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A SIMPLE ALGORITHM FOR DECIDING PRIMES IN $K[[x,y]]$

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ABSTRACT. The well-known Tschirnhausen transformation, $x \longrightarrow x - \frac{a}{n}$, eliminates the second term of the polynomial $x^n + ax^{n-1} + \cdots$. By a mere repeated application of this transformation, one can decide whether a given element of $K[[x,y]]$ is prime (irreducible) or not. Here K is an algebraically closed field of characteristic 0.

A generalised version of Hensel's Lemma is developed for the proofs. The entire paper can be understood by undergraduate students.

1. Basics.

Semigroups. In this paper, by a semigroup we always mean an additive subsemigroup of the positive rationals, $\mathbb{Q}^+$. Also, we assume they are finitely generated. Thus, a semigroup, S, has a minimal set of generators, $\omega_0, \ldots, \omega_N$, and we write

$$S = S(\omega_0, \ldots, \omega_N),$$

where $0 < \omega_0 < \cdots < \omega_N$, and

$$\omega_i \notin S(\omega_0, \ldots, \omega_{i-1}), \quad i \geq 1.$$

A (finitely generated) semigroup is isomorphic to one whose generators are integers.

Let $d_0 = 1$ and let d_i denote the smallest integer such that

$$d_i\omega_i \in S(\omega_0, \ldots, \omega_{i-1}), \quad i \geq 1.$$

We may call $d_N\omega_N$ the *last merging point* of S.

Let $S_N = S(\omega_0, \ldots, \omega_N)$ be given. We write

$$\boldsymbol{\omega} = (\omega_0, \ldots, \omega_N);$$

a typical element of S_N can then be written as an "inner product"

$$M \cdot \boldsymbol{\omega} = \sum_{i=0}^{N} m_i\omega_i \tag{1}$$

where $M = (m_0, \ldots, m_N)$ in an $(N+1)$-tuple of non-negative integers.

We call M *admissible* if $0 \leq m_i < d_i$ for $1 \leq i \leq N$. (Note that $m_0 \geq 0$ can be any integer.)

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DEFINITION. We say S_N is a Newton-Puiseux semigroup if $\omega_i > d_{i-1}\omega_{i-1}$ for all $i \geq 1$.

An element of a Newton-Puiseux semigroup S_N admits a unique expression (1), with M admissible. This is the Corollary to Lemma 2 in Section 6.

All semigroups arising in this paper are Newton-Puiseux.

Associated weight. Let S_N be given. Take indeterminants $Y_0, \ldots, Y_N$, and write

$$\mathbf{Y} = (Y_0, \ldots, Y_N), \quad \mathbf{Y}^M = Y_0^{m_0} \cdots Y_N^{m_N},$$

so that an element of the formal power series ring $K[[\mathbf{Y}]]$ is expressed as

$$f(\mathbf{Y}) = \sum_M a_M \mathbf{Y}^M, \quad a_M \in K.$$

Define a weight function on $K[[\mathbf{Y}]]$,

$$v_N : K[[\mathbf{Y}]] \longrightarrow \mathbb{Q}^+ \cup \{\infty\}$$

by

$$v_N(f) = \begin{cases} \min\{M \cdot \boldsymbol{\omega} \mid a_M \neq 0\}, & \text{if } f \neq 0, \\ \infty, & \text{if } f = 0. \end{cases}$$

Note that $v_N(Y_i) = \omega_i$.

We call $v_N(f)$ the *weighted order* of f associated to S_N.

Associated Newton polygon. Let S_N, v_N be as above.

Take an element in $K[[\mathbf{Y}, X]]$,

$$P(X; \mathbf{Y}) = \sum a_{M,d} \mathbf{Y}^M X^d, \quad a_{M,d} \in K.$$

In a coordinate plane, $\mathbb{R}^2$, let us plot a dot at the point $(d, M \cdot \boldsymbol{\omega})$ for each monomial term $a_{M,d}\mathbf{Y}^M X^d$, $a_{M,d} \neq 0$, of P. Note that the second component $M \cdot \boldsymbol{\omega}$ is an element of S_N. We call this dot a Newton dot.

When all M are admissible, there is at most one dot at a given point.

DEFINITION. The Newton polygon of $P(X; \mathbf{Y})$ associated to S_N is the boundary of the convex hull spanned by the set

$$\{(u, v) \mid \exists \text{ a Newton dot } (d, M \cdot \boldsymbol{\omega}) \text{ such that } u \geq d, v \geq M \cdot \boldsymbol{\omega}\}.$$

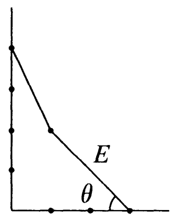
Suppose $P(X; \mathbf{Y})$ is regular in X, say of order k; that is,

$$P(X; 0) = X^k + \text{ higher order terms.}$$

Then, of course, $(k, 0)$ is a vertex of the Newton polygon. We call it the *first vertex*.

Let E denote the non-horizontal edge of the polygon at the first vertex, and θ the angle it makes with the negative horizontal direction, as indicated in the following example. We call E the *first edge*, and θ the *first angle*, of the Newton polygon of P, or simply of P.

EXAMPLE. $P(X, Y) = X^3 + XY^2 + Y^4$



G-adic bases. We follow Abhyankar-Moh ([1], [3]), who defined the notion.

Consider, as in [2], a sequence

$$\Gamma_N = \{G_0(x,y), \ldots, G_N(x,y), G_{N+1}(x,y)\}, \quad N \geq 0.$$

where $G_0 = y$, and for each $i \geq 1$, $G_i(x,y)$ is an element of $K[[y]][x]$, monic in x, say of degree D_i, such that each D_i properly divides D_{i+1}:

$$D_{i+1} = d_i D_i, \quad d_i > 1, \ 1 \leq i \leq N.$$

EXAMPLE. $\Gamma_1 = \{y, x, x^2 - y^3\}, \Gamma_2 = \{y, x, x^2 - y^3, (x^2 - y^3)^2 - xy^5\}$.

A repeated application of the Euclidean Division Algorithm shows that Γ_N is a G-adic base in the sense of Abhyankar-Moh ([1]): Given $F(x,y)$, there is a unique expression

$$F(x,y) = \sum a_{M,d} \mathbf{G}_N^M G_{N+1}^d \tag{2}$$

where $M = (m_0, \ldots, m_N)$ are admissible exponents, and $\mathbf{G}_N^M$ is a shorthand for $G_{\mathbf{0}}^{m_0} \cdots G_N^{m_N}$.

Let Γ_N be given a G-adic base. We define the *associated linear injection*

$$\ell_N : K[[x,y]] \longrightarrow K[[\mathbf{Y}, X]]$$

via (2) by:

$$\ell_N\big(F(x,y)\big) = \sum a_{M,d} \mathbf{Y}^M X^d. \tag{3}$$

Note that ℓ_N may not preserve multiplication. All exponents M in (3) are admissible. There is also an *associated substitution* map, which is a left inverse of ℓ_N,

$$\sigma_N : K[[\mathbf{Y}, X]] \longrightarrow K[[x,y]]$$

defined by

$$\sigma_N(Y_i) = G_i(x,y), \quad \sigma_N(X) = G_{N+1}(x,y),$$

preserving both the linear and multiplicative structures.

REMARK. When ℓ_N is given. A weighted order v_N is induced on $K[[x,y]]$ such that

$$v_N\big(F(x,y)\big) = v_n\Big(\ell_N\big(F(x,y)\big)\Big).$$

DEFINITION ([2]). When each G_i is a prime in $K[[x,y]]$, we say Γ_N is a Γ-adic base. All G-adic bases used in this paper are Γ-adic.

The Tschirnhausen transform. Let S_N, Γ_N and $P(X;\mathbf{Y})$ be given. Suppose P is in the image of ℓ_N, regular in X, of order k.

Suppose $\tan\theta \in S_N$, where θ is the first angle. We can write, as in (1),

$$\tan\theta = M\cdot\boldsymbol{\omega}, \quad M \text{ admissible.}$$

The *Tschirnhausen transform* of the pair (P,Γ_N) is defined as follows.

Consider the point $(k-1, M\cdot\boldsymbol{\omega})$, which lies on the first edge, E, next to the first vertex $(k,0)$. There is a Newton dot at this point if, and only if, P has a monomial term $a\mathbf{Y}^M X^{k-1}$, $a\neq 0$.

This dot can be eliminated by a Tschirnhausen transformation. Namely, we replace X by $X-\frac{a}{k}\mathbf{Y}^M$ in P to give

$$P'(X;\mathbf{Y}) = P\Big(X-\frac{a}{k}\mathbf{Y}^M;\mathbf{Y}\Big),$$

which no longer has a Newton dot at this point.

In the mean time, we replace G_{N+1} by

$$G^{(1)}_{N+1}(x,y) = G_{N+1}(x,y) + \frac{a}{k}\mathbf{G}^M_N. \tag{4}$$

Then, we define

$$\Gamma^{(1)}_N \equiv \{G_0,\ldots,G_N,G^{(1)}_{N+1}\},$$

and

$$P^{(1)}(X;\mathbf{Y}) = \ell_N\circ\sigma_N\big(P'(X;\mathbf{Y})\big).$$

The pair $\{P^{(1)},\Gamma^{(1)}_N\}$ is called the *Tschirnhausen transform* of $\{P,\Gamma_N\}$.

Observe that $(k,0)$ remains the first vertex of $P^{(1)}$; and also, clearly, $\theta^{(1)}\geq\theta$. (We use $\theta^{(1)}$ to denote the first angle of $P^{(1)}$.)

When $a=0$, the Tschirnhausen transformation is the identity transformation. We say it is *stationary*.

The following example shows that both cases $\theta^{(1)}>\theta$ and $\theta^{(1)}=\theta$ can happen. In either cases, however, there is no Newton dot at $(k-1,M\cdot\boldsymbol{\omega})$.

EXAMPLE. Take $\Gamma_0=\{y,x\}$. For $X^2+2XY+Y^2$, $\theta=\frac{\pi}{4}$, $\theta^{(1)}=\frac{\pi}{2}$. For $X^2+2XY+2Y^2$, $\theta^{(1)}=\theta=\frac{\pi}{4}$.

When $\tan\theta\notin S_N$, we say the transformation is *not applicable*. (Example: $\Gamma_0=\{y,x\}$, $P=x^2-y^3$.)

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2. **The algorithm.** The Assertions in this section will be proved in later sections. Take a non-zero element of $K[[x,y]]$,

$$F(x,y) = H_k(x,y) + H_{k+1}(x,y) + \cdots,$$

where H_k is the initial (homogeneous) form.

By applying a suitable linear transformation, if necessary, we can assume $H_k(1,0) = 1$. An application of a Tschirnhausen transformation will then reduce H_k to

$$H_k(x,y) = x^k + a_2 x^{k-2} y^2 + \cdots + a_k y^k. \tag{5}$$

Let us describe the initial stage of the algorithm, assuming (5).

Take any $\omega_0 \in \mathbb{Q}^+$. (Indeed, we can take $\omega_0 = 1$.) Let $S_0 = S(\omega_0)$, and let v_0 be defined by $v_0(Y_0) = \omega_0$. Take the first Γ-adic base to be

$$\Gamma_0 = \{G_0 = y, G_1 = x\}.$$

The associated maps ℓ_0, σ_0 are defined accordingly. Finally, let

$$P_0(X; Y_0) = \ell_0\big(F(x,y)\big),$$

which is regular in X, of order $k_0 = k$.

Now assume, inductively, that we are at stage N, $N \geq 0$, having defined a Newton-Puiseux semigroup, $S_N = S(\omega_0, \ldots, \omega_N)$, a Γ-adic base Γ_N, together with v_N, ℓ_N, σ_N, and

$$P_N(X; \mathbf{Y}) = \ell_N\big(F(x,y)\big), \quad \mathbf{Y} = (Y_0, \ldots, Y_N).$$

where P_N is regular in X, say, of order k_N.

ASSERTION 1. If $k_N = 1$, then $F(x,y)$ is prime.

In case $k_N > 1$, we apply the Tschirnhausen transformation recursively to the pair $\{P_N, \Gamma_N\}$, as long as it is applicable. This yields a sequence $\{P_N^{(s)}, \Gamma_N^{(s)}\}$, where $P_N^{(0)} = P_N$, $\Gamma_N^{(0)} = \Gamma_N$, and $\{P_N^{(s)}, \Gamma_N^{(s)}\}$ is the Tschirnhausen transform of $\{P_N^{(s-1)}, \Gamma_N^{(s-1)}\}$, for all s. Four cases may arise:

CASE 1. The transformation is always applicable, yielding an infinite sequence $\{P_N^{(s)}, \Gamma_N^{(s)}\}$.

CASE 2. We arrive at $\{P_N^{(s)}, \Gamma_N^{(s)}\}$, and find $\tan\theta_N^{(s)} = \infty$. (Here, $\theta_N^{(s)}$ denotes the first angle of $P_N^{(s)}$.)

CASE 3. Or, here we find that the Tschirnhausen transformation is stationary, with

$$\tan\theta_N^{(s)} \in S_N, \quad (\tan\theta_N^{(s)} < \infty).$$

CASE 4. Or, we have, $\tan\theta_N^{(s)} \notin S_N$, (so that it is no longer applicable).

ASSERTION 2. $\Gamma_N^{(s)}$ are Γ-adic bases.

ASSERTION 3. In Cases 1 and 2, ($k_N > 1$), $F(x,y)$ is the k-th power of a prime, hence reducible.

ASSERTION 4. In Case 3, $F(x,y)$ is reducible.

When Case 4 happens, we move on to define the $(N+1)$-st stage. Let $w_{N+1} = \tan \theta_N^{(s)}$, $S_{N+1} = S(w_0, \ldots, w_{N+1})$, and let d_{N+1} be the smallest integer such that

$$d_{N+1}\omega_{N+1} \in S_N, \quad (d_{N+1} > 1).$$

We shall see, in Section 4, that S_{N+1} is Newton-Puiseux.

When k_N is divisible by d_{N+1}, we define k_{N+1} and an admissible exponent $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_N)$ by

$$k_N = k_{N+1}d_{N+1}, \quad \boldsymbol{\alpha} \cdot \boldsymbol{\omega} = d_{N+1}\omega_{N+1}.$$

ASSERTION 6. Consider the monomial term, $a\mathbf{Y}^\alpha X^{k_N - d_{N+1}}$ of $P_N^{(s)}$. If $a = 0$, then $F(x,y)$ is reducible.

Now, suppose $a \neq 0$. We define

$$G_{N+2} = G_{N+1}^{(s)} + \frac{a}{k_{N+1}}\mathbf{G}_N^\alpha, \qquad (6)$$
$$\Gamma_{N+1} = \{G_0, \ldots, G_{N+1}^{(s)}, G_{N+2}\},$$

and

$$P_{N+1}(X; \mathbf{Y}_{N+1}) = \ell_N\big(F(x,y)\big)$$

which is regular in X, of order k_{N+1}, where

$$\ell_{N+1}(G_{N+1}) = Y_{N+1}, \quad \mathbf{Y}_{N+1} = (Y_0, \ldots, Y_{N+1}).$$

ASSERTION 7. G_{N+1} is prime, whence Γ_{N+1} is a Γ-adic base.

This completes the description of the algorithm.

Since $\{k_N\}$ is a strictly decreasing sequence of positive integers, Case 4 can not happen infinitely many times. The algorithm terminates in finitely many steps.

ATTENTION. Since G_{N+1} has been replaced by $G_{N+1}^{(s)}$ when G_{N+2} is defined, Γ_N is not necessarily a subset of Γ_{N+1}. However, note that

$$G_{N+1}^{(s)} = G_{N+1} + \text{ terms of higher weight.}$$

CONVENTION. When Γ_{N+1} has been defined. We shall use Γ_N to denote $\Gamma_N^{(s)}$, abusing notations, and then forget about the original Γ_N. In this new system of notations, Γ_N is a subset of Γ_{N+1}, for all N.

3. **Illustrative examples.** A simple example for Case 1 is:

$$x^2 + 2xy^2 + 2xy^3 + y^4 + 2xy^4 + 2y^5 + \cdots = (x + y^2 + y^3 + y^4 + \cdots)^2.$$

For Case 2, we can take

$$(x^2 + 2xy + y^2) + (xy^2 + y^3) + \frac{1}{4}y^4 = \left[(x+y) + \frac{1}{2}y^2\right]^2.$$

For Case 3, consider

$$F = (x^2 - y^3)^2 - y^7.$$

Here, we find

$$\begin{aligned} \nu(x) &= \nu(G_1) = 3/2, \\ S_1 &= S(1, 3/2), \\ G_2 &= x^2 - y^3, \\ P_1 &= X^2 - Y^7, \\ \tan\theta_1 &= 7/2 \in S_1. \end{aligned}$$

The Tschirnhausen transformation is stationary, F is reducible by Assertion 4. (The term $G_1 Y^2$ is missing from P_1.) A factorization is given at the end of Section 8.

For Case 4, our first example is $F = x^3 - xy^3 + y^5$. Here we have,

$$\begin{aligned} N &= 0, \\ P_0 &= X^3 - XY^3, \\ k_0 &= 3, \\ d_1 &= 2. \end{aligned}$$

Since k_0 is not divisible by d_1, F is reducible (Assertion 5).

Next, consider

$$F = (x^2 - y^3)^4 + y^{13}.$$

Here, we find

$$\begin{aligned} N &= 1, \\ G_1 &= x, \\ G_2 &= x^2 - y^3, \\ P_1 &= X^4 + Y^{13}, \\ \tan\theta_1 &= 3\frac{1}{4}, \\ d_1 &= 2. \end{aligned}$$

By Assertion 6, F is reducible. (The term $G_1 Y^5$ is missing from P_1.)

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Now let us consider

$$F = (x^2 - y^3)^4 + 2xy^5(x^2 - y^3)^2 + 2y^{13} + \cdots .$$

Here,

$$P_1 = X^4 + 2G_1Y^5X^2 + 2Y^{13}.$$

Following the algorithm, we define

$$G_3 = (x^2 - y^3)^2 + xy^5$$

which, by Assertion 7, is prime.

Finally, let us consider

$$(x^2 - y^3)^4 + 2xy^5(x^2 - y^3)^2 + y^{13} + \text{ higher weighted terms.}$$

This time,

$$\begin{aligned} P_1 &= X^4 + 2G_1Y^5X^2 + Y^{13} \\ &= (X^2 + G_1Y^5)^2 + \cdots , \end{aligned}$$

so that we move on to the next stage of the algorithm.

4. **Induction hypothesis.** We make two induction hypothesis at Stage N; they will be proved for $N+1$ at the end of Section 9.

(H_P) For the first angle θ_N of P_N, we have

$$\tan \theta_N \geq d_N\omega_N,$$

and if equality holds then there is no Newton dot at $(k_N - 1, \tan\theta_N)$.

(H_G) For $N \geq 1$, $G_{N+1}(x,y)$ has the form

$$G_{N+1} = G_N^{d_N} + c\mathbf{G}_{N-1}^{\boldsymbol{\alpha}_{N-1}}, \quad c \neq 0,$$

where $\boldsymbol{\alpha}_{N-1} = (\alpha_0, \ldots, \alpha_{N-1})$ is an admissible exponent such that

$$\sum_{i=0}^{N-1} \alpha_i\omega_i = d_N\omega_N.$$

When $N = 0$, (H_P) follows from (5); (H_G) says nothing, hence true.

5. **Stage $N = 0$.** We can assume $\omega_0 = 1$.

If $k = k_0 = 1$, $F(x,y)$ is obviously prime. So let us suppose $k > 1$.

In Case 1, where the Tschirnhausen transformation is always applicable, we find an infinite series $\sum C_ny^n$ such that

$$F(x,y) = \left(x - \sum C_ny^n\right)^k \cdot \text{unit}.$$

In Case 2, there is a finite series with the same property.

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Therefore Assertions 2 and 3 are true when $N = 0$.

For Assertion 4, let us first assume $\tan\theta_0 = 1$. By (5), the initial form of $P_0(X; Y)$ has the form

$$I(X, Y) = X^k + a_2X^{k-2}Y^2 + \cdots + a_kY^k.$$

Since at least one $a_i \neq 0$, $I(X, 1) = 0$ has at least two distinct roots, and so $I(X, Y)$ factors:

$$I(X, Y) = H_p(X, Y) \cdot K_q(X, Y), \quad p + q = k,$$

H_p, K_q are relatively prime (homogeneous) forms of degree p, q respectively, both monic in X.

LEMMA 1. *Every $(p + q - 1)$-form $L_{p+q-1}(X, Y)$ is in the ideal generated by H_p and K_q. That is, there exist forms A_{p-1}, B_{q-1} such that*

$$L_{p+q-1}(X, Y) = B_{q-1}(X, Y)H_p(X, Y) + A_{p-1}(X, Y)K_q(X, Y). \tag{7}$$

Consequently, every r-form, $r \geq p + q - 1$, is in this ideal.

The proof is well-known. Since H_p, K_q are relatively prime, polynomials A_{p-1}, B_{q-1}, of degree $p - 1, q - 1$, respectively, can be found such that

$$L_{p+q-1}(X, 1) = B_{q-1}(X)H_p(X, 1) + A_{p-1}(X)K_q(X, 1).$$

Then (7) follows by homogenizing this expression.

Now, consider any power series $P(X, Y)$, such as $P_0(X, Y)$, whose initial form is $I(X, Y)$. By a repeated application of Lemma 1, we can recursively find forms A_i, B_i such that

$$P(X, Y) = [H_p + A_{p+1} + A_{p+2} + \cdots][K_q + B_{q+1} + B_{q+2} + \cdots].$$

An application of σ_0 to $P_0(X, Y)$ then yields a factorization of $F(x, y)$.

Now, suppose $\tan\theta_0 > 1$ in Case 3. Let us define weights by

$$\nu(X) = \tan\theta_0, \quad \nu(Y) = 1.$$

Since there is no Newton dot at $(k - 1, \tan\theta_0)$, the weighted initial form of $P_0(X, Y)$ factors into two relatively prime weighted forms. The same reasoning as before will then lead to a factorization of P_0.

This is known as the weighted Hensel Lemma.

We shall consider stage $N = 0$ of Case 4 with the general case.

6. **More on Newton-Puiseux semigroups.** Consider a (finitely generated) semigroup S_N. The abelian group generated by S_N is generated by a single element, say g. There is a smallest integer r such that $(r + i)g \in S_N$ for all $i \geq 0$. Call rg the *conductor* of S_N.

When a semigroup is generated by two positive integers p, q, the conductor is $(p' - 1)(q' - 1)D$, where

$$D = \text{G.C.D.}(p, q), \quad p = p'D, \quad q = q'D.$$

When there are more than two generators, there is no simple formula for calculating the conductor. However, by an easy induction on N we can prove the following

LEMMA 2. *In a Newton-Puiseux semigroup S_N, the conductor is $\leq d_N\omega_N$. (Thus, beyond the last merging point, S_N coincides with the abelian group it generates).*

COROLLARY. *Every element in S_N admits a unique expression (1) with M admissible.*

PROOF. Suppose M, M' are admissible and $M \cdot \boldsymbol{\omega} = M' \cdot \boldsymbol{\omega}$, $m_N > m'_N$. Then $(m_N - m'_N)\omega_N$ belongs to the abelian group generated by S_{N-1}, hence to S_{N-1} itself. This is absurd, hence $m_N = m'_N$. Similarly, $m_i = m'_i$ for all other i.

7. **Construction of primes.** We are in stage N, having defined S_N, Γ_N, *etc.* The induction hypothesis H_P and H_G are also at our disposal.

Take any rational number $\omega_{N+1} \geq d_N\omega_N$. Let d_{N+1} denote the smallest integer such that

$$d_{N+1}\omega_{N+1} \in S_N, \quad (d_{N+1} = 1 \text{ if } \omega_{N+1} \in S_N).$$

Let $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_N)$ be the admissible exponent such that

$$\boldsymbol{\alpha} \cdot \boldsymbol{\omega} = d_{N+1}\omega_{N+1} \tag{8}$$

Take an integer $r \geq 2$. Note that $r\boldsymbol{\alpha} = (r\alpha_0, \dots, r\alpha_N)$ may not be admissible. When this happens, we like to investigate the expansion (3) for $\mathbf{G}_N^{r\boldsymbol{\alpha}}$.

For this purpose, it is convenient to define a weight on X and G_{N+1}:

$$v(X) = v(G_{N+1}) = \omega_{N+1}. \tag{9}$$

LEMMA 3. *Let $E = (e_0, \dots, e_n)$ be a given exponent.*

(i) Suppose $\omega_{N+1} > d_N\omega_N$. Then the weighted initial form of $\ell_N(\mathbf{G}_N^E)$ consists of only one monomial term $a\mathbf{Y}^\beta$, where $a \neq 0$, $\boldsymbol{\beta}$ is admissible, and

$$\boldsymbol{\beta} \cdot \boldsymbol{\omega} = E \cdot \boldsymbol{\omega}.$$

(ii) Suppose $\omega_{N+1} = d_N\omega_N$, and suppose $e_N < d_N$. Then the same is true.

(iii) Suppose $\omega_{N+1} = d_N\omega_N$. Then $\alpha_N = 0$. Hence if $E = r\boldsymbol{\alpha}$, again the same is true.

EXAMPLE. Consider $\Gamma_1 = \{y, x, x^2 - 2y^3\}$. Here $\omega_1 = 3/2$.
Let us compute the expansion of $(yx)^2$:

$$(yx)^2 = y^2[(x^2 - 2y^3) + 2y^3] = 2y^5 + y^2(x^2 - 2y^3).$$

In case we take $\omega_2 > d_1\omega_1 = 3$, $2y^5$ has the lowest weight,

$$v(2y^5) = 5 < v\big(y^2(x^2 - 2y^3)\big),$$

confirming (i).

However, if we take $\omega_2 = 3$, then both terms have weight 5; this explains why we assume $e_N < d_N$ in (ii).

PROOF. For an admissible exponent, E, there is nothing to prove.

Define an ordering of the exponents as follows:

$$(e'_0, \ldots, e'_N) < (e_0, \ldots, e_N)$$

if $\exists j$, $e'_j < e_j$ and $e'_i = e_i$ $\forall i > j$.

Now, suppose E is not admissible. Let $j > 0$ be the largest integer such that $e_j \geq d_j$. By (H_G), we can write

$$G_j^{d_j} = -c\mathbf{G}_{j-1}^{\alpha_{j-1}} + G_{j+1}. \tag{10}$$

Note that the first two terms both have weight $d_j\omega_j$; the third term, G_{j+1}, has higher weight for all j, $1 \leq j \leq N$, in case (i). In case (ii), this is true for all j, $1 \leq j \leq N-1$.

By a repeated application of (10), it follows that there is an exponent $E' < E$, such that

$$\mathbf{G}_N^E = C^*\mathbf{G}_N^{E'} + \text{ higher weighted terms},$$

where $C^* \neq 0$, $E \cdot \boldsymbol{\omega} = E' \cdot \boldsymbol{\omega}$.

Take an E' with this property which is minimal in the ordering. This E' must be admissible.

The proof of (iii) is easy. Since $d_N\omega_N \in S_{N-1}$, and $\boldsymbol{\alpha}$ is admissible, we must have $\alpha_N = 0$.

We introduce a terminology. Let g be the generator of the abelian group generated by S_N. Given an integer m, let $\mathcal{M}_{mg}$ denote the ideal in $K[[\mathbf{Y}, X]]$ of elements with weighted order $\geq mg$.

Given $P(X;\mathbf{Y})$, $P'(X;\mathbf{Y})$ in $\mathcal{M}_{mg}$, we say they are congruent modulo higher weighted terms, if

$$\ell_N \circ \sigma_N(P - P') \in \mathcal{M}_{(m+1)g}.$$

When $\mathcal{M}_{mg}$ is understood, we simply write

$$P(X;\mathbf{Y}) \equiv P'(X;\mathbf{Y}) \text{ m. h. w. t.}$$

Let $f(x,y) = \sigma_N(P)$, $f'(x,y) = \sigma_N(P')$; we also write

$$f(x,y) \equiv f'(x,y) \text{ m. h. w. t.}$$

Recall that ℓ_N may not preserve multiplication. However, we shall show it preserves the weighted initial form modulo higher weighted terms in the following sense.

Again, let us take any $\omega_{N+1} \geq d_N\omega_N$, and defined weights as in (9). Then the weighted initial form of a given $P(X;\mathbf{Y})$ is defined.

Let $F_i(x,y)$, $i = 1, 2, 3$, be given with

$$F_3(x,y) = F_1(x,y)F_2(x,y).$$

LEMMA 4. *Let $W_i(X;\mathbf{Y})$ denote the weighted initial form of $\ell_N(F_i)$. Then*

$$W_1(X;\mathbf{Y})W_2(X;\mathbf{Y}) \equiv W_3(X;\mathbf{Y}) \text{ m. h. w. t.}$$

PROOF. Recall that σ_N preserves multiplication. Hence

$$\ell_N \circ \sigma_N\big(\ell_N(F_1)\ell_N(F_2)\big) = \ell_N(F_3). \tag{11}$$

Consider a typical term in $\ell_N(F_1)\ell_N(F_2)$,

$$\xi \equiv C_1 C_2 \mathbf{Y}^{E_1+E_2} X^{d_1+d_2},$$

where $C_1 \mathbf{Y}^{E_i} X^{d_i}$ is a monomial term in $\ell_N(F_i)$.

By Lemma 3, $\ell_N \circ \sigma_N(\xi)$ has the same weighted order as ξ.

Now, comparing terms of minimal weighted order on both sides of (11), we find

$$\ell_N \circ \sigma_N(W_1 \cdot W_2) \equiv W_3 \text{ m. h. w. t.}$$

Several important consequences can be derived from this lemma. First, let us take $\omega_{N+1} = d_N \omega_N$. Let $H(G_0, \ldots, G_{N+1})$ be any series with weighted order $> \omega_{N+1}$.

LEMMA 5. *$G_{N+1} + H(G_0, \ldots, G_{N+1})$ is prime.*

This is clear: $\ell_N(G_{N+1} + H)$ has weighted initial form X, which is irreducible.

EXAMPLE. Consider $\Gamma_1 = \{y, x, x^2 - y^3\}$. Here, $G_2 + y^3 = x^2$ is not prime. Note that $v(y^3) = \omega_2$, and hence Lemma 5 does not apply.

Using the inductive hypothesis (H_P), we see that $G_{N+1}^{(i)}$, $i \geq 0$, are all primes.

Assertion 1 follows. Indeed, when $k_N = 1$, there is a finite, or infinite, series H, such that

$$F(x, y) = (G_{N+1} + H) \cdot \text{unit}.$$

Assertions 2 and 3 are immediate consequences too.

Now, let us take $\omega_{N+1} \notin S_N$, $\omega_{N+1} > d_N \omega_N$, and $\boldsymbol{\alpha}$ satisfying (8). Take a constant $c \neq 0$.

LEMMA 6. *Let $H(G_0, \ldots, G_{N+1})$ be a series with weighted order $> d_{N+1}\omega_{N+1}$. Then*

$$G_{N+1}^{d_{N+1}} - c\mathbf{G}_N^{\boldsymbol{\alpha}} + H(G_0, \ldots, G_{N+1})$$

is a prime.

The corresponding weighted initial form is $X^{d_{N+1}} - c\mathbf{Y}^{\boldsymbol{\alpha}}$, having weight $d_{N+1}\omega_{N+1}$. It has to be irreducible, since any weighted form of lower weighted form of lower weight consists of at most one monomial, and the product of two such is a single monomial term.

8. **Proof of Assertion 4** ($N \geq 1$). We are at stage N, having defined P_N, θ_N, *etc.* and then, being in Case 3, arrived at $P_N^{(s)}$, $\theta_N^{(s)}$. We shall write $P_N^{(s)}$, $\theta_N^{(s)}$ as P_N, θ_N, for simplicity of notation.

Let $\boldsymbol{\alpha}$ denote the admissible exponent such that $\boldsymbol{\alpha} \cdot \boldsymbol{\omega} = \tan\theta_N\big(= v(X)\big)$. Take a Newton dot on the first edge, representing a term of P_N of the form

$$a_r \mathbf{Y}^E X^{k_N - r}, \quad a_r \neq 0.$$

Then Lemma 3 can be applied to $r\boldsymbol{\alpha}$, giving a constant $c_r \neq 0$ such that

$$a_r \mathbf{Y}^E \equiv c_r \mathbf{Y}^{r\boldsymbol{\alpha}} \text{ m. h. w. t.}$$

These c_r can be used to define a homogeneous form in two variables

$$W(X, Y) = X^{k_N} + c_2 X^{k_N - 2} Y^2 + \cdots.$$

Attention should be paid to the absence of c_1; this is because we are in Case 3, there is no Newton dot at the corresponding point.

Observe that

$$W(X, \mathbf{Y}^{\boldsymbol{\alpha}}) \equiv P_N(X; \mathbf{Y}) \text{ m. h. w. t.}$$

Hence we can consider $W(X, \mathbf{Y}^{\boldsymbol{\alpha}})$, as the weighted initial form of P_N.

Since at least one $c_r \neq 0$, $(r > 1)$, $W(X, Y)$ factors:

$$W(X, Y) = H_p(X, Y) K_q(X, Y), \quad p + q = k_N, \tag{12}$$

where H_p, K_q are relatively prime homogeneous forms, monic in X.

Take any monomial $\mathbf{Y}^E X^d$ with weight

$$E \cdot \boldsymbol{\omega} + d \tan\theta_N > k_N \tan\theta_N.$$

Choose an integer $j \geq 0$ and a rational number t such that

$$E \cdot \boldsymbol{\omega} = j \tan\theta_N + t, \quad 0 \leq t < \tan\theta_N.$$

By Lemma 2, there is an admissible exponent J such that

$$\tan\theta_N + t = J \cdot \boldsymbol{\omega}.$$

Let us first consider the case

$$E \cdot \boldsymbol{\omega} \geq \tan\theta_N \quad (\text{hence } j \geq 1).$$

By Lemma 3, there exists a constant $C^* \neq 0$ such that

$$\mathbf{Y}^E \equiv C^* \mathbf{Y}^J \mathbf{Y}^{(j-1)\boldsymbol{\alpha}} \text{ m. h. w. t.}$$

Since $d + j = k_N$, Lemma 1 can be applied to $Y^{j-1} X^{d^*}$, for (12), giving

$$Y^{j-1} X^d = B_{s-p}(X, Y) H_p(X, Y) + A_{s-q}(X, Y) K_q(X, Y)$$

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where $s = d + j - 1$.

On substituting Y by $\mathbf{Y}^{\boldsymbol{\alpha}}$, we find that $\mathbf{Y}^E X^d$ is in the ideal generated by $H_p(X, \mathbf{Y}^{\boldsymbol{\alpha}})$, $K_q(X, \mathbf{Y}^{\boldsymbol{\alpha}})$, modulo higher weighted terms.

Now, consider the case

$$E \cdot \boldsymbol{\omega} < \tan\theta_N.$$

In this case, $j = 0$ and hence $d \geq 1$. Lemma 1 applies to X^{d-1}. Again, $\mathbf{Y}^E X^d$ is in the ideal generated by $H_p(X, \mathbf{Y}^{\boldsymbol{\alpha}})$, $K_q(X, \mathbf{Y}^{\boldsymbol{\alpha}})$.

The rest of the argument is standard for Hensel's Lemma. We can recursively find weighted forms A', B', A'', B'', *etc.* with increasing weights, such that

$$[H_p + A' + A'' + \cdots][K_q + B' + B'' + \cdots]$$

has $F(x, y)$ as its image under σ_N. Thus $F(x, y)$ is reducible, proving Assertion 4.

EXAMPLE.

$$(x^2 - y^3)^2 - y^7 = \left[(x^2 - y^3 + xy^2) + \frac{1}{2}y^4 + \frac{1}{4}xy^3 + \cdots\right] \cdot \left[(x^2 - y^3 - xy^2) + \frac{1}{2}y^4 - \frac{1}{4}xy^3 + \cdots\right].$$

9. **Proofs of Assertions 5 to 7.** We are in Case 4. Define $v(X) = \omega_{N+1}$ and let $\boldsymbol{\alpha}$ be an admissible exponent satisfying (8).

By an argument similar to that in Section 8, we can define

$$W(X, Y) = X^{k_N} + C_1 X^{k_N - d_{N+1}} Y + C_2 X^{k_N - 2d_{N+1}} Y^2 + \cdots$$

such that

$$W(X, \mathbf{Y}^{\boldsymbol{\alpha}}) \equiv P_N(X; \mathbf{Y}) \text{ m. h. w. t.}$$

Now, suppose k_N is not divisible by d_{N+1}. Let k_N be divided by d_{N+1}:

$$k_N = Q d_{N+1} + R, \quad 0 < R < d_{N+1}, \tag{13}$$

so that

$$W(X, Y) = X^R H(\xi, \eta),$$

where

$$\xi \equiv X^{d_{N+1}}, \quad \eta \equiv Y,$$

and $H(\xi, \eta)$ is a homogeneous Q-form, monic in ξ. The equation $H(\xi, 1) = 0$ may, or may not, have $\xi = 0$ as a root. Let $\mu \geq 0$ denote the multiplicity.

First, assume $\mu = 0$.

Let $I\big(\xi, H(\xi, \eta)\big)$ denote the ideal generated by ξ and $H(\xi, \eta)$. Then, clearly.

$$\eta^Q \in I\big(\xi, H(\xi, \eta)\big). \tag{14}$$

Take a monomial $\mathbf{Y}^E X^d$ such that

$$E \cdot \boldsymbol{\omega} + d\omega_{N+1} > k_N \omega_{N+1}. \tag{15}$$

We claim that

$$\mathbf{Y}^E X^d \in I\big(X^R, H(X^{d_{N+1}}, \mathbf{Y}^{\boldsymbol{\alpha}})\big). \tag{16}$$

This is obvious if $d \geq R$. In case $d < R$, it suffices to show that $\mathbf{Y}^E$ is divisible by $\mathbf{Y}^{Q\boldsymbol{\alpha}}$. Then (16) follows from (14).

By (13),

$$E \cdot \boldsymbol{\omega} - Qd_{N+1}\omega_{N+1} > (R - d)\omega_{N+1} \geq d_N \omega_N.$$

Hence the left-hand side, being an element of the abelian group generated by S_N, is actually in S_N, by Lemma 2. There is an exponent E^* such that

$$E \cdot \boldsymbol{\omega} = E^* \cdot \omega + Qd_{N+1}\omega_{N+1},$$

whence

$$\mathbf{Y}^E = \mathbf{Y}^{E^*}(\mathbf{Y}^{\boldsymbol{\alpha}})^Q.$$

Now, suppose $\mu \geq 1$. Let us write

$$H(\xi, \eta) = \xi^\mu K(\xi, \eta), \quad K(0, 1) \neq 0.$$

Lemma 1 is applicable to the pair ξ^μ, $K(\xi, \eta)$, so that

$$\xi^{i-1}\eta^{Q-i} \in I\big(\xi^\mu, K(\xi, \eta)\big), \quad 1 \leq i \leq Q. \tag{17}$$

Take a monomial $\mathbf{Y}^E X^d$ with property (15). We claim that

$$\mathbf{Y}^E X^d \in I\big(X^{\mu d_{N+1}+R}, K(X^{d_{N+1}}, \mathbf{Y}^{\boldsymbol{\alpha}})\big). \tag{18}$$

In case $d \geq \mu d_{N+1} + R$, this is obvious. Otherwise, let μ' denote the largest integer such that

$$(\mu' - 1)d_{N+1} + R \leq d < \mu' d_{N+1} + R.$$

Then, by a similar argument, we can show that $\mathbf{Y}^E X^d$ is divisible by

$$X^{(\mu'-1)d_{N+1}+R}\mathbf{Y}^{(Q-\mu')\boldsymbol{\alpha}}.$$

Hence (18) follows from (17).

Both for $\mu = 0$ and for $\mu > 0$, we can now use an argument similar to that for Assertion 4 to conclude that $F(x, y)$ is reducible, proving Assertion 5.

Finally, let us assume k_N is divisible by d_{N+1}, so that $R = 0$ in (13).

Coefficient $C_1, C_2, \ldots,$ can be determined so that

$$W(\xi, Y) = \xi^Q + C_1\xi^{Q-1}Y + \cdots + C_Q Y^Q$$

has the property that

$$W(X^{d_{N+1}}, \mathbf{Y}^{\boldsymbol{\alpha}}) \equiv P_N(X; \mathbf{Y}) \text{ m. h. w. t.}$$

Suppose $C_1 = 0$. Since at least one other $C_i \neq 0$, $W(\xi, Y)$ factors into two relatively prime factors, monic in ξ. Let us consider

$$\{G_0, \ldots, G_{N+1}, G_{N+2}\}, \quad G_{N+2} = G_{N+1}^{d_{N+1}}.$$

This is G-adic base, but not a Γ-adic base. Consider the expansion (3) of $F(x,y)$ with respect to this base. Since $W(\xi, Y)$ factors, by repeating the argument for Assertion 4, we come to the conclusion that $F(x,y)$ is reducible. This completes the proof of Assertion 6.

Now assume $C_1 \neq 0$. Define

$$G_{N+2} = G_{N+1}^{d_{N+1}} + \frac{C_1}{Q} \mathbf{G}_N^{\boldsymbol{\alpha}},$$

which, by Lemma 6, is prime, proving Assertion 7.

Note that the induction hypothesis (H_G) has also been proved for $N+1$.

As for (H_P), using $\Gamma_{N+1} = \{G_0, \ldots, G_{N+2}\}$ as the Γ-adic base, $P_{N+1}(X; Y_0, \ldots, Y_{N+1})$ is defined, having first vertex at $(k_{N+1}, 0)$.

In case

$$W(\xi, 1) = \left(\xi + \frac{C_1}{Q}\right)^Q \tag{19}$$

we clearly have

$$\tan \theta_{N+1} > d_{N+1} \omega_{N+1}.$$

In case (19) does not hold, we will have

$$\tan \theta_{N+1} = d_{N+1} \omega_{N+1},$$

but there will be no Newton dot at $(k_{N+1} - 1, \tan \theta_{N+1})$.

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