Patterns within
The ramblings of a former editor

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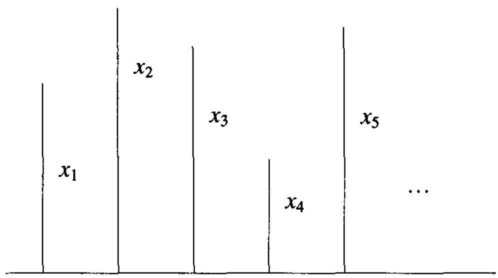
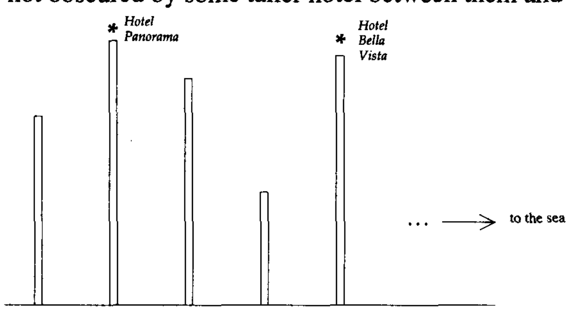
You have heard about the monkeys who will eventually type the complete works of Shakespeare, but that's just one example of the fact that eventually patterns will emerge out of chaos. As a simple example let us prove the well-known result that any sequence of real numbers must have an increasing or a decreasing subsequence. To avoid any ambiguity about what these terms mean let us assume that the numbers in the sequence are all different and, for the sake of illustration, let us assume that they are all positive. Then illustrate the sequence $x_1, x_2, x_3, x_4, x_5, \ldots$ by vertical lines in the following way:



Now imagine that these are hotels stretching down to the sea on the Costa Little: mark those hotels '*' if they have a sea-view, i.e. if their view of the sea is not obscured by some taller hotel between them and the sea:



Now either there will exist an infinite number of hotels with a sea-view (in which case they clearly form a decreasing sequence) or from some point onwards (the kth, say) no hotel has a sea-view. But then the kth's view is obscured by some higher one, whose view is obscured by some higher one, whose view is ... giving, an increasing subsequence.

For a different example consider the well-known fact that if there are six people in a room then either three of them will know each other or three will be mutual strangers. To make this more precise and to prepare the ground for generalisations consider the six people as points in a plane and join two

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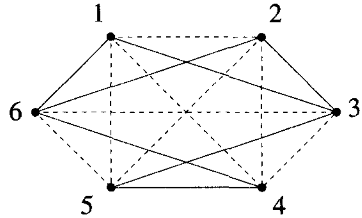
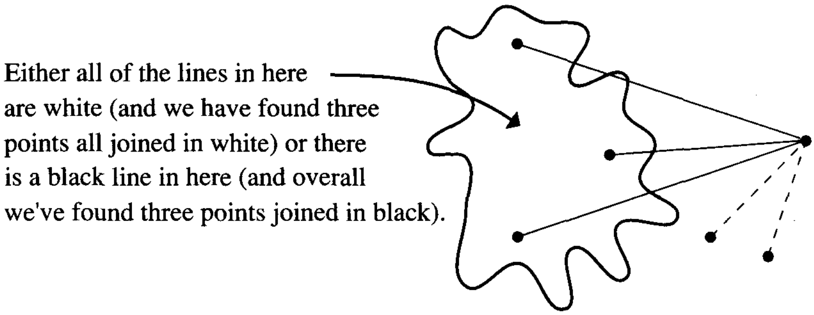
by a black line if they know each other and by a white line (shown as dotted in the picture) if they don't. Then there will exist three points all joined to each other in the same colour; e.g. 1, 2 and 5 in the following example:



The proof of that result for any colouring is delightful. Consider any one point and the five lines leading from it (each in black or white). There must be at least three lines of one colour: assume, say, that there are three black. Now consider the three points at the other ends of those three black lines:



This can be extended in all sorts of ways: see, for example, Keith Austin's article [1] – which also shows some links with the previous problem – or the Chapter on Ramsey theory in my book [2]. One interesting way of extending this result is to prove that *if* 17 points are *all* joined in pairs by lines in any one of three colours, then again there will be three of them all joined in lines of the same colour. Can you see how the proof will go? Imagine the 17 have all been joined up in black, white or red lines. Then consider any one point and the 16 lines coming from it: there must be at least of six of the same colour, so assume that there are six red lines and look at the points at the other end of those six red lines. If any pair of them is joined in red we have found three all joined in red. On the other hand if all six of those points are joined in black and white then we are back to the earlier problem and there are bound to be three joined in black or white.

Where did the '17' come from? We had to make sure that the 16 lines included at least 6 of one colour. This followed since $16 = 5 + 5 + 5 + 1$ and so 5 of each colour would not be enough. So how many points will have to be joined up in four colours to ensure that we can use similar arguments to show that there exist three all joined in the same colour? When we consider one point we would like its lines to include 17 of the same colour, so $65 (= 4 \times 16 + 1)$ lines will ensure this, and we must look at 66 points all joined together in one of four colours, etc. The table shows the values obtained inductively by this sort of argument, and the interested reader can prove it:

Number of colours	1	2	3	4	5	...	n	...
Number of points	3	6	17	66	327	...	$[n!e] + 1$	...

It's a pleasant surprise that the number e has emerged from that finite process. The result also enables you to deduce Schur's theorem which says that if the integers from 1 to $[n!e]$ are each coloured in one of n colours then there are bound to exist x, y and z, not necessarily distinct, of the same colour and with $x + y = z$. To prove it, imagine that you are given the integers coloured. Then take $[n!e] + 1$ points in the plane, numbered 1, 2, 3, ..., and join each one to each of the others using the same colour for the line joining i to j ($i < j$) as was used for the number $j - i$. There must exist three, i, j and k say with $i < j < k$, joined in the same colour: what does that tell you about the numbers $j - i$, $k - j$ and $k - i$?

That result leads us on to another related problem. We illustrate this first by making the following observation: if you evenly space out nine balls in a row, each of them being black or white, then there are bound to be three of the same colour with one exactly mid-way between the other two; e.g. 3, 6 and 9 in the following array:

○	●	○	●	●	○	●	○	○
1	2	3	4	5	6	7	8	9

However it is certainly possible to place eight balls in a row without that property (for example the eight obtained by omitting number 9 above). Another way of saying this is that if you colour each of the integers 1 to 9 black or white then three of them of the same colour will have one which is the average of the other two. In order to prepare the way for generalisations, an even grander way of saying the same thing is that:

> If you colour the integers 1 to 9 in either of 2 colours there will exist a 3-term arithmetic progression in one colour.

It is amazing that we can generalise that result to any number of colours instead of just 2 and any length arithmetic progression instead of 3 that we like: of course the number corresponding to the 9 might be very much larger and indeed, in general, quite hard to find. For example there exist numbers M and N such that:

> If you colour the integers 1 to M in any of 3 colours there will exist a 3-term arithmetic progression in one colour.
>
> If you colour the integers 1 to N in any of 3 colours there will exist a 4-term arithmetic progression in one colour.
>
> Etc.

With a bit of computer help you might like to try to find the lowest M and N which work. But if you're not actually looking for the lowest number which works, then some delightful counting arguments can be used. For example changing the 9 to 325 in the first result gives

> If you colour the integers 1 to 325 in either of 2 colours there will exist a 3-term arithmetic progression in one colour

and enables us to use the following proof. Imagine the coloured 1 – 325 as a row of evenly-spaced black and white balls and imagine them in 65 clumps of 5. So for example the first clump might be

○ ○ ● ● ○

and the second clump might be

● ○ ● ● ○

As there are only 32 different patterns of this type there must be two clumps amongst the first 33 which have the same pattern, the mth and nth clumps, say, where $m < n$. Then look also at the $(2n - m)$th clump (which is still amongst the 65 clumps). Then either one of these clumps itself contains a 3-term arithmetic progression of one colour (and we've finished) or overall we'll look at our three special clumps; e.g.

... ● ○ ● ● ○ ... ● ○ ● ● ○ ... ○ ○ ○ ○ ○ ...
? ? ? ? ??

Now imagine the ball marked '??' as white and black in turn: can you see in the example illustrated that in either case it is possible to pick out a 3-term arithmetic progression in one colour? Can you see how the proof generalises to all cases?

The advantage of that proof is that it generalises to higher numbers of colours. For example, if you want to find your own proof of the corresponding result for a 3-term arithmetic progression in one colour when 3 colours are allowed, try replacing $325\left(= (2\times 2+1)\times(2\times 2^{2\times 2+1}+1)\right)$ by

$$(2\times 3+1)\times(2\times 3^{2\times 3+1}+1)\times\left(2\times 3^{(2\times 3+1)\times(2\times 3^{2\times 3+1}+1)}+1\right)$$

and look at clumps of clumps of 7(!). This idea leads to the *van der Waerden numbers.*

All the above examples are part of Ramsey theory which covers a whole family of results illustrating in some specific way that 'infinite disorder is impossible'. When I remember my desk during the ten years in which I edited the *Gazette* I am tempted to think that Ramsey theory is wrong. That was partly the excuse for including these items in an article for this centenary edition. But no excuse is really needed: the delightfulness of the results and the surprise emergence of a pattern in each case will, I hope, have entertained you. Indeed, during my editorship one of the real joys of the job was the discovery by various contributors of some pattern which I had not seen before. Ones that stick in mind include 'Strike it out – add it up' [3] which observed the following patterns obtained by deleting some integers and forming subtotals of what is left:

Delete every second	1 ~~2~~ 3 ~~4~~ 5 ~~6~~ 7 ~~8~~ 9 ~~10~~ 11 …
Form subtotals of the rest	1 4 9 16 25 36 …

That was expected, but what about:

Delete every third	1 2 ~~3~~ 4 5 ~~6~~ 7 8 ~~9~~ 10 11 …
Form subtotals and delete every second	1 ~~3~~ 7 ~~12~~ 19 ~~27~~ 37 ~~48~~ …
Form subtotals of the rest	1 8 27 64 …

That delightful emergence of squares and cubes generalises in all sorts of ways.

I also remember with pleasure the articles on self-descriptive strings [4, 5, 6]. What is special about the list

$$6 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0?$$

How many 0's does it have? How many 1's? How many 2's? ... As subsequent articles and correspondence showed, this topic created a lot of classroom interest. Other ideas which I learned in that era and have used time and again in my teaching are the neat proofs that $\cos(\sin x) > \sin(\cos x)$ [7], combinations of postage stamps [8] (and many subsequent notes), the convergence of $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ [9], and so on and so on.

I also had my fair share of world scoops. Perhaps the best 'proof' of Fermat's last theorem out of the many I received was the following: 'If you take a cube of side 3 and a cube of side 4 and break them down into little 1 × 1 × 1 cubes you get 91 cubes altogether. No matter how hard you try these do not form another cube. *A similar argument works in all cases.*'

Another gem missed by *Gazette* readers was the following 'proof' of Goldbach's conjecture that every even number greater than 4 can be written as the sum of two odd primes. Bearing in mind the simplicity of the following proof, it is amazing that it defeated great mathematicians for so long: 'Let the two odd primes be $2n + 1$ and $2m + 1$: their sum is $2n + 2m + 2$ which *is* even'.

All in all, editing the *Gazette* for part of its first 100 years was a privilege which made me many friends throughout the world and it greatly enriched the pattern of my mathematical life.

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Sir William McCrea was a Scholar at Trinity College, Cambridge, receiving his BA in 1926 and PhD in 1929. He lectured at Edinburgh University before moving to Imperial College as a Reader in 1932. In 1936 he was appointed to a Chair in Belfast, moving to Royal Holloway College in 1944 and to the University of Sussex in 1972, where he was Professor of Astronomy. He was awarded the higher degree of ScD in 1957 and has five honorary degrees. His books include Relativity physics, Analytical geometry of three dimensions and Physics of the sun and stars. A fellow of the Royal Society, his research interests include analysis, mathematical physics, cosmogony and cosmology. He was President of the Royal Astronomical Society 1961-63 and has served as a governor of Royal Holloway College. He joined the Association in the 1930s and was President 1973-74. He first contributed to the Gazette in 1933.

Victor Bryant has been at the University of Sheffield since beginning his PhD in 1966. Now a senior lecturer, he is author of several undergraduate books including *Yet another introduction to analysis* and *Aspects of combinatorics*. His first *Gazette* article was in 1971 and he was Editor from 1980 to 1990. He enjoys brainteasers and regularly sets them for the *Sunday Times* and for *New Scientist*. He has forgotten when he first joined the Association 'but it was a long time ago'.

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