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FINITE-DIMENSIONAL EXTENSIONS OF CERTAIN SYMMETRIC OPERATORS(1)

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1. Let H be a Hilbert space with inner product $\langle\ ,\ \rangle$. A well-known theorem of von Neumann states that, if S is a symmetric operator in H, then S has a self-adjoint extension in H if and only if S has equal deficiency indices. This result was extended by Naimark, who proved that, even if the deficiency indices of S are unequal, there always exists a Hilbert space H_1 such that $H \subseteq H_1$ and S has a self-adjoint extension in H_1. Such an extension is called *finite-dimensional* if $\dim(H_1 \ominus H) < \infty$.

Recent extension theorems of Coddington [1] can be applied to non-densely defined operators to generalise the von Neumann and Naimark theorems. Coddington's results, summarised in §2 below, are used in §3 to prove simply that a closed symmetric operator with finite but unequal deficiency indices has no finite-dimensional selfadjoint extensions. This result is contained in work by Gilbert [3, Theorem 7], but the method of proof in §3 is new.

2. Let H^2 be the space $H \oplus H$ (Cf. [2, pp. 255–256, IV. 4.16]), and let T be a closed subspace of H^2. The adjoint of T, T^*, is defined by

$$T^* = \{(h, k) \in H^2 : \langle g, h\rangle = \langle f, k\rangle \quad \text{for all} \quad (f, g) \in T\}.$$

T is said to be *symmetric* if $T \subseteq T^*$ and to be *selfadjoint* if $T = T^*$.

For a symmetric subspace T, let M^+ and M^- be defined by

$$M^{\pm} = \{(h, k) \in T^* : k = \pm ih\}.$$

(If T is the graph of a densely-defined symmetric operator, M^+ and M^- are the usual deficiency subspaces.) Coddington proves [1, Theorem 4] that T has a selfadjoint extension in H^2 if and only if $\dim M^+ = \dim M^-$.

3. Let S be the graph of a Hermitian operator in H, i.e. $\langle Sf, g\rangle = \langle f, Sg\rangle$ for all f and g in $D(S)$, such that $D(S)$ is not necessarily dense in H but such that $R(S) \subseteq \overline{D(S)}$. ($D(S)$, $R(S)$ denote the domain and range of S respectively.) Then S is symmetric in the sense of §2 above and Coddington's theorem gives a criterion for the existence of a selfadjoint subspace of H^2 which extends S. If such an extension, A say, exists, then A_s, in the notation of [1], is the graph of an operator,

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selfadjoint in $\overline{D(A_s)}$, which extends S in H. Moreover, all the selfadjoint extensions of S in H are described in this way, since we have the

LEMMA. *Let K be a closed subspace of* H, *and let A be the graph of an operator such that $\overline{D(A)}=K$ and A is selfadjoint in K. In particular, $R(A) \subseteq K$. Let $B=A\oplus\{(0, g) \in H^2 : g \in K^{\perp}\}$. Then B is a selfadjoint subspace of* H^2.

To describe the relationship between the possible selfadjoint extensions of S in H^2 and the possible selfadjoint extensions of S in $\overline{D(S)}$, let S' denote the adjoint of S in $\overline{D(S)}$, and let D_+, D_- be the deficiency subspaces of S in $\overline{D(S)}$, i.e. $D_{\pm}=\{(g, S'g) \in S' : S'g=\pm ig\}$. Then S has a selfadjoint extension in $\overline{D(S)}$ if and only if $\dim D_+=\dim D_-$.

If X^+ and X^- are defined by

$$X^{\pm} = \{(h, \pm ih) \in H^2 : h \in H\ominus D(S)\},$$

a computation shows that

$$M^+ = D_+\oplus X^+,\ M^- = D_-\oplus X^-.$$

But $\dim X^+=\dim X^-=\dim(H\ominus D(S))$. Thus, if $\dim D_+$ and $\dim D_-$ are finite but unequal, S does not have a selfadjoint extension in H^2 if $\dim(H\ominus D(S))$ is finite. In particular, we have the

THEOREM. *Let S be a (densely-defined) closed symmetric operator in* H *with finite but unequal deficiency indices, and let H_1 be a Hilbert space such that $H \subset H_1$ and such that S has a selfadjoint extension in H_1. Then* $\dim(H_1\ominus H)=\infty$.

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