Operators on Anti-dual pairs: Self-adjoint Extensions and the Strong Parrott Theorem

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Abstract. The aim of this paper is to develop an approach to obtain self-adjoint extensions of symmetric operators acting on anti-dual pairs. The main advantage of such a result is that it can be applied for structures not carrying a Hilbert space structure or a normable topology. In fact, we will show how hermitian extensions of linear functionals of involutive algebras can be governed by means of their induced operators. As an operator theoretic application, we provide a direct generalization of Parrott's theorem on contractive completion of 2 by 2 block operator-valued matrices. To exhibit the applicability in noncommutative integration, we characterize hermitian extendibility of symmetric functionals defined on a left ideal of a C^*-algebra.

Introduction

The question whether a self-adjoint extension exists arises naturally in various situations when a partially defined (bounded or unbounded) symmetric operator is given. For classical results, we refer the reader to [2, 6, 8, 16] and the references therein; for more recent results, see, for example, [3,11]. In our previous paper [19], we developed a Krein–von Neumann type extension theory for positive operators acting on anti-dual pairs. That technique is general enough to not only overcome the lack of a Hilbert space structure, but also the lack of a normable topology. Our running example in [19]—illustrating the applicability of the general setting—came from noncommutative integration theory. Namely, we have demonstrated how functional extensions can be governed by their induced operators. The aim of this paper is to continue these investigations and to discuss the problem of self-adjoint extendibility.

Below, we briefly describe the content. Section 1 contains a short overview of concepts and earlier results that help the reader to follow the proofs. In particular, we briefly sketch the construction of the generalized Krein–von Neumann extension that serves as the basis for our treatment. In Section 2, we examine the extension problem

in our anti-dual pair setting. The main result, Theorem 2.1, generalizes Krein's theorem on the existence of a norm preserving self-adjoint extension of a bounded symmetric operator [10, Theorem 5.33]. Due to the lack of norm, we are going to consider extensions bounded by a fixed positive operator A. It will turn out that extensions preserving the A-bound form an operator interval. As a nice application of Theorem 2.1, in Section 3, we will generalize Yamada's recent result [21], which is an extension of the Strong Parrott Theorem [7,12]. We will close the paper by demonstrating that Theorem 2.1 on self-adjoint extensions is an effective generalization. Namely, we will see in Section 4 how this result can be applied to obtain hermitian extensions of functionals of an involutive algebra.

1 Preliminaries

In this section, we briefly summarize all the notions and tools to make the presentation self-contained. For more details we refer the reader to [19, Section 2 and 3]. An anti-dual pair denoted by $\langle F, E \rangle$ is a system of two complex vector spaces E and F intertwined by a separating sesquilinear map $\langle \cdot, \cdot \rangle \colon F \times E \to \mathbb{C}$, *i.e.,* $\langle \cdot, \cdot \rangle$ is linear in its first, and conjugate linear in its second argument. Let D be a linear subspace of E. We call a linear operator $A \colon D \to F$ *symmetric*, if $\langle Ax, y \rangle = \overline{\langle Ay, x \rangle}$ holds for all $x, y \in D$. In analogy with the Hilbert space case, an operator $A \colon D \to F$ is said to be *positive* if

$$\langle Ax, x \rangle \geq 0, \qquad x \in D.$$

Just as in the dual pair case, we endow E and F with the corresponding weak topologies $\sigma(E, F)$ (resp. $\sigma(F, E)$), induced by the families $\{\langle f, \cdot \rangle \mid f \in F\}$ (resp. $\{\langle \cdot, x \rangle \mid x \in E\}$). Both $\sigma(E, F)$ and $\sigma(F, E)$ are locally convex Hausdorff topologies with duality properties

(1.1) $$\bar{E}' = F \qquad \text{and} \qquad F' = E.$$

Here, F' and $\bar{E}'$ refer to the topological dual and anti-dual space of F and E, respectively, and the vectors $f \in F$ and $x \in E$ are identified with $\langle f, \cdot \rangle$, and $\langle \cdot, x \rangle$, respectively. We will call the anti-dual pair $\langle F, E \rangle$ weak-$\star$ sequentially complete if the topological vector space $(F, \sigma(F, E))$ is sequentially complete.

One of the useful properties of weak topologies is the following: for a topological vector space (V, τ), a linear operator $T \colon V \to F$ is continuous with respect to τ and $\sigma(F, E)$ if and only if the linear functionals

$$T_x(v) \coloneqq \langle Tv, x \rangle, \qquad v \in V,$$

are continuous for every $x \in E$.

This fact and (1.1) enable us to define the adjoint (that is, the topological transpose) of a weakly continuous operator. Let $\langle F_1, E_1 \rangle$ and $\langle F_2, E_2 \rangle$ be anti-dual pairs and let $T \colon E_1 \to F_2$ be a weakly continuous linear operator. Then the weakly continuous linear operator $T^* \colon E_2 \to F_1$ satisfying

$$\langle Tx_1, x_2 \rangle_2 = \overline{\langle T^* x_2, x_1 \rangle_1}, \qquad x_1 \in E_1, x_2 \in E_2,$$

is called the adjoint of T. The set of *everywhere defined* weakly continuous (*i.e.,* $\sigma(E, F)$-$\sigma(F, E)$ continuous) linear operators $T \colon E \to F$ will be denoted by $\mathscr{L}(E, F)$. In the case when H and K are Hilbert spaces, $\mathscr{L}(H, K)$ coincides with the set $\mathscr{B}(H, K)$

of bounded linear operators from H to K. An operator $T \in \mathscr{L}(E, F)$ is called self-adjoint if $T^* = T$. Obviously, everywhere defined symmetric operators (and hence everywhere defined positive operators) are automatically weakly continuous and self-adjoint.

Now we proceed by recalling the construction of the Krein–von Neumann extension of a positive operator. We will use the notations of this section without further notice. For more details, see [19, Theorem 3.1 $(iv) \Rightarrow (i)$]. Let $\langle F, E \rangle$ be a w^*-sequentially complete anti-dual pair and let $A \colon E \supseteq \operatorname{dom} A \to F$ be a positive operator with domain $\operatorname{dom} A$. Assume further that for any y in E, there is $M_y \geq 0$ such that

$$(1.2) \qquad |\langle Ax, y \rangle|^2 \leq M_y \langle Ax, x \rangle \qquad \text{for all } x \in \operatorname{dom} A.$$

This assumption guarantees that one can build a Hilbert space H_A by taking the Hilbert space completion of the inner product space $\big(\operatorname{ran} A, (\cdot \,|\, \cdot)_A \big)$, where

$$(1.3) \qquad (Ax \,|\, Ax')_A := \langle Ax, x' \rangle, \qquad x, x' \in \operatorname{dom} A.$$

Again, by (1.2), the canonical embedding operator

$$(1.4) \qquad J_0 \colon H_A \supseteq \operatorname{ran} A \to F, \qquad J_0(Ax) := Ax$$

is weakly continuous, and thus admits a unique continuous extension J to H_A by w^*-sequentially completeness of F. Since $J \in \mathscr{L}(H_A; F)$, we have $J^* \in \mathscr{L}(E; H_A)$. From

$$(Ax' \,|\, J^* x)_A = \langle J(Ax'), x \rangle = \langle Ax', x \rangle = (Ax' \,|\, Ax)_A, \qquad x, x' \in \operatorname{dom} A,$$

it follows that $J^* x = Ax$ for all $x \in \operatorname{dom} A$. As for any $x \in \operatorname{dom} A$, we have

$$J J^* x = J(Ax) = Ax,$$

the operator $J J^* \in \mathscr{L}(E, F)$ is a positive extension of A. We will refer to $A_N := J J^*$ as the Krein–von Neumann extension of A.

We remark that the extension result above is closely related to the theory of reproducing kernels (see, for example, [13, 14, 21]). In fact, one can say that the operator $A \colon E \supseteq D \to F$ is a restriction of a reproducing kernel if and only if (1.2) holds. Finally, we mention that our assumption that F be w^*-sequentially complete is weaker than the quasi-completeness imposed by Schwartz in [14].

2 Self-adjoint Extensions of Symmetric Operators

M. G. Krein [10] proved that every bounded symmetric Hilbert space operator possesses a norm preserving self-adjoint extension. The problem of constructing self-adjoint extensions of a symmetric operator arises in our anti-dual pair setting naturally. Since we cannot speak about norm preservation due to the lack of norm, we need to find a suitable notion to generalize Krein's theorem. Observe that the norm of a self-adjoint operator $S \in \mathscr{B}(H)$ can be expressed by means of the partial order induced by positivity. Namely, $\|S\|$ is the smallest constant $\alpha \geq 0$ such that $-\alpha I \leq S \leq \alpha I$. Based on this observation, a symmetric operator $S_0 \colon E \supseteq \operatorname{dom} S_0 \to F$ is called *A-bounded* for a fixed positive operator $A \in \mathscr{L}(E, F)$ if

$$|\langle S_0 x, y \rangle|^2 \leq \alpha^2 \cdot \langle Ax, x \rangle \langle Ay, y \rangle, \qquad x \in \operatorname{dom} S_0, y \in E,$$

holds. The smallest constant α is called the *A-bound* of S_0 and is denoted by $\alpha_A(S_0)$. We will call the extension $S \supset S_0$ *A-bound preserving* if $\alpha_A(S_0) = \alpha_A(S)$.

In the next theorem, which is the main result of this section, we will present a sufficient condition to guarantee for a symmetric linear operator that it possesses a self-adjoint extension. Moreover, we describe the set of all *A*-bound preserving extensions of a given symmetric operator.

Theorem 2.1 *Let $\langle F, E \rangle$ be a weak-* sequentially complete anti-dual pair and let $S_0 \colon \operatorname{dom} S_0 \to F$ be a symmetric operator, i.e.,*

$$\langle S_0 x, y \rangle = \overline{\langle S_0 y, x \rangle}, \qquad x, y \in \operatorname{dom} S_0.$$

Suppose that S_0 is A-bounded with some positive operator $A \in \mathscr{L}(E, F)$. Then there exist two distinguished self-adjoint extensions $S_m, S_M \in \mathscr{L}(E, F)$ of S_0 such that

$$\alpha_A(S_m) = \alpha_A(S_M) = \alpha_A(S_0).$$

In fact, the interval $[S_m, S_M]$ consists exactly of all self-adjoint extensions $S \supset S_0$ such that $\alpha_A(S) = \alpha_A(S_0)$:

$$(2.1) \qquad [S_m, S_M] = \{S \in \mathscr{L}(E, F) \mid S_0 \subset S = S^*, \ \alpha_A(S) = \alpha_A(S_0)\}.$$

Proof Introduce the following linear manifold

$$\operatorname{dom} \widehat{S_0} := \{Ax \mid x \in \operatorname{dom} S_0\} \subseteq H_A,$$

and fix an $x \in \operatorname{dom} S_0$. Let us define the conjugate linear functional f_x as

$$f_x \colon \operatorname{ran} A \subseteq H_A \to \mathbb{C}, \qquad f_x(Ay) := \langle S_0 x, y \rangle, \qquad y \in E.$$

Observe that f_x is continuous, because

$$(2.2) \qquad |f_x(Ay)|^2 \leq \alpha^2 \cdot \langle Ax, x \rangle, \qquad y \in E, (Ay \mid Ay)_A \leq 1$$

holds for some α by *A*-boundedness. According to the Riesz representation theorem, there exists a unique representing vector $\zeta_x \in H_A$ such that

$$\langle S_0 x, y \rangle = (\zeta_x \mid Ay)_A, \qquad y \in E.$$

If $Ax = Ax'$ for some x, x' in $\operatorname{dom} S_0$, then $f_x = f_{x'}$, and thus $\zeta_x = \zeta_{x'}$. Therefore, the mapping $\widehat{S_0} \colon \operatorname{dom} \widehat{S_0} \to H_A$,

$$\widehat{S_0}(Ax) := \zeta_x$$

is well defined and linear. By (2.2), we have

$$(\widehat{S_0}(Ax) \mid \widehat{S_0}(Ax))_A \leq \alpha^2 \cdot (Ax \mid Ax)_A, \qquad x \in \operatorname{dom} S_0,$$

whence we infer that $\widehat{S_0}$ is bounded with $\|\widehat{S_0}\| = \alpha_A(S_0)$. For x, y in $\operatorname{dom} S_0$,

$$(\widehat{S_0}(Ax) \mid Ay)_A = \langle S_0 x, y \rangle = \overline{\langle S_0 y, x \rangle} = (Ax \mid \widehat{S_0}(Ay))_A,$$

hence $\widehat{S_0}$ is symmetric. Introduce the operators

$$\widehat{T_m} := \|\widehat{S_0}\| + \widehat{S_0}, \qquad \widehat{T_M} := \|\widehat{S_0}\| - \widehat{S_0}.$$

Clearly, $\widehat{T}_m$ and $\widehat{T}_M$ are both positive operators on dom $\widehat{S}_0$. Furthermore, we have for all $h \in \text{dom}\,\widehat{S}_0$ that

$$\|\widehat{T}_m h\|_A^2 = \|\widehat{S}_0 h\|_A^2 + 2(\widehat{S}_0 h \,|\, h)_A + \|\widehat{S}_0\|^2 \|h\|_A^2 \le 2\|\widehat{S}_0\|(\widehat{T}_m h \,|\, h)_A,$$

and similarly,

$$\|\widehat{T}_M h\|_A^2 \le 2\|\widehat{S}_0\|(\widehat{T}_M h \,|\, h)_A.$$

By [19, Theorem 4.2], there exist two minimal positive extensions $\widehat{A}_m, \widehat{A}_M \in \mathscr{B}(H_A)$ of $\widehat{T}_m, \widehat{T}_M$, respectively. Note also that

$$\|\widehat{A}_M k\|_A^2 \le 2\|\widehat{S}_0\|(\widehat{A}_M k \,|\, k)_A \qquad \text{and} \qquad \|\widehat{A}_M k\|_A^2 \le 2\|\widehat{S}_0\|(\widehat{A}_M k \,|\, k)_A$$

hold for all $k \in H_A$. Now set

$$\widehat{S}_m := \widehat{A}_m - \|\widehat{S}_0\| \qquad \text{and} \qquad \widehat{S}_M := \|\widehat{S}_0\| - \widehat{A}_M.$$

Clearly, $\widehat{S}_m, \widehat{S}_M \in \mathscr{B}(H_A)$ are both self-adjoint extensions of $\widehat{S}_0$. For $k \in H_A$,

$$\|\widehat{S}_m k\|_A^2 = \|\widehat{T}_m k\|_A^2 - 2\|\widehat{S}_0\|^2(\widehat{T}_m k \,|\, k)_A + \|\widehat{S}_0\|\|k\|_A^2 \le \|\widehat{S}_0\|\|k\|_A^2,$$

and therefore $\|\widehat{S}_m\| = \|\widehat{S}_0\| = \alpha_A(S_0)$. Similarly, $\|\widehat{S}_M\| = \alpha_A(S_0)$. Letting

$$S_m := J\widehat{S}_m J^* \qquad \text{and} \qquad S_M := J\widehat{S}_M J^*,$$

we conclude that $S_m, S_M \in \mathscr{L}(E, F)$ are self-adjoint operators such that

$$\alpha_A(S_m) = \alpha_A(S_M) = \alpha_A(S_0).$$

Finally, for $x \in \text{dom}\,S_0$ and $y \in E$,

$$\langle S_m x, y \rangle = (\widehat{S}_m(Ax) \,|\, Ay)_A = (\widehat{S}_0(Ax) \,|\, Ay)_A = \langle S_0 x, y \rangle,$$

hence $S_0 \subset S_m$. A similar calculation shows that $S_0 \subset S_M$ holds as well.

To prove (2.1) let $S \in [S_m, S_M]$ and take $x \in \text{dom}\,S_0$ and $y \in E$. Since $S - S_m \ge 0$ it follows that

$$\begin{aligned}
|\langle (S - S_m)x, y \rangle|^2 &\le \langle (S - S_m)x, x \rangle \langle (S - S_m)y, y \rangle \\
&\le \langle (S_M - S_m)x, x \rangle \langle (S - S_m)y, y \rangle = 0,
\end{aligned}$$

hence $Sx = S_m x = S_0 x$, that is, $S_0 \subset S$. On the other hand $S_m \le S \le S_M$ implies that

$$\begin{aligned}
|\langle Sx, y \rangle| &\le \langle (S - S_m)x, x \rangle^{1/2} \langle (S - S_m)y, y \rangle^{1/2} + |\langle S_m x, y \rangle| \\
&\le 3\alpha_A(S_0)\langle Ax, x \rangle^{1/2}\langle Ay, y \rangle^{1/2},
\end{aligned}$$

hence there is a symmetric operator $\widehat{S} \in \mathscr{B}(H_A)$ with $\|\widehat{S}\| \le 3\alpha_A(S_0)$ such that $S = J\widehat{S}J^*$. It is clear that $\widehat{S}_m \le \widehat{S} \le \widehat{S}_M$, and thus $\alpha_A(S_0) = \|\widehat{S}\| = \alpha_A(S)$. Assume conversely that $S_0 \subset S$ is any self-adjoint extension such that $\alpha_A(S) = \alpha_A(S_0)$. Then it is clear that $\alpha_A(S_0) \pm \widehat{S}$ are bounded positive extensions of $\alpha_A(S_0) \pm \widehat{S}_0$, hence $\widehat{A}_m \le \alpha_A(S_0) + \widehat{S}$ and $\widehat{A}_M \le \alpha_A(S_0) - \widehat{S}$. Consequently, $\widehat{S}_m \le \widehat{S} \le \widehat{S}_M$ and also $S_m \le S \le S_M$. The proof is complete. ∎

In the following corollary, we recover the classical result of Krein on self-adjoint norm-preserving extensions.

Corollary 2.2   *Let H be a Hilbert space and let $S_0 : \operatorname{dom} S_0 \to H$ be a bounded symmetric operator. Then S admits two self-adjoint norm-preserving extensions S_m, $S_M \in \mathscr{B}(H)$ such that the interval $[S_m, S_M]$ consists exactly of all self-adjoint norm-preserving extensions of S_0 :*

$$[S_m, S_M] = \{S \in \mathscr{B}(H) \mid S_0 \subset S = S^*, \|S_0\| = \|S\|\}.$$

If a self-adjoint operator $B \in \mathscr{B}(H)$ leaving $\operatorname{dom} S_0$ invariant satisfies $BS_0 \subset S_0 B$, then

(2.3) $$S_m B = B S_m, \qquad S_M B = B S_M.$$

Proof   From the proof of the previous theorem, we see that $S_m = (S_0 + \|S_0\|)_N - \|S_0\|$ and that $S_M = \|S_0\| - (\|S_0\| - S_0)_N$. Here we have

$$B(\|S_0\| - S_0) \subset (\|S_0\| - S_0)B \qquad \text{and} \qquad B(S_0 + \|S_0\|) \subset (S_0 + \|S_0\|)B,$$

hence $B(S_0 + \|S_0\|)_N = (S_0 + \|S_0\|)_N B$ and $B(\|S_0\| - S_0)_N = (\|S_0\| - S_0)_N B$ due to [19, Corollary 4.3]. This clearly gives (2.3).   ∎

3   A Generalized Strong Parrott Theorem

The aim of this section is to generalize Parrott's famous theorem [12] on contractive extensions of 2 by 2 block operator-valued matrices, which is one of the crucial results in extension and dilation theory. As an application, we will deduce Yamada's recent result [21, Theorem 4] on the extension of the Strong Parrott Theorem [7, 20].

Theorem 3.1   *Let $\langle F_1, E_1 \rangle_1$ and $\langle F_2, E_2 \rangle_2$ be two w^*-sequentially complete anti-dual pairs and let $T_1 : E_1 \supseteq \operatorname{dom} T_1 \to F_2$ and $T_2 : E_2 \supseteq \operatorname{dom} T_2 \to F_1$ be linear operators such that*

$$\langle T_1 x_1, x_2 \rangle_2 = \overline{\langle T_2 x_2, x_1 \rangle_1}, \qquad x_1 \in \operatorname{dom} T_1, x_2 \in \operatorname{dom} T_2.$$

Assume, furthermore, that there exist two positive operators $A_i \in \mathscr{L}(E_i; F_i)$ and constants $\alpha_i \geq 0$, $(i = 1, 2)$ such that the following estimates hold true:

$$|\langle T_1 x_1, y_2 \rangle_2|^2 \leq \alpha_1 \langle A_1 x_1, x_1 \rangle_1 \langle A_2 y_2, y_2 \rangle_2, \qquad x_1 \in \operatorname{dom} T_1, y_2 \in E_2,$$
$$|\langle T_2 x_2, y_1 \rangle_1|^2 \leq \alpha_2 \langle A_1 y_1, y_1 \rangle_1 \langle A_2 x_2, x_2 \rangle_2, \qquad x_2 \in \operatorname{dom} T_2, y_1 \in E_1.$$

Then there exists a $T \in \mathscr{L}(E_1; F_2)$ such that $T_1 \subseteq T$ and $T_2 \subseteq T^*$ and that

$$|\langle T y_1, y_2 \rangle_1|^2 \leq \max\{\alpha_1, \alpha_2\} \cdot \langle A_1 y_1, y_1 \rangle_1 \langle A_2 y_2, y_2 \rangle_2, \qquad y_1 \in E_1, y_2 \in E_2.$$

Proof   Let us consider the pair $(E_1 \times E_2, F_1 \times F_2)$ with antiduality

$$[(f_1, f_2), (e_1, e_2)] = \langle f_1, e_1 \rangle_1 + \langle f_2, e_2 \rangle_2,$$

and introduce the following linear operator:

$$S_0 : E_1 \times E_2 \supseteq \operatorname{dom} T_1 \times \operatorname{dom} T_2 \to F_1 \times F_2, \qquad S_0(x_1, x_2) := (T_2 x_2, T_1 x_1).$$

An easy calculation shows that S_0 is symmetric: indeed, for $x_1, z_1 \in \operatorname{dom} T_1$ and $x_2, z_2 \in \operatorname{dom} T_2$,

$$\left[S_0(x_1, x_2), (z_1, z_2)\right] = \langle T_2 x_2, z_1 \rangle_1 + \langle T_1 x_1, z_2 \rangle_2 = \overline{\langle T_1 x_z, z_1 \rangle_1} + \overline{\langle T_2 z_2, x_1 \rangle_2}$$
$$= \overline{\left[S_0(z_1, z_2), (x_1, x_2)\right]}.$$

Consider the positive operator $\Lambda := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ acting between $E_1 \times E_2$ and $F_1 \times F_2$; then

$$\left|\left[S_0(x_1, x_2), (y_1, y_2)\right]\right| \le \left|\langle T_2 x_2, y_1 \rangle_1\right| + \left|\langle T_1 x_1, y_2 \rangle_2\right|$$
$$\le \alpha \cdot \left(\langle A_1 x_1, x_1 \rangle_1^{1/2} \langle A_2 y_2, y_2 \rangle_2^{1/2} + \langle A_1 y_1, y_1 \rangle_1^{1/2} \langle A_2 x_2, x_2 \rangle_2^{1/2} \right)$$
$$\le \alpha \cdot \left[\Lambda(x_1, x_2), (x_1, x_2)\right]^{1/2} \left[\Lambda(y_1, y_2), (y_1, y_2)\right)\right]^{1/2}$$

for $x_i \in \operatorname{dom} T_i$, $y_i \in E_i$ $(i = 1, 2)$ and with $\alpha := \max\{\sqrt{\alpha_1}, \sqrt{\alpha_2}\}$. This means that S_0 and Λ fulfill all the conditions of Theorem 2.1. Hence, we conclude that there exists a self-adjoint operator $S \in \mathscr{L}(E_1 \times E_2; F_1 \times F_2)$, which extends S_0 and that

$$\left|\left[S(y_1, y_2), (w_1, w_2)\right]\right|^2 \le \alpha^2 \cdot \left[\Lambda(y_1, y_2), (y_1, y_2)\right]\left[\Lambda(w_1, w_2), (w_1, w_2)\right],$$

$y_i, w_i \in E_i$ $(i = 1, 2)$. Let us interpret S as an operator matrix of the form

$$S = \begin{bmatrix} B_1 & T^* \\ T & B_2 \end{bmatrix},$$

where $B_i \in \mathscr{L}(E_i; F_i)$ are self-adjoint operators and $T \in \mathscr{L}(E_1; F_2)$. We claim that T possesses the desired properties. Indeed,

$$\langle T x_1, y_2 \rangle_2 = \left[S(x_1, 0), (0, y_2)\right] = \left[S_0(x_1, 0), (0, y_2)\right] = \langle T_1 x_1, y_2 \rangle_2,$$

for $x_1 \in \operatorname{dom} T_1$, $y_2 \in E_2$, and similarly,

$$\langle T^* x_2, y_1 \rangle_1 = \left[S(0, x_2), (y_1, 0)\right] = \left[S_0(0, x_2), (y_1, 0)\right] = \langle T_2 x_2, y_1 \rangle_1,$$

for $x_2 \in \operatorname{dom} T_2$, $y_1 \in E_1$, thus we conclude that $T_1 \subset T$ and $T_2 \subset T^*$. Finally,

$$\left|\langle T y_1, y_2 \rangle_2\right|^2 = \left|\left[S(y_1, 0), (0, y_2)\right]\right|^2$$
$$\le \alpha^2 \cdot \left[\Lambda(y_1, 0), (y_1, 0)\right]\left[\Lambda(0, y_2), (0, y_2)\right]$$
$$= \alpha^2 \cdot \langle A_1 y_1, y_1 \rangle_1 \langle A_2 y_2, y_2 \rangle_2,$$

for $y_i \in E_i$, $i = 1, 2$, which completes the proof. ∎

Using the generalized Parrott theorem above, we obtain a new proof for a recent result of A. Yamada [21, Theorem 4].

Corollary 3.2 *Let $\langle F_1, E_1 \rangle_1$, $\langle F_2, E_2 \rangle_2$ be anti-dual pairs and let H, K be Hilbert spaces. For $S_1 \in \mathscr{L}(E_1, H)$, $S_2 \in \mathscr{L}(E_1, K)$, $T_1 \in \mathscr{L}(H, F_2)$, and $T_2 \in \mathscr{L}(K, F_2)$, the following conditions are equivalent:*

(i) *$T_1 S_1 = T_2 S_2$, $S_2^* S_2 \le S_1^* S_1$, and $T_1 T_1^* \le T_2 T_2^*$;*
(ii) *there exists $X \in \mathscr{B}(H, K)$, $\|X\| \le 1$, such that $XS_1 = S_2$ and $T_2 X = T_1$, i.e., X makes the following diagram commutative:*

$$\begin{array}{ccc}
 & E_1 & \\
S_1 \swarrow & & \searrow S_2 \\
H & \xdashrightarrow{X} & K \\
T_1 \searrow & & \swarrow T_2 \\
 & F_2. &
\end{array}$$
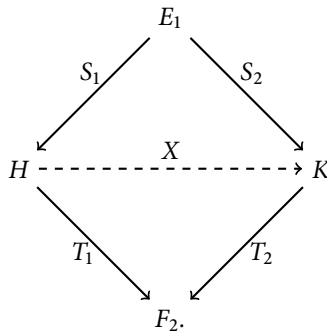
Proof   Implication (ii)⇒(i) is straightforward, so we only prove that (i) implies (ii). Consider the anti-dual pairs $(H\,|\,H)$ and $(K\,|\,K)$ and the operators

$$X_0\colon H \supseteq \operatorname{ran} S_1 \to K \qquad \text{and} \qquad X_1\colon K \supseteq \operatorname{ran} T_2^* \to H,$$

defined by

$$X_0(S_1 x_1) \coloneqq S_2 x_1, \qquad X_1(T_2^* x_2) \coloneqq T_1^* x_2.$$

From (i), we see that X_0, X_1 are well defined contractions such that

$$\bigl(X_0(S_1 x_1) \,|\, T_2^* x_2\bigr) = (S_2 x_1 \,|\, T_2^* x_2) = (S_1 x_1 \,|\, T_1^* x_2) = \bigl(S_1 x_1 \,|\, X_1(T_2^* x_2)\bigr),$$

for every $x_1 \in E_1$ and $x_2 \in E_2$. Hence, the pair X_0, X_1 fulfills the conditions of Theorem 3.1 with $A_1 = I_H$, $A_2 = I_K$ and $\alpha_1 = \alpha_2 = 1$. Consequently, there exists $X \in \mathscr{B}(H, K)$, $\|X\| \le 1$ such that $X_0 \subset X$, $X_1 \subset X^*$, and therefore $X S_1 = S_2$ and $X^* T_2^* = T_1^*$, which yields (ii).   ∎

Yamada's work itself generalizes Parrott's theorem [12] and the Strong Parrott Theorem (see [1, 4, 7]). So we get the following classical result automatically.

Corollary 3.3   *Let H and K be Hilbert spaces, let $H_1 \subseteq H$ and $K_1 \subseteq K$ be closed linear subspaces, and denote by P_{K_1} the orthogonal projection onto K_1. For given contractions $T_1\colon H_1 \to K$ and $T_1'\colon H \to K_1$, the following conditions are equivalent:*

(i) $P_{K_1} T_1 = T_1'|_{H_1}$;

(ii)   *there exists a contraction $T \in \mathscr{B}(H, K)$ such that*

$$T_1 = T|_{H_1} \quad \text{and} \quad T_1' = P_{K_1} T.$$

4 Hermitian Extensions of Linear Functionals

Positive functionals play an important role in the representation theory of algebras. Extension of such functionals has been investigated in many different settings. For example, if f is a positive linear functional defined on a closed ideal in a C^*-algebra, then f always admits an extension with the same norm (see [5, II.6.4.16]). Positive functionals defined on left-ideals of the full operator algebra possessing normal extension were characterized in [18], while positive extendibility of positive functionals defined on left ideals of general *-algebras was studied in [19]. The aim of this section is to demonstrate how our anti-dual pair setting can be used to construct hermitian extensions of linear functionals in the unital *-algebra setting.

Let $\mathscr{A}$ be a unital $*$-algebra with unit 1. If $\mathscr{I} \subseteq \mathscr{A}$ is a left ideal, then we call a linear functional $g \colon \mathscr{I} \to \mathbb{C}$ *symmetric* if

$$g(b^*a) = \overline{g(a^*b)}, \qquad a, b \in \mathscr{I}.$$

A linear functional $g \in \mathscr{A}^*$ is called *hermitian* if it satisfies

$$g(x^*) = \overline{g(x)}, \qquad x \in \mathscr{A}.$$

It is easy to check that a linear functional $g \in \mathscr{A}^*$ is hermitian if and only if it is symmetric. Note that for a $*$-algebra without unit element, this equivalence no longer holds.

Assume that a positive linear functional $f \colon \mathscr{A} \to \mathbb{C}$ is given. We say that the symmetric functional $g \colon \mathscr{I} \to \mathbb{C}$ is *f-bounded* if

$$(4.1) \qquad |g(x^*a)|^2 \le \alpha^2 f(x^*x) f(a^*a), \qquad x \in \mathscr{A}, a \in \mathscr{I},$$

holds for some $\alpha > 0$. The *f-bound* $\alpha_f(g)$ is defined as the smallest constant α that fulfills (4.1). If $\ell \colon \mathscr{I} \to \mathbb{C}$ is a linear functional, then the correspondence

$$\langle La, x \rangle \coloneqq \ell(x^*a), \qquad a \in \mathscr{I}, x \in \mathscr{A},$$

defines a linear operator $L \colon \mathscr{I} \to \bar{\mathscr{A}}^*$. Clearly, L is positive if ℓ is positive and L is symmetric if ℓ is so. Suppose now that $f \in \mathscr{A}^*$ is a positive functional and denote by A the positive operator associated with f, *i.e.*,

$$\langle Ax, y \rangle = f(y^*x), \qquad x, y \in \mathscr{A}.$$

Let H_A denote the corresponding Hilbert space, that is, H_A is the completion of ran A endowed with the inner product (1.3). Observe that in that case, we have

$$(Ax \,|\, Ay)_A = f(y^*x), \qquad x, y \in \mathscr{A}.$$

Consider the canonical embedding $J \colon H_A \to \bar{\mathscr{A}}^*$ in (1.4) and recall its useful properties

$$J^*x = Ax, \qquad x \in \mathscr{A},$$

and $JJ^* = A$. Assume in addition that

$$(4.2) \qquad |f(y^*x^*xy)| \le M_x f(y^*y), \qquad x, y \in \mathscr{A}$$

for some $M_x \ge 0$. This assures that the operators $\pi_f(x)$ defined by

$$\pi_f(x)(J^*y) \coloneqq J^*(xy), \qquad y \in \mathscr{A},$$

are continuous on H_A by norm bound $M_x^{1/2}$. Thus, for every $x \in \mathscr{A}$, we can extend $\pi_f(x)$ to an element of $\mathscr{B}(H_A)$. It is then immediate that $\pi_f \colon \mathscr{A} \to \mathscr{B}(H_A)$ is a $*$-homomorphism such that

$$(4.3) \qquad f(x) = (\pi_f(x)(J^*1) \,|\, J^*1)_A, \qquad x \in \mathscr{A}.$$

A positive functional satisfying (4.2) (and hence (4.3)) will be called *representable* [15].

Theorem 4.1 *Let $\mathscr{A}$ be a unital $*$-algebra, $\mathscr{I} \subseteq \mathscr{A}$ a left ideal and $f \in \mathscr{A}^*$ a representable positive functional. If $g_0 \colon \mathscr{I} \to \mathbb{C}$ is an f-bounded symmetric functional with f-bound $\alpha_f(g_0)$, then there exist two distinguished f-bounded hermitian functionals $g_m, g_M \in \mathscr{A}^*$ with f-bound $\alpha_f(g_m) = \alpha_f(g_M) = \alpha_f(g_0)$ extending g_0. Furthermore,*

$g_m \leq g_M$ and the interval $[g_m, g_M]$ consists of all hermitian f-bound preserving extensions of g_0:

$$[g_m, g_M] = \{g \in \mathscr{A}^* \mid g_0 \subset g = g^*, \alpha_f(g) = \alpha_f(g_0)\}.$$

Proof Along the lines of the proof of Theorem 2.1, let us introduce a symmetric operator S_0 on $\mathrm{dom}\, S_0 := \{J^* a \mid a \in \mathscr{I}\}$ such that

$$(S_0(J^* a) \mid J^* x)_A = g_0(x^* a), \qquad a \in \mathscr{I}, x \in \mathscr{A}.$$

A straightforward calculation shows that $S_0 \colon \mathrm{dom}\, S_0 \to H_A$ is bounded with norm $\|S_0\| = \alpha_f(g_0)$. For $x, y \in \mathscr{A}$ and $a \in \mathscr{I}$, we have

$$(S_0 \pi_f(x)(J^* a) \mid J^* y)_A = (S_0 J^*(xa) \mid J^* y)_A = g(y^* xa) = (S_0(J^* a) \mid J^*(x^* y))_A$$
$$= (S_0 J^* a \mid \pi_f(x^*) J^* y)_A = (\pi_f(x) S_0(J^* a) \mid J^* y)_A;$$

hence, we infer that $\mathrm{dom}\, S_0$ is π_f-invariant and $\pi_f(x) S_0 \subset S_0 \pi_f(x)$ for all $x \in \mathscr{A}$. By Corollary 2.2, it follows that there exist two norm preserving self-adjoint extensions $S_m, S_M \in \mathscr{B}(H_A)$ of S_0 such that

$$(4.4) \qquad \pi_f(x) S_m = S_m \pi_f(x), \qquad \text{and} \qquad \pi_f(x) S_M = S_M \pi_f(x),$$

whenever x is self-adjoint, and hence also for every $x \in \mathscr{A}$. We claim that the functionals

$$g_m(x) := (S_m J^* x \mid J^* 1)_A, \qquad \text{and} \qquad g_M(x) := (S_m J^* x \mid J^* 1)_A, \qquad x \in \mathscr{A},$$

fulfill all conditions of the statement. First, we observe that g_m and g_M are hermitian; indeed, by (4.4), we have for every $x \in \mathscr{A}$ that

$$g_m(x^*) = (S_m J^* x^* \mid J^* 1)_A = (\pi_f(x^*) S_m J^* 1 \mid J^* 1)_A = (S_m J^* 1 \mid J^* x)_A = \overline{g_m(x)}.$$

A similar argument shows that g_M is hermitian. Next, observe that g_m and g_M extend g_0, because for every $a \in \mathscr{I}$,

$$g_m(a) = (S_m J^* a \mid J^* 1)_A = (S_0 J^* a \mid J^* 1)_A = g(a)$$

holds, and similarly, $g_M(a) = g(a)$. Finally, we have

$$\|S_m\| = \|S_M\| = \|S\| = \alpha_f(g_0),$$

whence it follows readily that g_m and g_M have f-bound $\alpha_f(g_0)$.

Now, let $g \in \mathscr{A}^*$ be an arbitrary hermitian extension of g_0 having f-bound $\alpha_f(g_0)$. Then there is a self-adjoint operator $S \in \mathscr{B}(H_A)$, $\|S\| = \alpha_f(g_0)$ such that

$$(S J^* x \mid J^* y)_A = g(y^* x), \qquad x, y \in \mathscr{A}.$$

It is clear that $S_0 \subset S$, and therefore, $S_m \leq S \leq S_M$, due to Corollary 2.2. As a straightforward consequence, we conclude that $g_m \leq g \leq g_M$. Suppose conversely that $g \in \mathscr{A}^*$ is a hermitian functional such that $g_m \leq g \leq g_M$. First, observe that g is f-bounded as it satisfies

$$|g(y^* x)| \leq (g - g_m)(x^* x)^{1/2} (g - g_m)(y^* y)^{1/2} + |g_m(y^* x)|$$
$$\leq 3\alpha_f(g_0) f(x^* x)^{1/2} f(y^* y)^{1/2}.$$

Hence, there exists a self-adjoint operator $S \in \mathscr{B}(H_A)$, such that

$$g(y^*x) = (JSJ^*x \mid y)_A, \qquad x, y \in \mathscr{A}.$$

From $g_m \leq g \leq g_M$, it follows that $S_m \leq S \leq S_M$, and therefore, $S_0 \subset S$ by Corollary 2.2. Consequently,

$$g_0(a) = (S_0 J^*a \mid J^*1)_A = (SJ^*a \mid J^*1)_A = g(a), \qquad a \in \mathscr{I};$$

thus, $g_0 \subset g$. The proof is complete. ∎

We remark that Theorem 4.1 provides only a sufficient condition for the existence of hermitian extensions. On C^*-algebras, the statement of Theorem 4.1 may be improved in two ways; first, the condition on f of being representable can be replaced by the formally weaker one of being positive. On the other hand, the existence of a dominating positive functional is both necessary and sufficient.

Corollary 4.2 *Let $\mathscr{A}$ be a unital C^*-algebra and $\mathscr{I} \subseteq \mathscr{A}$ a left ideal. A linear functional $g_0 \colon \mathscr{I} \to \mathbb{C}$ possesses a continuous hermitian extension g if and only if g_0 is symmetric and f-bounded for some positive functional $f \in \mathscr{A}^*$.*

Proof On a C^*-algebra, every positive functional f is representable. If g_0 is symmetric and f-bounded, then g_0 has a hermitian extension g that is of the form

$$g(a) = (SJ^*a \mid J^*1), \qquad a \in \mathscr{A},$$

where S is a bounded self-adjoint operator on H_A. It follows therefore that g is continuous. For the converse, assume that $g \in \mathscr{A}^*$ is a continuous hermitian extension of g_0. Let $g = g_+ - g_-$ be the Hahn–Jordan decomposition of g, with $g_+, g_- \in \mathscr{A}^*$ positive functionals (see [9]). Letting $f := g_+ + g_-$, it is easy to check that g_0 is f-bounded with bound $\alpha_f(g_0) = 4$. ∎

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