mathring{J}$ denotes the interior of J.

THEOREM 1.3. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume. Then, the following statements are equivalent*:

(i) *($\mathcal{RS}$) holds true;*

(ii) *$\{\ell^\wedge > 0\}$ is a (possibly unbounded) interval J and $\ell \in W^{1,1}(\mathring{J})$.*

As we have already pointed out, the proof of the direction $(ii) \Longrightarrow (i)$ of theorem 1.3 relies on the proof of [**2**, Theorem 1.2]. We will prove that the converse $(i) \Longrightarrow (ii)$ is also true. We highlight the fact that our approach is not based on a comprehensive use of a general perimeter formula for sets $E \subset \mathbb{R}^n$ satisfying equality in (1.6), as it appears in [**6**]. On the contrary, inspired by the techniques developed in [**7**], we analyse the properties of the function ℓ and we provide a careful study of the transformations that can be applied on the symmetric set F_ℓ, without creating any perimeter contribution.

To this end, the rest of the paper is structured as follows. In § 2, we fix the notation, build the necessary background and we gather some preliminary results that appeared in the literature. In § 3, we show the direction $(i) \Longrightarrow (ii)$ of theorem 1.3, by studying the properties of the distribution function ℓ, and exploiting counterexamples where rigidity is violated.

2. Background and proof of the theorem 1.3 $(ii) \Longrightarrow (i)$

In this section, we will recall the necessary machinery, which will be used throughout the paper. The interested reader could refer to [**1**, **2**, **12**, **13**, **17**, **23**].

We fix $n \in \mathbb{N}$ with $n \geqslant 2$. For each $x \in \mathbb{R}^n$, we write $x = (z, w)$, with $z \in \mathbb{R}$ and $w \in \mathbb{R}^{n-1}$. The standard Euclidean norm will be denoted by $|\cdot|$ in $\mathbb{R}$, $\mathbb{R}^{n-1}$ or $\mathbb{R}^n$ depending on the context. For $1 \leqslant m \leqslant n$, we will denote the m-dimensional Hausdorff measure in $\mathbb{R}^n$ by $\mathcal{H}^m$. For every radius $\rho > 0$ and $x \in \mathbb{R}^n$ we write $B_\rho(x)$ for the open ball of $\mathbb{R}^n$ with radius ρ and centred at x. The volume of the unit ball in $\mathbb{R}^n$ is denoted as ω_n, i.e., $\omega_n := \mathcal{H}^n(B_1(0))$. Note that throughout the paper, in case of balls in different dimensions, we will denote the corresponding ball in dimension m with radius ρ centred at $w \in \mathbb{R}^m$ by writing $B^m(w, \rho)$.

Now, for $x \in \mathbb{R}^n$ and $\nu \in \partial B_1(0)$, we set

$$H^+_{x,\nu} = \{y \in \mathbb{R}^n : \langle (y - x), \nu \rangle \geqslant 0\}$$

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and

$$H^-_{x,\nu} = \{y \in \mathbb{R}^n : \langle (y-x), \nu \rangle \leqslant 0\}.$$

Let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence of Lebesgue measurable sets in $\mathbb{R}^n$ with $\mathcal{H}^n(E_j) < \infty$ for every $j \in \mathbb{N}$, and let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set with $\mathcal{H}^n(E) < \infty$. We say that $\{E_j\}_{j\in\mathbb{N}}$ *converges to* E as $j \to \infty$ and we write

$$E_j \to E \quad \text{if} \quad \mathcal{H}^n(E_j \triangle E) \to 0 \text{ as } j \to \infty,$$

where $\triangle$ stands for the symmetric difference of sets. Additionally, if E_1, $E_2 \subset \mathbb{R}^n$ are Lebesgue measurable sets, we say that

$$E_1 \subset_{\mathcal{H}^n} E_2 \quad \text{if} \quad \mathcal{H}^n(E_1 \backslash E_2) = 0,$$

and

$$E_1 =_{\mathcal{H}^n} E_2 \quad \text{if} \quad \mathcal{H}^n(E_1 \triangle E_2) = 0.$$

Moreover, the characteristic function of a Lebesgue measurable set $E \subset \mathbb{R}^n$ will be denoted by χ_E.

2.1. Density points

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and $x \in \mathbb{R}^n$. We define the *lower* and *upper* n*-dimensional densities of* E *at* x as

$$\theta_*(E,x) = \liminf_{\rho\to 0^+} \frac{\mathcal{H}^n(E \cap B_\rho(x))}{\omega_n \rho^n}, \quad \text{and} \quad \theta^*(E,x) = \limsup_{\rho\to 0^+} \frac{\mathcal{H}^n(E \cap B_\rho(x))}{\omega_n \rho^n},$$

respectively. The maps $x \longmapsto \theta_*(E, x)$ and $x \longmapsto \theta^*(E, x)$ are Borel functions (even in case where E is Lebesgue non-measurable, see [**23**, Chapter 1, Remark 3.1]) and they coincide $\mathcal{H}^n$-a.e. in $\mathbb{R}^n$. Hence, the n-dimensional density of E at x is defined as the Borel function

$$\theta(E,x) = \lim_{\rho\to 0^+} \frac{\mathcal{H}^n(E \cap B_\rho(x))}{\omega_n \rho^n}, \text{ for } \mathcal{H}^n\text{-a.e. } x \in \mathbb{R}^n.$$

For each $s \in [0, 1]$, we define the *set of points of density* s *with respect to* E as

$$E^{(s)} := \{x \in \mathbb{R}^n \ : \ \theta(E,x) = s\}$$

The *essential boundary* $\partial^e E$ of E is defined as the set

$$\partial^e E := \mathbb{R}^n \backslash (E^{(0)} \cup E^{(1)})$$

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2.2. Approximate limits of measurable functions

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a Lebesgue measurable function. We define the *approximate upper limit* $g^\vee(x)$ and the *approximate lower limit* $g^\wedge(x)$ of g at $x \in \mathbb{R}^n$ as

$$g^\vee(x) = \inf\left\{s \in \mathbb{R} : x \in \{g > s\}^{(0)}\right\} = \inf\left\{s \in \mathbb{R} : x \in \{g < s\}^{(1)}\right\} \tag{2.1}$$

and

$$g^\wedge(x) = \sup\left\{s \in \mathbb{R} : x \in \{g < s\}^{(0)}\right\} = \sup\left\{s \in \mathbb{R} : x \in \{g > s\}^{(1)}\right\}, \tag{2.2}$$

respectively. We highlight the fact that both $g^\vee$ and $g^\wedge$ are Borel functions and they are defined *for every* $x \in \mathbb{R}^n$ with values in $\mathbb{R} \cup \{\pm\infty\}$. In addition, if $g_1 : \mathbb{R}^n \to \mathbb{R}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}$ are measurable functions such that $g_1 = g_2$ $\mathcal{H}^n$-a.e. on $\mathbb{R}^n$, then it turns out that

$$g_1^\wedge(x) = g_2^\wedge(x) \quad \text{and} \quad g_1^\vee(x) = g_2^\vee(x) \quad \text{for every } x \in \mathbb{R}^n.$$

The *approximate discontinuity set* S_g of g is defined as

$$S_g := \{g^\wedge \neq g^\vee\},$$

and satisfies $\mathcal{H}^n(S_g) = 0$. Moreover, even if $g^\wedge,\ g^\vee$ could take values $\pm\infty$ on S_g, it turns out that the difference $g^\vee - g^\wedge$ is well-defined in $\mathbb{R} \cup \{\pm\infty\}$ for every point $x \in S_g$. In the light of the above considerations, the *approximate jump* $[\,g\,]$ of g is the Borel function $[\,g\,] : \mathbb{R}^n \to [0, \infty]$ defined as

$$[\,g\,](x) := \begin{cases} g^\vee(x) - g^\wedge(x), & \text{if } x \in S_g \\ 0, & \text{elsewhere.} \end{cases}$$

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. We will say that $s \in \mathbb{R} \cup \{\pm\infty\}$ is the approximate limit of g at x with respect to E, denoted by $s = \mathrm{aplim}(g, E, x)$, if

$$\theta(\{|g - s| > \epsilon\} \cap E, x) = 0, \quad \text{for every } \epsilon > 0 \quad (s \in \mathbb{R}), \tag{2.3a}$$

$$\theta(\{g < M\} \cap E, x) = 0, \quad \text{for every } M > 0 \quad (s = +\infty), \tag{2.3b}$$

and

$$\theta(\{g > -M\} \cap E, x) = 0, \quad \text{for every } M > 0 \quad (s = -\infty). \tag{2.3c}$$

We will say that $x \in S_g$ is a *jump point* of g if there exist $\nu \in \partial B_1(0)$ such that

$$g^\vee(x) = \mathrm{aplim}(g, H^+_{x,\nu}, x) \quad \text{and} \quad g^\wedge(x) = \mathrm{aplim}(g, H^-_{x,\nu}, x).$$

In this spirit, we define the *approximate jump direction* $\nu_g(x)$ of g at x as $\nu_g(x) := \nu$. The set of approximate jump points of g is denoted by J_g. Note that $J_g \subset S_g$ and $\nu_g : J_g \to \partial B_1(0)$ is a Borel function.

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2.3. Functions of bounded variation

Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $C^1_c(\Omega;\mathbb{R}^n)$ and by $C_c(\Omega;\mathbb{R}^n)$ the class of C^1 functions with compact support and the class of all continuous functions with compact support from Ω to $\mathbb{R}^n$, respectively. We also recall the Sobolev space $W^{1,1}(\Omega)$, that is, the space of all functions $g \in L^1(\Omega)$, whose distributional derivative Dg belongs to $L^1(\Omega)$.

Given $g \in L^1(\Omega)$, the *total variation* of g in Ω is defined as

$$|Dg|(\Omega) = \sup\left\{\int_\Omega g(x)\ \mathrm{div}\ T(x)\,\mathrm{d}x : T \in C^1_c(\Omega;\mathbb{R}^n),\ |T| \leqslant 1\right\}.$$

We then define the space of functions of bounded variation in Ω, denoted by $BV(\Omega)$, as the set of functions $g \in L^1(\Omega)$ such that $|Dg|(\Omega) < \infty$. In addition, we will say that $g \in BV_{loc}(\Omega)$, if $g \in BV(\Omega')$ for every $\Omega' \subset\subset \Omega$. If $g \in BV(\Omega)$, due to Radon-Nikodym decomposition of Dg with respect to $\mathcal{H}^n$, we have

$$Dg = D^a g + D^s g,$$

where $D^a g$ and $D^s g$ are mutually singular measures and $D^a g \ll \mathcal{H}^n$. The density of $D^a g$ with respect to $\mathcal{H}^n$ will be denoted as ∇g, and we have that $\nabla g \in L^1(\Omega, \mathbb{R}^n)$ with $D^a g = \nabla g\ d\mathcal{H}^n$. Additionally, it turns out that $\mathcal{H}^{n-1}(S_g \backslash J_g) = 0$ and $[\,g\,] \in L^1_{loc}(\mathcal{H}^{n-1} \llcorner J_g)$, see [**1**, Theorem 3.78]. The *jump part of* g is the $\mathbb{R}^n$-valued Radon measure given by

$$D^j g = [\,g\,]\nu_g d\mathcal{H}^{n-1} \llcorner J_g. \tag{2.4}$$

Finally, the *Cantorian part* $D^c g$ of Dg is defined as the $\mathbb{R}^n$-valued Radon measure

$$D^c g = D^s g - D^j g,$$

and is such that $|D^c g|(N) = 0$ for every set $N \subset \mathbb{R}^n$, which is σ-finite with respect to $\mathcal{H}^{n-1}$, see [**1**, Proposition 3.92].

Note, that in the special case $n = 1$, if $(a,\,b) \subset \mathbb{R}$ is an open interval, every $g \in BV(a,\,b)$ can be decomposed as the sum

$$g = g^a + g^j + g^c, \tag{2.5}$$

where $g^a \in W^{1,1}(a,\,b)$, g^j is a purely jump function (that is, $Dg^j = D^j g^j$) and g^c is a purely Cantorian function (that is, $Dg^c = D^c g^c$), see [**1**, Corollary 3.33]. Moreover, if g is a *good representative* (see [**1**, Theorems 3.27, 3.28]), then the total variation $|Dg|$ of Dg can be written as

$$|Dg|(a,b) = \sup\left\{\sum_{i=1}^{M} |g(x_{i+1}) - g(x_i)| :\ a < x_1 < x_2 < \cdots < x_M < b\right\} \tag{2.6}$$

where the supremum is taken over all $M \in \mathbb{N}$ and over all possible partitions of the interval $(a,\,b)$ with $a < x_1 < x_2 < \cdots < x_M < b$.

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2.4. Sets of locally finite perimeter in the Euclidean space

Let n, $m \in \mathbb{N}$ with $1 \leqslant m \leqslant n$. Let also $E \subset \mathbb{R}^n$ be an $\mathcal{H}^m$-measurable set. We say that E is a *countably $\mathcal{H}^m$-rectifiable* set if there exists a countable family of Lipschitz functions $(g_j)_{j\in\mathbb{N}}$, where $g_j : \mathbb{R}^m \to \mathbb{R}^n$, such that $E \subset_{\mathcal{H}^m} \bigcup_{j\in\mathbb{N}} g_j(\mathbb{R}^m)$. In addition, if $\mathcal{H}^m(E \cap K) < \infty$ for every compact set $K \subset \mathbb{R}^n$, we say that E is a *locally $\mathcal{H}^m$-rectifiable* set.

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. We say that E is a set of *locally finite perimeter* in $\mathbb{R}^n$ if there exists an $\mathbb{R}^n$-valued Radon measure μ_E, such that

$$\int_E \nabla\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}\mu_E, \qquad \text{for every } \psi \in C_c^1(\mathbb{R}^n).$$

Note that, E is a set of locally finite perimeter if and only if $\chi_E \in BV_{loc}(\mathbb{R}^n)$. If $G \subset \mathbb{R}^n$ is a Borel set, then the relative perimeter of E in G is defined as

$$P(E;G) := |\mu_E|(G).$$

When $G = \mathbb{R}^n$, we ease the notation to $P(E) := P(E;\mathbb{R}^n)$.

The *reduced boundary* $\partial^* E$ of E is the set of all $x \in \mathbb{R}^n$ such that

$$\nu_E(x) = \lim_{\rho\to 0^+} \frac{\mu_E(B_\rho(x))}{|\mu_E|(B_\rho(x))} \quad \text{exists and belongs to } \partial B_1(0).$$

The Borel function $\nu_E : \partial^* E \to \partial B_1(0)$ is usually referred to as the *measure-theoretic outer normal* to E. Due to Lebesgue-Besicovitch derivation theorem and [**1**, Theorem 3.59], it holds that the reduced boundary $\partial^* E$ of E is a locally $(n-1)$-rectifiable set in $\mathbb{R}^n$ and

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E,$$

so that

$$\int_E \nabla\psi(x)\; dx = \int_{\partial^* E} \phi(x)\nu_E(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \quad \text{ for every } \psi \in C_c^1(\mathbb{R}^n).$$

Thus, for every Borel set $G \subset \mathbb{R}^n$ we have that

$$P(E;G) = |\mu_E|(G) = \mathcal{H}^{n-1}(G \cap \partial^* E).$$

Finally, if E is a set of locally finite perimeter, it holds

$$\partial^* E \subset E^{(1/2)} \subset \partial^e E, \tag{2.7}$$

and additionally, thanks to *Federer's theorem* (see e.g., [**1**, Theorem 3.61] or [**17**, Theorem 16.2]), we have that

$$\mathcal{H}^{n-1}(\partial^e E \setminus \partial^* E) = 0, \tag{2.8}$$

which implies that the essential boundary $\partial^e E$ of E is locally $\mathcal{H}^{n-1}$-rectifiable in $\mathbb{R}^n$.

2.5. Preliminary results

In this final subsection, we state some results which will be useful in the following.

The first significant result relates to the set E_z defined in (1.1). Namely, as it turns out, for $\mathcal{H}^1$-a.e $z \in \mathbb{R}$, E_z is a set of finite perimeter and its reduced boundary $\partial^*(E_z)$ enjoys an advantageous property. These facts follow due to a variant of a result by Vol'pert [**26**], which is provided in [**2**, Theorem 2.4].

PROPOSITION 2.1 Vol'pert. *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then for $\mathcal{H}^1$-a.e. $z \in \mathbb{R}$ the following hold true*:

(i) *E_z is a set of finite perimeter in $\mathbb{R}^{n-1}$;*

(ii) *$\mathcal{H}^{n-2}((\partial^* E)_z \triangle \partial^*(E_z)) = 0$.*

Thanks to (ii) above, we will often write $\partial^* E_z$ instead of $(\partial^* E)_z$ or $\partial^*(E_z)$. The next result presents a crucial regularity property of the function ℓ, and it can be found in [**2**, Lemma 3.1].

PROPOSITION 2.2. *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then either $\ell(z) = \infty$ for $\mathcal{H}^1$-a.e. $z \in \mathbb{R}$, or $\ell(z) < \infty$ for $\mathcal{H}^1$-a.e. $z \in \mathbb{R}$ and $\mathcal{H}^n(E) < \infty$. In the latter case, we have $\ell \in BV(\mathbb{R})$.*

We present the following auxiliary inequality, which is a special case of [**2**, Proposition 3.4].

PROPOSITION 2.3. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume. Let $E \subset \mathbb{R}^n$ be an ℓ-distributed set and let $f : \mathbb{R} \to [0, \infty]$ be a Borel measurable function. Then*

$$\int_{\partial^* E} f(z)\,\mathrm{d}\mathcal{H}^{n-1}(x) \geqslant \int_{\mathbb{R}} f(z)\sqrt{(\mathcal{H}^{n-2}(\partial^* E_z))^2 + |\nabla \ell(z)|^2}\,\mathrm{d}z + \int_{\mathbb{R}} f(z)\,\mathrm{d}|D^s \ell|(z), \tag{2.9}$$

Moreover, if $E = F_\ell$, the equality holds in (2.9).

A straightforward consequence of the above result is the following.

COROLLARY 2.4. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume. Then*

$$P(F_\ell; B \times \mathbb{R}^{n-1}) = \int_B \sqrt{(\mathcal{H}^{n-2}(\partial^* (F_\ell)_z))^2 + |\nabla \ell(z)|^2}\,\mathrm{d}z + |D^s \ell|(B), \tag{2.10}$$

for every Borel set $B \subset \mathbb{R}$.

For sake of completeness, we close this preliminary section by presenting the proof of theorem 1.3 $(ii) \Longrightarrow (i)$.

Proof of theorem 1.3 $\boldsymbol{(ii) \Longrightarrow (i)}$. Suppose that (ii) holds. Since $\ell \in W^{1,1}(\mathring{J})$, by proposition 1.1, the condition $(\mathcal{RS})$ is satisfied with $\Omega = \mathring{J}$. In addition, since J is

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one-dimensional, ℓ is absolutely continuous in $\mathring{J}$ and therefore,

$$\ell^{\wedge}(z) = \ell^{\vee}(z) = \ell^*(z) > 0 \text{ for all } z \in \mathring{J},$$

where ℓ^* stands for the Lebesgue representative of ℓ. Thus, it turns out that (1.11) is true. Therefore, due to theorem 1.2, (i) follows. $\square$

3. Proof of the theorem 1.3 $(i) \Longrightarrow (ii)$

We start our analysis with the following lemma, which will be extensively used in the sequel.

Lemma 3.1. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume. Let also r_ℓ be defined as in (1.3) and consider $\bar{z} \in \mathbb{R}$. Then*

$$(\partial^* F_\ell)_{\bar{z}} =_{\mathcal{H}^{n-1}} B^{n-1}(0, r_\ell^{\vee}(\bar{z})) \backslash B^{n-1}(0, r_\ell^{\wedge}(\bar{z})). \tag{3.1}$$

Proof. The proof is divided into two steps.

Step 1: We prove

$$(\partial^* F_\ell)_{\bar{z}} \subset \overline{B^{n-1}(0, r_\ell^{\vee}(\bar{z}))} \backslash B^{n-1}(0, r_\ell^{\wedge}(\bar{z})). \tag{3.2}$$

To this end, it is enough to prove that

$$r_\ell^{\wedge}(\bar{z}) \leqslant |w| \quad \text{for every } w \in (\partial^* F_\ell)_{\bar{z}} \tag{3.3a}$$

and

$$r_\ell^{\vee}(\bar{z}) \geqslant |w| \quad \text{for every } w \in (\partial^* F_\ell)_{\bar{z}}. \tag{3.3b}$$

First, we prove (3.3a). To achieve that, we observe that (3.3a) will follow by proving the implication:

$$r_\ell^{\wedge}(\bar{z}) > |w| \Longrightarrow (\bar{z}, w) \in F_\ell^{(1)},$$

or equivalently,

$$r_\ell^{\wedge}(\bar{z}) > |w| \Longrightarrow (\bar{z}, w) \in (\mathbb{R}^n \backslash F_\ell)^{(0)}.$$

To this aim, let $w \in \mathbb{R}^{n-1}$ be such that $r_\ell^{\wedge}(\bar{z}) > |w|$, and let $\delta > 0$ be such that

$$r_\ell^{\wedge}(\bar{z}) = |w| + \delta.$$

Let now $\bar{\rho} \in (0,\, \delta/2]$. Then,

$$|w - w'| < \bar{\rho} \leqslant \frac{\delta}{2} \quad \text{for every } (z', w') \in B_{\bar{\rho}}((\bar{z}, w)).$$

By virtue of the triangle inequality, we have

$$r_\ell^{\wedge}(\bar{z}) = |w| + \delta \geqslant |w'| - |w - w'| + \delta > |w'| + \frac{\delta}{2},$$

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so that

$$r_\ell^\wedge(\bar z) - \frac{\delta}{2} > |w'| \quad \text{for every } (z', w') \in B_{\bar\rho}((\bar z, w)). \tag{3.4}$$

Now, thanks to (3.4) and the definition of F_ℓ, we have

$$\begin{aligned}&(\mathbb{R}^n \backslash F_\ell) \cap B_{\bar\rho}((\bar z, w))\\ &\quad \subset \left\{(z', w') \in \mathbb{R} \times \mathbb{R}^{n-1} : r_\ell^\wedge(\bar z) - \frac{\delta}{2} > |w'| \geqslant r_\ell(z')\right\} \cap B_{\bar\rho}((\bar z, w)).\end{aligned}$$

Hence, for every $\rho \in (0,\, \bar\rho)$, we have

$$\begin{aligned}&\mathcal{H}^n((\mathbb{R}^n \backslash F_\ell) \cap B_\rho((\bar z, w)))\\ &\quad = \int_{\bar z-\rho}^{\bar z+\rho} \mathcal{H}^{n-1}((\mathbb{R}^n \backslash F_\ell) \cap B_\rho((\bar z, w)) \cap \{z = \zeta\})\,\mathrm{d}\zeta\\ &\quad \leqslant \int_{\bar z-\rho}^{\bar z+\rho} \chi_{\left\{r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2}\right\}}(\zeta) \mathcal{H}^{n-1}((\mathbb{R}^n \backslash F_\ell) \cap B_\rho((\bar z, w)) \cap \{z = \zeta\})\,\mathrm{d}\zeta\\ &\quad = \int_{(\bar z-\rho, \bar z+\rho) \cap \left\{r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2}\right\}} \mathcal{H}^{n-1}((\mathbb{R}^n \backslash F_\ell) \cap B_\rho((\bar z, w)) \cap \{z = \zeta\})\,\mathrm{d}\zeta.\end{aligned}$$

Now, for $\rho \in (0,\, \bar\rho)$ and for every $\zeta \in (\bar z - \rho,\, \bar z + \rho)$, we observe that,

$$B_\rho((\bar z, w)) \cap \{z = \zeta\} \subset \left\{(z, w_0) \in \mathbb{R} \times \mathbb{R}^{n-1} : z = \bar z \text{ and } w_0 \in B^{n-1}(w, \rho)\right\}.$$

Therefore, for $\rho \in (0,\, \bar\rho)$ we obtain

$$\begin{aligned}&\mathcal{H}^n((\mathbb{R}^n \backslash F_\ell) \cap B_\rho((\bar z, w)))\\ &\quad \leqslant \int_{(\bar z-\rho, \bar z+\rho) \cap \left\{r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2}\right\}} \mathcal{H}^{n-1}(B^{n-1}(w, \rho))\,\mathrm{d}\zeta\\ &\quad = \omega_{n-1}\rho^{n-1} \int_{(\bar z-\rho, \bar z+\rho) \cap \left\{r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2}\right\}} 1\,\mathrm{d}\zeta\\ &\quad = \omega_{n-1}\rho^{n-1}\ \mathcal{H}^1\left((\bar z - \rho, \bar z + \rho) \cap \left\{r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2}\right\}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}&\lim_{\rho \to 0^+} \frac{\mathcal{H}^n((\mathbb{R}^n \backslash F_\ell) \cap B_\rho((\bar z, w)))}{\omega_n \rho^n}\\ &\leqslant \frac{\omega_{n-1}}{\omega_n} \lim_{\rho \to 0^+} \frac{\mathcal{H}^1\left((\bar z - \rho, \bar z + \rho) \cap \left\{r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2}\right\}\right)}{\rho}\\ &= 0,\end{aligned}$$

where in the last equality we make use of the definition of $r_\ell^\wedge(\bar z)$, see (2.2). This shows (3.3a). Employing an analogous argument, it can be shown that

$$r_\ell^\vee(\bar z) < |w| \Longrightarrow (\bar z, w) \in F_\ell^{(0)}.$$

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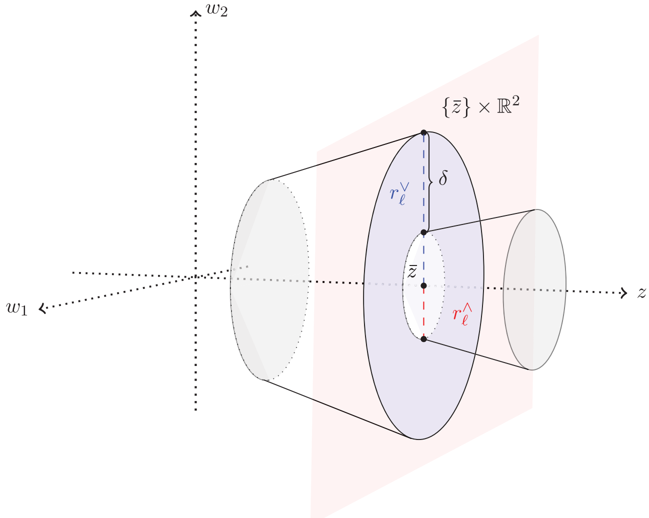

Figure 4. A graphical illustration of Step 1 for $n = 3$.

This implies (3.3b), and finally proves (3.2). For a graphical illustration of Step 1, see Figure 4.

Step 2: We conclude the proof. We first observe that by corollary 2.4 with $B = \{\bar{z}\}$, we obtain

$$\begin{aligned}
\mathcal{H}^{n-1}((\partial^* F_\ell)_{\bar{z}}) &= \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z = \bar{z}\}) \\
&= \mathcal{H}^{n-1}(\partial^* F_\ell \cap (\{\bar{z}\} \times \mathbb{R}^{n-1})) \\
&= P(F_\ell; \{\bar{z}\} \times \mathbb{R}^{n-1}) \\
&= \ell^\vee(\bar{z}) - \ell^\wedge(\bar{z}) \\
&= \mathcal{H}^{n-1}(\overline{B^{n-1}(0, r_\ell^\vee(\bar{z}))}) - \mathcal{H}^{n-1}(B^{n-1}(0, r_\ell^\wedge(\bar{z}))) \\
&= \mathcal{H}^{n-1}(\overline{B^{n-1}(0, r_\ell^\vee(\bar{z}))} \setminus (B^{n-1}((0, r_\ell^\wedge(\bar{z})))).
\end{aligned}$$

Finally, recalling that, by Step 1

$$(\partial^* F_\ell)_{\bar{z}} \subset \overline{B^{n-1}(0, r_\ell^\vee(\bar{z}))} \setminus B^{n-1}(0, r_\ell^\wedge(\bar{z})),$$

we obtain

$$\begin{aligned}
(\partial^* F_\ell)_{\bar{z}} &=_{\mathcal{H}^{n-1}} \overline{B^{n-1}(0, r_\ell^\vee(\bar{z}))} \setminus B^{n-1}(0, r_\ell^\wedge(\bar{z})) \\
&=_{\mathcal{H}^{n-1}} B^{n-1}(0, r_\ell^\vee(\bar{z})) \setminus B^{n-1}(0, r_\ell^\wedge(\bar{z})),
\end{aligned}$$

which concludes the proof. □

Now, we can show that if the set $\{\ell^\wedge > 0\}$ fails to be a (possibly unbounded) interval, then rigidity is violated.

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PROPOSITION 3.2. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume, and let r_ℓ be defined as in (1.3). Suppose that the set $\{\ell^\wedge > 0\}$ is not an interval. That is, suppose that there exists $\bar z \in \{\ell^\wedge = 0\}$ such that*

$$(-\infty, \bar z) \cap \{\ell^\wedge > 0\} \neq \emptyset \quad \textit{and} \quad (\bar z, +\infty) \cap \{\ell^\wedge > 0\} \neq \emptyset.$$

Then, rigidity is violated. More precisely, setting $E_1 := F_\ell \cap \{z < \bar z\}$ and $E_2 = F_\ell \backslash E_1$, then

$$E := E_1 \cup ((0, \tau) + E_2) \in \mathcal{K}(\ell) \quad \textit{for every } \tau \in \mathbb{R}^{n-1}.$$

Proof. Let E_1, E_2 and E be as in the statement. Let also $\tau \in \mathbb{R}^{n-1}$. First of all, note that, since $\{z < \bar z\}$ is open and $E \cap \{z < \bar z\} = F_\ell \cap \{z < \bar z\}$, we have

$$E^{(s)} \cap \{z < \bar z\} = (E \cap \{z < \bar z\})^{(s)} = (F_\ell \cap \{z < \bar z\})^{(s)} = F_\ell^{(s)} \cap \{z < \bar z\}$$

for every $s \in [0, 1]$. In accordance of that, we infer

$$\partial^* E \cap \{z < \bar z\} = \partial^* F_\ell \cap \{z < \bar z\}. \tag{3.5}$$

In the same fashion, for every $\tau \in \mathbb{R}^{n-1}$, we obtain

$$\begin{aligned} \partial^* E \cap \{z > \bar z\} &= \partial^*((0, \tau) + F_\ell) \cap \{z > \bar z\} \\ &= ((0, \tau) + \partial^* F_\ell) \cap \{z > \bar z\} \\ &= (0, \tau) + (\partial^* F_\ell \cap \{z > \bar z\}). \end{aligned} \tag{3.6}$$

Hence, due to (3.5) and (3.6), we have

$$\begin{aligned} P(E) &= \mathcal{H}^{n-1}(\partial^* E \cap \{z < \bar z\}) + \mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) + \mathcal{H}^{n-1}(\partial^* E \cap \{z > \bar z\}) \\ &= \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z < \bar z\}) + \mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) \\ &\quad + \mathcal{H}^{n-1}((0, \tau) + (\partial^* F_\ell \cap \{z > \bar z\})) \\ &= \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z < \bar z\}) + \mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) + \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z > \bar z\}). \end{aligned}$$

As a consequence, in order to complete the proof, we need to show that

$$\mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) = \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z = \bar z\}). \tag{3.7}$$

In what will follow, without loss of generality we assume that

$$r_\ell^\vee(\bar z) = \operatorname{aplim}(r_\ell, (-\infty, \bar z), \bar z) \quad \text{and} \quad r_\ell^\wedge(\bar z) = \operatorname{aplim}(r_\ell, (\bar z, +\infty), \bar z) = 0. \tag{3.8}$$

We divide the proof of (3.7) in several steps.

Step 1: We show that

$$(\partial^* E)_{\bar z} \subset \overline{B^{n-1}(0, r_\ell^\vee(\bar z))} \cup \{\tau\}. \tag{3.9}$$

To this end, it suffices to prove that

$$|w| \leqslant r_\ell^\vee(\bar z) \quad \text{for every } w \in (\partial^* E)_{\bar z} \backslash \{\tau\}. \tag{3.10}$$

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Step 1a: We show that

$$|w| > r_\ell^\vee(\bar z) \Longrightarrow (\bar z, w) \ \in E_1^{(0)}.$$

To this aim, suppose that there exists $\delta > 0$ such that

$$|w| = r_\ell^\vee(\bar z) + \delta.$$

Then, by arguing as in Step 1 of lemma 3.1, for every $\bar\rho \in (0,\, \delta/2)$ we obtain

$$|w'| > r_\ell^\vee(\bar z) + \frac{\delta}{2} \quad \text{for every } (z', w') \in B_{\bar\rho}((\bar z, w)).$$

So, by the definition of E_1, we have

$$\begin{aligned}
&E_1 \cap B_{\bar\rho}((\bar z, w)) = F_\ell \cap \{z < \bar z\} \cap B_{\bar\rho}((\bar z, w)) \\
&\subset \left\{(z', w') \in \mathbb{R} \times \mathbb{R}^{n-1} : z' < \bar z \text{ and } r_\ell^\vee(\bar z) + \frac{\delta}{2} < |w'| < r_\ell(z')\right\} \cap B_{\bar\rho}((\bar z, w)).
\end{aligned}$$

Thus, for every $\rho \in (0,\, \bar\rho)$, by similar calculations as in Step 1 of lemma 3.1, we obtain

$$\begin{aligned}
&\lim_{\rho \to 0^+} \frac{\mathcal{H}^n(E_1 \cap B_\rho((\bar z, w)))}{\omega_n \rho^n} \\
&\leqslant \frac{1}{\omega_n} \lim_{\rho \to 0^+} \int_{(\bar z - \rho, \bar z) \cap \{r_\ell > r_\ell^\vee(\bar z) + \frac{\delta}{2}\}} \mathcal{H}^{n-1}(F_\ell \cap B_\rho((\bar z, w)) \cap \{z = \zeta\})\, \mathrm{d}\zeta \\
&\leqslant \frac{\omega_{n-1}}{\omega_n} \lim_{\rho \to 0^+} \frac{\mathcal{H}^1((\bar z - \rho, \bar z) \cap \{r_\ell > r_\ell^\vee(\bar z) + \frac{\delta}{2}\})}{\rho} \\
&= 0,
\end{aligned}$$

where in the latter inequality (3.8) has been used.

Step 1b: We show that

$$\{z = \bar z\} \backslash \{(\bar z, \tau)\} \subset ((0, \tau) + E_2)^{(0)}. \tag{3.11}$$

To this aim, suppose that $\epsilon := |w - \tau| > 0$. We will prove that $(\bar z,\, w) \in ((0,\, \tau) + E_2)^{(0)}$. Recalling the argument which was used in the proof of (3.4), we choose $\bar\rho \in (0,\, \epsilon/2)$ such that

$$|w' - \tau| > \frac{\epsilon}{2} \quad \text{for every } (z', w') \in B_{\bar\rho}((\bar z, w)).$$

Then, we have

$$\begin{aligned}
&((0, \tau) + E_2) \cap B_{\bar\rho}((\bar z, w)) \\
&= ((0, \tau) + (F_\ell \cap \{z \geqslant \bar z\})) \cap B_{\bar\rho}((\bar z, w)) \\
&\subset \left\{(z', w') \in \mathbb{R} \times \mathbb{R}^{n-1} : z' \geqslant \bar z,\ \ \frac{\epsilon}{2} < |w' - \tau| < r_\ell(z')\right\} \cap B_{\bar\rho}((\bar z, w)).
\end{aligned}$$

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Now, for $\rho \in (0, \bar\rho)$ and for every $\zeta \in (\bar z - \rho, \bar z + \rho)$, we note that,

$$B_\rho((\bar z, w)) \cap \{z = \zeta\} \subset \left\{(z, w_0) \in \mathbb{R} \times \mathbb{R}^{n-1} : z = \bar z \text{ and } w_0 \in B^{n-1}(w, \rho)\right\}.$$

Thus, for every $\rho \in (0, \bar\rho)$,

$$\begin{aligned}\mathcal{H}^n((0,\tau) + E_2) \cap B_\rho((\bar z, w))) &\leqslant \int_{(\bar z, \bar z+\rho)\cap\{r_\ell > \frac{\epsilon}{2}\}} \mathcal{H}^{n-1}(B^{n-1}(w,\rho))\,\mathrm{d}\zeta \\ &= \omega_{n-1}\rho^{n-1} \int_{(\bar z, \bar z+\rho)\cap\{r_\ell > \frac{\epsilon}{2}\}} 1\,\mathrm{d}\zeta \\ &= \omega_{n-1}\rho^{n-1}\ \mathcal{H}^1((\bar z, \bar z + \rho) \cap \left\{r_\ell > \frac{\epsilon}{2}\right\}).\end{aligned}$$

Based on this, by (3.8) we infer that

$$\begin{aligned}\lim_{\rho\to 0^+} \frac{\mathcal{H}^n((0,\tau)) + E_2) \cap B_\rho((\bar z, w)))}{\omega_n \rho^n} &\leqslant \frac{\omega_{n-1}}{\omega_n} \lim_{\rho\to 0^+} \frac{\mathcal{H}^1((\bar z, \bar z+\rho) \cap \left\{r_\ell > \frac{\epsilon}{2}\right\})}{\rho} \\ &= 0,\end{aligned}$$

which proves (3.11).

Step 1c: To conclude the proof of Step 1, we observe that, by Step 1a and 1b, as well as by the definition of E, it follows that

$$\left\{(\bar z, w) \in \mathbb{R} \times \mathbb{R}^{n-1} : |w| > r_\ell^\vee(\bar z)\right\} \Big\backslash \{(\bar z, \tau)\} \subset E_1^{(0)} \cap ((0,\tau) + E_2)^{(0)} = E^{(0)}.$$

Therefore,

$$\begin{aligned}(\partial^* E)_{\bar z} &\subset \mathbb{R}^{n-1} \Big\backslash (\left\{w \in \mathbb{R}^{n-1} : |w| > r_\ell^\vee(\bar z)\right\} \Big\backslash \{\tau\}) \\ &= \overline{B^{n-1}(0, r_\ell^\vee(\bar z))} \cup \{\tau\},\end{aligned}$$

which shows (3.9).

Step 2: Finally, we show (3.7). Note that, thanks to Step 1, lemma 3.1 and perimeter inequality (1.6), we have

$$\begin{aligned}P(E; \{z = \bar z\}) &= \mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) = \mathcal{H}^{n-1}((\partial^* E)_{\bar z}) \\ &\leqslant \mathcal{H}^{n-1}(B^{n-1}((0, r_\ell^\vee(\bar z))) \\ &= \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z = \bar z\}) \\ &= P(F_\ell; \{z = \bar z\}) \leqslant P(E; \{z = \bar z\}),\end{aligned}$$

which makes our proof complete. □

We will now show that, if the jump part $D^j\ell$ of $D\ell$ is non-zero, then rigidity is violated.

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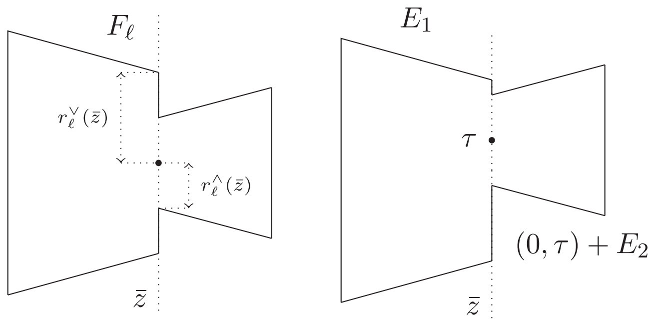

Figure 5. A graphical illustration of Step 1 for $n = 2$.

Proposition 3.3. *Let $\ell : \mathbb{R} \to [0, \infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume, and let r_ℓ be defined as in (1.3). Suppose that ℓ has a jump at some point $\bar z \in \mathbb{R}$. Then rigidity is violated. More precisely, setting $E_1 := F_\ell \cap \{z < \bar z\}$ and $E_2 := F_\ell \backslash E_1$, then*

$$E := E_1 \cup ((0, \tau) + E_2) \in \mathcal{K}(\ell)$$

for every $\tau \in \mathbb{R}^{n-1}$ such that

$$0 < |\tau| < r_\ell^\vee(\bar z) - r_\ell^\wedge(\bar z). \tag{3.12}$$

Proof. Let $E_1,\ E_2$ and E be as in the statement. Let also $\tau \in \mathbb{R}^{n-1}$ be such that (3.12) is satisfied. It is not restrictive to assume that

$$r_\ell^\vee(\bar z) = \mathrm{aplim}(r_\ell, (-\infty, \bar z), \bar z) \quad \text{and} \quad r_\ell^\wedge(\bar z) = \mathrm{aplim}(r_\ell, (\bar z, +\infty), \bar z). \tag{3.13}$$

By an analogous argument as in the beginning of the proof of proposition 3.2, we obtain

$$P(E) = \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z < \bar z\}) + \mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) + \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z > \bar z\}).$$

Hence, in order to complete the proof, we finally need to show that

$$\mathcal{H}^{n-1}(\partial^* E \cap \{z = \bar z\}) = \mathcal{H}^{n-1}(\partial^* F_\ell \cap \{z = \bar z\}). \tag{3.14}$$

We divide the proof of (3.14) into further steps.

Step 1: We prove that

$$(\partial^* E)_{\bar z} \subset \overline{B^{n-1}(0, r_\ell^\vee(\bar z))} \backslash B^{n-1}(\tau, r_\ell^\wedge(\bar z)). \tag{3.15}$$

In order to show (3.15), it suffices to prove that

$$r_\ell^\wedge(\bar z) \leqslant |w - \tau| \quad \text{for every } w \in (\partial^* E)_{\bar z} \tag{3.16a}$$

and

$$r_\ell^\vee(\bar z) \geqslant |w| \quad \text{for every } w \in (\partial^* E)_{\bar z}. \tag{3.16b}$$

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First, let us prove (3.16a). To achieve that, we observe, due to (2.7), our claim will follow if we prove that

$$|w-\tau| < r_\ell^\wedge(\bar z) \Longrightarrow (\bar z, w) \in (\mathbb{R}^n \backslash E)^{(0)}. \tag{3.17}$$

To this end, suppose that $w \in \mathbb{R}^{n-1}$ is such that $|w-\tau| < r_\ell^\wedge(\bar z)$. Then, we observe that

$$\mathbb{R}^n \backslash E = ((\mathbb{R}^n \backslash E) \cap \{z < \bar z\}) \cup ((\mathbb{R}^n \backslash E) \cap \{z \geqslant \bar z\}).$$

Now, arguing as in Step 1 of proposition 3.2, we infer that

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n((\mathbb{R}^n \backslash E) \cap B_\rho((\bar z, w) \cap \{z < \bar z\})}{\omega_n \rho^n} = 0. \tag{3.18}$$

Hence, to complete the proof of the claim, it remains to show that

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}^n((\mathbb{R}^n \backslash E) \cap B_\rho((\bar z, w) \cap \{z \geqslant \bar z)\}}{\omega_n \rho^n} = 0. \tag{3.19}$$

Then there exists $\delta > 0$ such that

$$|w-\tau| + \delta = r_\ell^\wedge(\bar z).$$

Let now $\bar\rho \in (0,\, \delta/2]$. Then, for each $(z',\, w') \in B_{\bar\rho}((\bar z,\, w))$

$$r_\ell^\wedge(\bar z) \geqslant |w'-\tau| - |w-w'| + \delta > |w'-\tau| - \frac{\delta}{2} + \delta = |w'-\tau| + \frac{\delta}{2},$$

so that

$$r_\ell^\wedge(\bar z) - \frac{\delta}{2} > |w'-\tau| \quad \text{for every } (z', w') \in B_{\bar\rho}((\bar z, w)). \tag{3.20}$$

Then, employing (3.20) and the definition of the set E, we infer

$$\begin{aligned}
&(\mathbb{R}^n \backslash E) \cap B_{\bar\rho}((\bar z, w)) \cap \{z \geqslant \bar z\} \\
&\subset \left\{ (z', w') \in \mathbb{R} \times \mathbb{R}^{n-1} : z \geqslant \bar z,\ r_\ell^\wedge(\bar z) - \frac{\delta}{2} > |w'-\tau| \geqslant r_\ell(z') \right\} \cap B_{\bar\rho}((\bar z, w)).
\end{aligned}$$

Moreover, we note that for $\rho \in (0,\, \bar\rho)$ and for every $\zeta \in (\bar z - \rho,\, \bar z + \rho)$, we have

$$B_\rho((\bar z, w)) \cap \{z = \zeta\} \subset \left\{ (z, w_0) \in \mathbb{R} \times \mathbb{R}^{n-1} : z = \bar z \text{ and } w_0 \in B^{n-1}(w, \rho) \right\}.$$

As a consequence, for $\rho \in (0,\, \bar\rho)$ we obtain

$$\begin{aligned}
&\mathcal{H}^n((\mathbb{R}^n \backslash E) \cap B_\rho((\bar z, w)) \cap \{z \geqslant \bar z\}) \\
&\leqslant \int_{(\bar z, \bar z+\rho) \cap \left\{ r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2} \right\}} \mathcal{H}^{n-1}(B^{n-1}(w, \rho))\, \mathrm{d}\zeta \\
&= \omega_{n-1} \rho^{n-1} \int_{(\bar z, \bar z+\rho) \cap \left\{ r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2} \right\}} 1\, \mathrm{d}\zeta \\
&= \omega_{n-1} \rho^{n-1} \mathcal{H}^1((\bar z, \bar z + \rho) \cap \left\{ r_\ell < r_\ell^\wedge(\bar z) - \frac{\delta}{2} \right\}).
\end{aligned}$$

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Then, thanks to (3.13), we infer

$$\begin{aligned}
&\lim_{\rho\to 0^+} \frac{\mathcal{H}^n((\mathbb{R}^n\backslash E)\cap B_\rho((\bar z,w))\cap\{z\geqslant \bar z\})}{\omega_n\rho^n}\\
&\leqslant \frac{\omega_{n-1}}{\omega_n}\lim_{\rho\to 0^+}\frac{\mathcal{H}^1((\bar z,\bar z+\rho)\cap\left\{r_\ell<r_\ell^\wedge(\bar z)-\frac{\delta}{2}\right\})}{\rho}=0,
\end{aligned}$$

where (2.3b) has been employed. This proves (3.19). Then, combining (3.18) and (3.19), (3.17) follows, and thus the proof of (3.16a) is complete.
Now, for (3.16b), arguing again as in Step 1 of proposition 3.2, we have

$$|w|>r_\ell^\vee(\bar z)\Longrightarrow(\bar z,w)\in E_1^{(0)}. \tag{3.21}$$

Making use of similar arguments as above, it can be shown that

$$|w|>r_\ell^\vee(\bar z)\Longrightarrow(\bar z,w)\in((0,\tau)+E_2)^{(0)}, \tag{3.22}$$

which, shows (3.16b), and in turn (3.15). For a graphical illustration of Step 1, see Figure 5.

Step 2: We conclude the proof. From (3.12), we infer that

$$B^{n-1}(\tau,r_\ell^\wedge(\bar z))\subset B^{n-1}(0,r_\ell^\vee(\bar z)).$$

As a consequence, thanks to Step 1, lemma 3.1 and perimeter inequality (1.6), we have

$$\begin{aligned}
P(E;\{z=\bar z\})&=\mathcal{H}^{n-1}(\partial^*E\cap\{z=\bar z\})=\mathcal{H}^{n-1}((\partial^*E)_{\bar z})\\
&\leqslant\mathcal{H}^{n-1}(B^{n-1}((0,r_\ell^\vee(\bar z))\backslash B^{n-1}(\tau,r_\ell^\wedge(\bar z)))\\
&=\ell^\vee(\bar z)-\ell^\wedge(\bar z)\\
&=P(F_\ell;\{z=\bar z\})\\
&\leqslant P(E;\{z=\bar z\}).
\end{aligned}$$

From this, we deduce (3.14), which completes the proof. □

We are going to prove now that if the Cantorian part $D^c\ell$ of $D\ell$ is non-zero, then rigidity is violated.

Proposition 3.4. *Let $\ell:\mathbb{R}\to[0,\infty)$ be a measurable function, such that F_ℓ is a set of finite perimeter and finite volume. Let also r_ℓ be as in (1.3). Suppose that $D^c\ell\neq 0$. Then rigidity is violated.*

Proof. With no loss of generality, we assume that ℓ is a purely Cantorian function. Indeed, one can decompose ℓ as

$$\ell=\ell^a+\ell^j+\ell^c, \tag{3.23}$$

where $\ell^a\in W^{1,1}(\mathbb{R})$, ℓ^j is purely jump function and ℓ^c is purely Cantorian. In the case of $\ell^j\neq 0$, the result becomes trivial since, due to proposition 3.3, rigidity

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is violated. Now, note that, in the generic case where $\ell \neq \ell^c$, due to (3.23), the argument of the proof can be repeated just to the Cantorian part ℓ^c of ℓ. Thus, in what will follow, we assume that

$$D\ell = D\ell^c = D^c\ell.$$

In addition, due to proposition 3.2, we can assume that $\{\ell^\wedge > 0\}$ is an interval, otherwise, the result becomes trivial. Since ℓ is purely Cantorian, there exists a continuous representative of ℓ. From now on, we work with this continuous representative, which is still denoted by ℓ.

Note now that, since ℓ is continuous, there exists $a, b > 0$ such that $J := (a, b) \subset\subset \{\ell^\wedge > 0\}$ and

$$\ell(z) > 0, \qquad \text{for every } z \in J. \tag{3.24}$$

Since $D^c\ell \neq 0$, we can assume that $|D^c\ell|(J) > 0$.

We now fix $\lambda \in (0, 1)$, and we define the function $g : \mathbb{R} \to \mathbb{R}$ as

$$g(z) = \begin{cases} 0, & \text{if } z \in (-\infty, a) \\ \lambda(r_\ell(z) - r_\ell(a)), & \text{if } z \in [a, b] \\ \lambda(r_\ell(b) - r_\ell(a)), & \text{if } z \in (b, +\infty). \end{cases}.$$

Let us fix a unit vector $e \in \mathbb{R}^{n-1}$. We define the set

$$E := \left\{(z, w) \in \mathbb{R} \times \mathbb{R}^{n-1} : |w - g(z)e| < r_\ell(z)\right\}. \tag{3.25}$$

One can observe that we cannot obtain E using a single translation on F_ℓ along $\mathbb{R}^{n-1}$. We are going to prove now that $E \in \mathcal{K}(\ell)$. We divide the proof in several steps.

Step 1: We construct a sequence $\{\ell^k\}_{k\in\mathbb{N}}$, where $\ell^k : J \to [0, \infty)$, which satisfies the following properties:

(i) $r^k_\ell(z) \longrightarrow r_\ell(z)$, as $k \to \infty$ for every $z \in J$,

(ii) $D\ell^k = D^j\ell^k$ for every $k \in \mathbb{N}$,

(iii) $\lim\limits_{k\to\infty} P(F_{\ell^k}; J \times \mathbb{R}^{n-1}) = P(F_\ell; J \times \mathbb{R}^{n-1})$.

By (2.6) and since ℓ is continuous, we have

$$|D\ell|(J) = \sup\left\{\sum_{i=1}^{N-1} |\ell(z_{i+1}) - \ell(z_i)| : \; a < z_1 < z_2 < \cdots < z_N < b\right\},$$

where the supremum runs over $N \in \mathbb{N}$ and over all $z_1, z_2, \ldots, z_N$ with $a < z_1 < z_2 < \cdots < z_N < b$. From that, for every $k \in \mathbb{N}$ there exist $N_k \in \mathbb{N}$ and

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$z_1^k, \ldots, z_N^k$ with $a < z_1^k < \cdots < z_N^k < b$ such that

$$|D\ell|(J) \leqslant \sum_{i=1}^{N_k-1} |\ell(z_{i+1}^k) - \ell(z_i^k)| + \frac{1}{k} \tag{3.26}$$

and

$$|z_{i+1}^k - z_i^k| < \frac{1}{k}, \quad \text{for every } i = 1, \ldots, N_k - 1.$$

It is not restrictive to assume that the partitions are increasing in k, i.e.

$$\{z_1^k, \cdots, z_{N_k}^k\} \subset \{z_1^{k+1}, \cdots, z_{N_k+1}^{k+1}\} \quad \text{for every } k \in \mathbb{N}.$$

We define now for every $k \in \mathbb{N}$

$$\ell^k(z) := \sum_{i=0}^{N_k} \ell(z_i^k) \chi_{[z_i^k, z_{i+1}^k)}(z), \quad z \in J, \tag{3.27}$$

where we set $z_0^k := a$ and $z_{N_k+1}^k := b$. Moreover, we set

$$r_\ell^k(z) := \left(\frac{\ell^k(z)}{\omega_{n-1}}\right)^{1/n-1} \qquad \text{for every } z \in J \text{ and for every } k \in \mathbb{N}.$$

Note that, by definition, $r_\ell^k = r_{\ell^k}$ and $r_\ell^k \in BV(J)$. By the continuity of ℓ, we infer that

$$\ell^k(z) \longrightarrow \ell(z) \quad \text{uniformly for all } z \in \bar{J}. \tag{3.28}$$

Hence, since the map $\eta \longmapsto (\eta/\omega_{n-1})^{1/n-1}$ is continuous in $(0, \infty)$, we infer that (i) holds true. Moreover, by (3.27), (ii) holds also true.
For (iii), thanks to (3.28), we infer that

$$\lim_{k\to\infty} \mathcal{H}^{n-2}((\partial^* F_{\ell^k})_z) = \mathcal{H}^{n-2}((\partial^* F_\ell)_z) \text{ for all } z \in J. \tag{3.29}$$

In addition,

$$\begin{aligned} |D\ell^k|(J) &= \sum_{i=0}^{N_k} |\ell(z_{i+1}^k) - \ell(z_i^k)| \\ &= |\ell(z_1^k) - \ell(a)| + \sum_{i=0}^{N_k-1} |\ell(z_{i+1}^k) - \ell(z_i^k)| + |\ell(b) - \ell(z_{N_k}^k)|. \end{aligned} \tag{3.30}$$

Now, using (3.26), we obtain

$$|D\ell|(J) - \frac{1}{k} \leqslant \sum_{i=1}^{N_k-1} |\ell(z_{i+1}^k) - \ell(z_i^k)| \leqslant |D\ell|(J).$$

Combining the above inequality with (3.30) and recalling again the continuity of ℓ, we infer that

$$\begin{aligned}|D^c\ell|(J) = |D\ell|(J) &= \lim_{k\to\infty} \sum_{i=0}^{N_k-1} |\ell(z_{i+1}^k) - \ell(z_i^k)| \\ &= \lim_{k\to\infty} |D\ell^k|(J) = \lim_{k\to\infty} |D^s\ell^k|(J). \end{aligned} \tag{3.31}$$

Finally, recalling corollary 2.4 and employing (3.29), we obtain

$$\begin{aligned}\lim_{k\to\infty} P(F_{\ell^k}; J\times\mathbb{R}^{n-1}) &= \lim_{k\to\infty} \Big(\int_J \mathcal{H}^{n-2}((\partial^* F_{\ell^k})_z)\,\mathrm{d}z + |D^s\ell^k|(J)\Big) \\ &= \int_J \mathcal{H}^{n-2}((\partial^* F_\ell)_z\,\mathrm{d}z + |D^s\ell|(J) \\ &= P(F_\ell; J\times\mathbb{R}^{n-1}),\end{aligned}$$

which proves (iii).

Step 2: For $k\in\mathbb{N}$, we will construct a ℓ^k-distributed set E^k satisfying

$$P(E^k; J\times\mathbb{R}^{n-1}) = P(F_{\ell^k}; J\times\mathbb{R}^{n-1}).$$

As a consequence of (3.) in Step 1, for $k\in\mathbb{N}$ we infer that $Dr_\ell^k = D^j r_\ell^k$ and that the jump set of r_{ℓ^k} is a finite set. In particular,

$$Dr_\ell^k = \sum_{i=1}^{N_k} (r_\ell(z_i^k) - r_\ell(z_{i-1}^k))\delta_{z_i^k},$$

where, for each $i\in\{1, 2, \dots, N_k\}$, $\delta_{z_i^k}$ denotes the Dirac delta measure concentrated at the point z_i^k. Let us now fix $\lambda\in(0, 1)$ and we define iteratively the family of sets $\{E_i^k\}_{i=1}^{N_k}\subset J\times\mathbb{R}^{n-1}$ as

$$\begin{aligned} E_1^k &:= [F_{\ell^k}\cap(\{z<z_1^k\}\backslash\overline{\{z<a\}})]\cup[\lambda(r_\ell(z_1^k)-r_\ell(a))e \\ &\qquad + (F_{\ell^k}\cap(\{z<b\}\backslash\{z<z_1^k\}))] \\ E_2^k &:= [E_1^k\cap\{z<z_2^k\}]\cup[\lambda(r_\ell(z_2^k) \\ &\qquad - r_\ell(z_1^k))e + (E_1^k\backslash\{z<z_2^k\})] \\ &\ \ \vdots \\ E_{N_k}^k &:= [E_{N_k-1}^k\cap\{z<z_{N_k}^k\}]\cup[\lambda(r_\ell(z_{N_k}^k)-r_\ell(z_{N_k-1}^k))e \\ &\qquad + (E_{N_k-1}^k\backslash\{z<z_{N_k}^k\})],\end{aligned}$$

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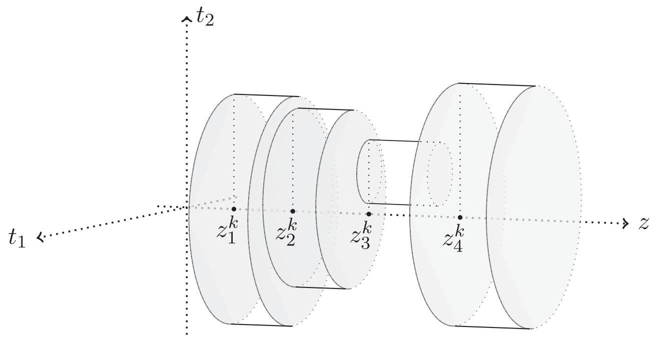

Figure 6. A graphical illustration of the set $E^k_{N_k}$ in Step 2.

see Figure 6. Applying proposition 3.3 for each $i \in \{1, \ldots, N_k\}$, we infer that

$$P(E^k_1; J \times \mathbb{R}^{n-1}) = P(E^k_2; J \times \mathbb{R}^{n-1}) = \cdots = P(E^k_{N_k}; J \times \mathbb{R}^{n-1}) \\ = P(F_{\ell^k}; J \times \mathbb{R}^{n-1}).$$

Note now, that for $i \in \{1, 2, \cdots, N_k\}$ the general term of the above family of sets can be written as

$$\begin{aligned} E^k_i = {} & [F_{\ell^k} \cap \{z < z^k_1\} \backslash \overline{\{z < a\}}] \cup [\lambda(r_\ell(z^k_1) - r_\ell(a))e \\ & + (F_{\ell^k} \cap (\{z < z^k_2\} \backslash \{z < z^k_1\}))] \\ & \cup [\lambda(r_\ell(z^k_2) - r_\ell(a))e + (F_{\ell^k} \cap (\{z < z^k_3\} \backslash \{z < z^k_2\}))] \\ & \cup \cdots \cup [\lambda(r_\ell(z^k_{N_k}) - r_\ell(a))e + (F_{\ell^k} \cap (\{z < b\} \backslash \{z < z^k_{N_k}\}))]. \end{aligned}$$

Therefore, if we set

$$E^k := E^k_{N_k} = \left\{(z, w) \in J \times \mathbb{R}^{n-1} : |w - \lambda(r_{\ell^k}(z) - r_{\ell^k}(a))e| < r_{\ell^k}(z)\right\}, \tag{3.32}$$

we conclude that

$$P(E^k; J \times \mathbb{R}^{n-1}) = P(F_{\ell^k}; J \times \mathbb{R}^{n-1}), \quad \text{for every } k \in \mathbb{N}.$$

Step 3: We claim now, that

$$E^k \longrightarrow \widetilde{E} \quad \text{in } J \times \mathbb{R}^{n-1}$$

for some ℓ-distributed set $\widetilde{E}$ satisfying

$$P(\widetilde{E}; J \times \mathbb{R}^{n-1}) = P(F_\ell; J \times \mathbb{R}^{n-1}).$$

Indeed, thanks to (3.) of Step 1, it turns out that

$$r^k_\ell(z) \longrightarrow r_\ell(z) \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in J.$$

As a result, recalling (3.32) and if $\widetilde{E}$ is defined as

$$\widetilde{E} := \left\{(z, w) \in J \times \mathbb{R}^{n-1} : |w - \lambda(r_\ell(z) - r_\ell(a))e| < r_\ell(z)\right\}, \tag{3.33}$$

it follows that $\widetilde{E}$ is ℓ-distributed and $E^k \longrightarrow \widetilde{E}$ in $J \times \mathbb{R}^{n-1}$.

Finally, by Step1, Step 2, lower semicontinuity of perimeter with respect to L^1 convergence (see e.g. [**17**, Theorem 12.15]) and perimeter inequality (1.6), we obtain

$$\begin{aligned} P(F_\ell; J \times \mathbb{R}^{n-1}) &\leqslant P(\widetilde{E}; J \times \mathbb{R}^{n-1}) \leqslant \liminf_{k\to\infty} P(E^k; J \times \mathbb{R}^{n-1}) \\ &= \liminf_{k\to\infty} P(F_{\ell^k}; J \times \mathbb{R}^{n-1}) = \lim_{k\to\infty} P(F_{\ell^k}; J \times \mathbb{R}^{n-1}) \\ &= P(F_\ell; J \times \mathbb{R}^{n-1}), \end{aligned}$$

and thus,

$$P(\widetilde{E}; J \times \mathbb{R}^{n-1}) = P(F_\ell; J \times \mathbb{R}^{n-1}).$$

Step 4: Now consider the set E defined in (3.25). By previous steps, it turns out that E is ℓ-distributed, and furthermore

$$\begin{aligned} E =_{\mathcal{H}^n} &(F_\ell \cap \{z < a\}) \cup [\widetilde{E} \cap (\{z < b\} \backslash \{z < a\})] \cup [\lambda(r_\ell(b) - r_\ell(a))e \\ &+ (F_\ell \backslash \{z < b\})]. \end{aligned}$$

Since $J \times \mathbb{R}^{n-1} = (a,\, b) \times \mathbb{R}^{n-1} = \{z < b\} \backslash \overline{\{z < a\}}$ and using similar argument as in the proof of proposition 3.2, we have

$$\begin{aligned} P(E) &= P(E; \{z < a\}) + P(E; \{z = a\}) + P(E; \{z < b\} \backslash \overline{\{z < a\}}) \\ &\quad + P(E; \{z = b\}) + P(E; \{z > b\}) \\ &= P(F_\ell; \{z < a\}) + P(E; \{z = a\}) + P(\widetilde{E}; \{z < b\} \backslash \overline{\{z < a\}}) \\ &\quad + P(E; \{z = b\}) + P(F_\ell; \{z > b\}) \\ &= P(F_\ell; \{z < a\}) + P(E; \{z = a\}) + P(F_\ell; \{z < b\} \backslash \overline{\{z < a\}}) \\ &\quad + P(E; \{z = b\}) + P(F_\ell; \{z > b\}), \end{aligned}$$

where Step 3 has been employed.
In addition, an analogous argument as in Step 1 of the proof of proposition 3.3 shows that

$$P(E; \{z = a\}) = P(E; \{z = b\}) = 0.$$

As a consequence, we infer

$$\begin{aligned} P(E) &= P(F_\ell; \{z < a\}) + P(F_\ell; \{z < b\} \backslash \overline{\{z < a\}}) + P(F_\ell; \{z > b\}) \\ &= P(F_\ell). \end{aligned}$$

Therefore, $E \in \mathcal{K}(\ell)$. In the light of this, the proof is completed. □

We can now show the implication $(i) \Longrightarrow (ii)$ of theorem 1.3.

Proof of theorem 1.3: $\boldsymbol{(i)} \Longrightarrow \boldsymbol{(ii)}$. We assume that (ii) is false. Namely, suppose that either the set $\{\ell^\wedge > 0\}$ is not an interval or $\{\ell^\wedge > 0\}$ is an interval and

$\ell \notin W^{1,1}(\{\ell^\wedge > 0\})$. Then, if $\{\ell^\wedge > 0\}$ is not an interval, by proposition 3.2, we have that rigidity is violated. On the other hand, if $\ell \notin W^{1,1}(\{\ell^\wedge > 0\})$ then, by propositions 3.3 and 3.4, rigidity is also violated. This contradiction completes our proof. □

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