SPHERICAL GEOMETRIES AND MULTIGROUPS

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1. Introduction. The notion *spherical geometry* is suggested by the familiar geometry of the Euclidean 2-sphere in which the role of path is played by "arc of great circle". The first postulational treatment of the subject seems to be that of Halsted [10] for the two-dimensional case. Kline [11] under the name *double elliptic geometry*, gave a greatly simplified foundation for the three-dimensional case based on the primitive notions *point* and *order*.[1] Halsted and Kline study not merely descriptive (that is positional, non-metrical) properties of figures but also introduce metrical notions by postulating or defining congruence. Kline includes a continuity postulate designed to yield real spherical geometry.

Our object is to study the descriptive properties of spherical geometries by general mathematical methods under the weakest possible hypotheses. Just as there exist affine or projective geometries of arbitrary dimension corresponding to any coefficient field (not necessarily commutative), we should like to define spherical geometries of arbitrary dimension corresponding to any *ordered* field. This is possible if we consider the prototype of a spherical geometry to be the set of rays emanating from a point of an ordered affine geometry. This model of course is suggested by the familiar isomorphic mapping of a Euclidean 2-sphere into the family of rays which emanate from its centre. The model does not always (that is for all underlying fields) enjoy all the metrical properties of a Euclidean sphere, but it does exhibit the familiar descriptive properties and it does yield, in a sense, a "topological" sphere for every ordered field.

In order to give spherical geometries an autonomous existence we characterize them abstractly by postulates taking *point* as primitive. To do justice to ordinary goemetrical intuition we follow Kline in adopting as the second primitive notion the 3-term relation *order* suggested by the relation of points a,b,c when b is interior to the minor arc of a great circle which joins a and c. However this relation, despite its intuitive salience, does not facilitate generalization—it is, so to speak, too strongly linear or one-dimensional. Thus we define from it the notion *join* of a pair of points, which can be generalized to sets and extended to n points and forms the basis of our treatment of spherical geometries.

Consider then the following postulates involving a set S of elements $a,b,c, \ldots$ called *points* and a 3-term relation *order* indicated (abc), which may be read *points a,b,c are in the order abc, or b lies between a and c*:

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[1]Hallett [9], Flanders [8] have also given treatments of the subject.

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O1. *If* (abc) *then* a,b,c *are distinct.*

O2. *If* (abc) *then* (cba).

O3. *For each point* a *there exists a unique point* p *such that* $p \neq a$ *and* (axy) *always implies* (xyp).

DEFINITION 1. The uniquely determined point p in O3 is called the *opposite* of a and is denoted functionally by a'.

O4. *If* $b \neq a,a'$ *there exists* x *such that* (axb).

DEFINITION 2. If a,b are points and $b \neq a,a'$ the set of all x for which (axb) is called the *join* or *sum* of a,b and is denoted operationally by $a + b$. The join of a and a, denoted $a + a$, we take to consist of a itself. For simplicity we shall identify element a and set (a) whose only member is a,[2] so that for example we may assert the idempotent law $a + a = a$. This also enables us to employ the inclusion signs $\subset$, $\supset$ for elements as well as sets. For the present we do not define $a + a'$ (see sec. **3**).

In order to iterate the operation $+$ we must define sum of *sets* of points. Thus we introduce

DEFINITION 3. If A,B are non-void sets of points $A + B$, the *join* or *sum* of A and B is the set union $\sum_{a \subset A,\, b \subset B} (a + b)$. Observe that this is consistent with the definition of join of *points* a,b just adopted since if A,B consist of single elements, say $A = a$, $B = b$ then $A + B$ as defined reduces to $a + b$.[3] Further note that if any element of B is the opposite of an element of A then $A + B$ is meaningless, since one of the summands in its definition is not significant.

O5. $(a + b) + c = a + (b + c)$ provided both members are defined.

Observe that O5 involves the restrictions $b \neq a'$, that c shall not be the opposite of any point of $a + b$, etc. We shall consider later (sec. **3**) the matter of removing these restrictions.

Now we formally define the sense in which the term spherical geometry is to be employed.

DEFINITION 4. A set S in which is defined a relation (abc) satisfying O1, . . . , O5 is called a *spherical geometry*.

If we take S to be a Euclidean n-sphere and (abc) to mean b is an interior point of the minor arc of a great circle which joins a and c then O1, . . . , O5 are satisfied, and we call S with order thus defined a *Euclidean* spherical geometry. A second type of spherical geometry (which includes the first in the sense of isomorphism) arises if S is the set of rays emanating from a point P of an ordered affine space of arbitrary (finite or infinite) dimension and (abc) means ray b is interior to the angle formed by the non-opposite rays a,c. We can form an analogous class of *analytic* spherical geometries as follows. In a

[2]In virtue of this agreement which is maintained throughout the paper (whether or not the elements are points) our definitions and theorems concerning non-void sets hold also for elements.

[3]A similar consistency principle holds throughout the paper whenever a notion defined for sets is apparently ambiguous when applied to elements.

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linear vector space L over an ordered division ring F, define $[\alpha \beta \gamma]$ to mean there exist X, Y in F satisfying

$$\beta = X\alpha + Y\gamma, \qquad X + Y = 1, \qquad 0 < X, Y.$$

Define $\vec{\alpha}$, the *ray* of L determined by α a non-zero element of L, to be the set of $X\alpha$ where $X > 0$ and ranges over F. Call rays $\vec{\alpha}$, $-\vec{\alpha}$ *opposite*. Then S, the set of all rays of L, is a spherical geometry if we define (abc) to mean rays a,c are distinct and not opposite and $a = \vec{\alpha}$, $b = \vec{\beta}$, $c = \vec{\gamma}$ where $[\alpha\beta\gamma]$. It is noteworthy that any spherical geometry of sufficiently high "dimension" is representable as such an *analytic* spherical geometry just as any projective geometry of dimension greater than 2 can be coordinatized—the proof of the former can be made to depend on the latter and will be given elsewhere.

The postulate set O1, . . . , O5 was evolved from that of Kline [11, Axioms I, . . . , X] with the object of formulating a simple and natural basis for spherical geometry and facilitating the study of the operation join. In O5 the only essential novelty is an *operational* formulation of a triangle transversal postulate used by Kline [11, Axiom VII][4] generalized so as to include as many degenerate cases as possible within the limits imposed by the restriction in our definition of $+$; this gives it increased deductive power since it covers linear as well as two-dimensional cases.

Our procedure in the study of spherical geometries will be to exploit consistently the algebraic properties of the operation join. We shall show that by adjoining an ideal element o to spherical geometry S to play the role of an identity element and by extending the definition of $+$ appropriately we can convert S into a generalized group with many-valued composition, called a *multigroup*.[5] The multigroups thus generated are in a class which we call *regular* multigroups; these bear close analogies to abelian groups since each element a has a unique inverse $-a$ and subtraction is related to addition by the familiar formula, $a - b = a + (-b)$. Thus we are able to subsume the theory of spherical geometries under that of regular multigroups, in fact we show it is equivalent in a certain sense to the theory of a particular class of regular multigroups (Theorems 12, 13).

It is known [14, 15] that projective and descriptive (ordered linear) geometries can be characterized and developed as multigroups,[6] which however do not bear close *formal* analogies to abelian groups or to the multigroups which have received most attention from algebraists. On the other hand our regular multigroups are covered by the multigroup theory of Dresher and Ore [6] which

[4]Compare Veblen [17, Assumption 5]; see also Flanders [8, Axiom 05] and his reference to Hallett [9].

[5]A multigroup is a system closed under an associative many-valued operation $\circ$, which contains elements x,y satisfying the relations $a \circ x \supset b$, $y \circ a \supset b$ when a,b are in the system; see [6, pp. 706, 707]. For references on multigroups see J. E. Eaton, "Associative multiplicative systems," *Amer. J. Math.*, vol. 62 (1940) 222-232; also see J. Kuntzmann, "Contribution à l'étude des systèmes multiformes," *Ann. Sciences Toulouse*, (4) vol. 3 (1939), 155-194.

[6]For a "simultaneous" formulation of these geometries as multigroups see [16].

was motivated by algebraic considerations suggested by group theory. Thus from the viewpoint of this paper it would seem that spherical geometries are more "regular" than descriptive or projective geometries and may possibly deserve a more central position in the comparative theory of geometrical systems.

We explicitly develop the theory of spherical geometries only to the point necessary to establish their equivalence to a class of regular multigroups. We then outline the theory of regular multigroups including: subsystems and their generation; cosets, homomorphisms and factor multigroups; linear independence and rank. These ideas cover the geometrical topics: linear (or spherical) subspaces, their alignment and intersection properties; half-spaces (for example in a Euclidean spherical geometry, semicircles, hemispheres, etc.); separation of linear subspaces; linear independence and dimension. For the sake of concreteness and familiarity we use Euclidean spherical geometries to exhibit the geometrical significance of the above algebraic ideas, although they are applicable to arbitrary spherical geometries with no essential change in the discussion.

2. Order properties. In this section we develop the theory of order in a spherical geometry S from postulates O1, . . . , O5 to prepare for the extension of the associative law for $+$. The main results (Theorems 7,8) give combinatory formulas for certain sums of points. Theorems 1, . . . , 6 are principally theorems or postulates of Kline [11] and are intuitively very familiar.

THEOREM 1. *(axa') and $(aa'x)$ are always false.*[7]

Proof. By O3, (axa') implies $(xa'a')$ which is contrary to O1. Similarly $(aa'x)$ implies $(a'xa')$ contrary to O1.

COROLLARY. $(a')' = a$; or equivalently $b = a'$ implies $a = b'$.

Proof. Let a'' denote $(a')'$. Suppose $a'' \neq a$. By O3 $a'' \neq a'$. Thus by O4 (axa'') holds for some x. Thus by O3 $(xa''a')$, which by O2 implies $(a'a''x)$. This contradicts Theorem 1, so that $a'' = a$.

The following result enables us to interpret order relations in "join" language and vice versa.

THEOREM 2. *(abc) implies $b \subset a + c$; conversely $b \subset a + c$ implies (abc) provided $a \neq c$.*

Proof. Suppose (abc). Then $c \neq a$ by O1 and $c \neq a'$ by Theorem 1. Thus $b \subset a + c$ by Definition 2. The remainder of the theorem is immediate by Definition 2.

Next we prove (Kline [11, Axiom V])

THEOREM 3. *(abc), (acd) imply (abd).*

Proof. Suppose (abc), (acd). By Theorem 2 we have

(1) $$b \subset a + c, \qquad c \subset a + d.$$

We wish to assert

(2) $$b \subset a + (a + d) = (a + a) + d.$$

[7]Compare Kline [11, Axiom I, Theorem 3].

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The first relation in (2) is implied by (1) in view of Definition 3, provided the expression $a + (a + d)$ is significant; that is, provided $a' \not\subset a + d$. Suppose $a' \subset a + d$. By O1, (acd) implies $a \neq d$. Thus the second part of Theorem 2 implies $(aa'd)$, contrary to Theorem 1. Thus the first relation in (2) is justified. By O5 the second relation in (2) is valid, provided the expression $(a + a) + d$ is significant. But by the idempotent law $(a + a) + d$ reduces to $a + d$ whose existence is involved in relation (1). Thus (2) is verified and it implies $b \subset a + d$. By the second part of Theorem 2 we have (abd) and the proof is complete.

COROLLARY 1. (abc) implies that (bca) is false.

Proof. Suppose (abc), (bca). Then (acb), which with (abc) implies by Theorem 3 (abb).

COROLLARY 2. (abc), (acd) imply (bcd).

Proof. By O2, O3 we have the following implications: $(acd) \rightarrow (dca) \rightarrow (cad')$. Also $(abc) \rightarrow (cba)$. By Theorem 3, O2, O3 and the corollary to Theorem 1 we have

$$(cba), (cad') \rightarrow (cbd') \rightarrow (d'bc) \rightarrow (bcd).$$

We continue with three theorems on order of four points [11, Theorems 22, 24, 25]. We dispense with the proofs—the first depends on the associative law like Theorem 3 and the latter two then follow by standard arguments of the foundations of geometry.

THEOREM 4. *If (abc), (bcd) and $d \neq a'$ then (abd) or $(ab'd)$.*

THEOREM 5. *If (abc), (abd) and $c \neq d$ then (acd) or (adc).*

THEOREM 6. *If (axb), (ayb), $x \neq y$ then (axy) or (ayx).*

We now prove the principal results of this section.

THEOREM 7. $a + (a' + b) = a + b \cup b \cup a' + b$ *provided*[8] $b \neq a, a'$.

Proof. Suppose $b \neq a, a'$. Then $a' + b$ is defined and is non-void. Furthermore $a' \subset a' + b$ implies $(a'a'b)$ since $a' \neq b$. Thus $a' \not\subset a' + b$ and $a + (a' + b)$ is significant. Clearly the right member of the relation to be established is significant. We shall complete the proof by showing the equivalence of the following relations:

$$(1) \qquad x \subset a + (a' + b),$$

$$(2) \qquad x \subset a + b \cup b \cup a' + b.$$

Suppose (1). Then by Definition 3

$$(3) \qquad x \subset a + y, \qquad y \subset a' + b$$

holds for some y. The second relation in (3) implies $(a'yb)$. From this we have (yba) and so (aby). Thus $a \neq y$ and the first relation in (3) implies (axy). If $x = b$ then (2) holds. Suppose $x \neq b$. Then by Theorem 6 (axy), (aby) imply (axb) or (abx). If (axb) then $x \subset a + b$. If (abx) then (bxa') so that $(a'xb)$ and $x \subset a' + b$. In either case (2) holds.

[8]We use the symbol $\cup$ to denote set theoretic addition. In expressions involving $+$, $\cup$ we adopt the convention that portions separated by $\cup$ signs are to be considered enclosed in parentheses unless the contrary is explicitly indicated.

Conversely we show (2) implies (1). First suppose $x \subset a + b$. Then (axb) since $a \neq b$. Since $b \neq a',a$ there exists z such that $(a'zb)$ by O4. Hence (zba) and (abz). By Theorem 3, (axb), (abz) imply (axz). Thus we have

$$(4) \qquad x \subset a + z, \qquad z \subset a' + b$$

and (1) follows by Definition 3. Now suppose $x = b$. Then (4) holds with the same choice of z and (1) follows as before. Finally suppose $x \subset a' + b$. Then $(a'xb)$ and $x \neq a',a$. Now choose z such that $(a'zx)$. Then (zxa) and (axz). Furthermore $(a'zx)$, $(a'xb)$ imply by Theorem 3 $(a'zb)$. Thus (4) holds and the theorem is established.

THEOREM 8. *If* $p \subset a + b$ *then* $a + b = a + p \cup p \cup b + p$.

Proof. Suppose $p \subset a + b$. If $a = b$ the result is trivial. Suppose $a \neq b$. Then (apb). Let $R = a + p \cup p \cup b + p$. Suppose $x \subset R$. If $x = p$ then $x \subset a + b$. Suppose $x \subset a + p$. Since $a \neq p$ we have (axp). This with (apb) implies by Theorem 3 (axb), so that $x \subset a + b$. Similarly $x \subset b + p$ implies $x \subset a + b$. Thus $R \subset a + b$. Conversely suppose $x \subset a + b$. Then (axb). If $x = p$ then $x \subset R$. Suppose $x \neq p$. Then (axb), (apb) imply by Theorem 6 (axp) or (apx). If (axp) then $x \subset a + p \subset R$. If (apx), then (axb) implies by Corollary 2 of Theorem 3 (pxb). Thus (bxp) and $x \subset b + p \subset R$. Hence $a + b = R$ and the theorem is proved.

3. The associative law. In this section we show how to extend the definition of $+$ in a spherical geometry S so as to obtain the unrestricted validity of the associative law. However this is impossible within the confines of S (Theorem 9), if S is non-trivial, but can be accomplished very simply in the system formed by the adjunction to S of an "ideal" element o which plays the role of an identity for the operation $+$.

THEOREM 9. *Let spherical geometry S contain at least three points. Then it is impossible to extend our definition of* $+$ (*Definition* 2) *to all pairs of points of S in such a way as to preserve the associative law.*[9]

Proof. Suppose such an extension of Definition 2 possible in S—it being understood of course that the iterated sums appearing in the associative law are defined by Definition 3. Suppose $p \neq a,a'$. The associative law and Theorem 7 imply

$$(1) \qquad (a + a') + p = a + p \cup p \cup a' + p.$$

Thus $(a + a') + p \supset p$ and by Definition 3 there exists o in S satisfying

$$(2) \qquad o + p \supset p, \qquad a + a' \supset o.$$

If $p \neq o,o'$ the first relation in (2) implies (opp). Thus $p = o$ or o' so that $o = p$ or p' and (2) implies $a + a' \supset p$ or p'. It is not restrictive to suppose

$$(3) \qquad p \subset a + a'.$$

[9]The numerical restriction is essential since the spherical geometry composed of points p,q satisfying $p = q'$, $q = p'$ with vacuous order relation satisfies the associative law if we define $a + a'$ to consist of a,a'.

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By O4, $(p'qa)$ for some q. Thus (qap) and $a \subset p + q$. By Definition 3 we may add q to both sides of (3) and we obtain

(4) $$a \subset (a + a') + q.$$

Since $q \neq a,a'$ we may replace p in (1) by q getting in view of (4), $a \subset a + q \cup q \cup a' + q$. This implies (aaq), $a = q$ or $(a'aq)$ which are false, and the proof is complete.[10]

This is not as disappointing as it might seem. The associative law fails in S because it implies (2) which requires that $a + a'$ contain for each p, a "relative" identity element o. This is impossible *in* S, and suggests the possibility of validating (2) and the associative law by going *outside* S. The simplest way to do this is to assign to $a + a'$ an ideal element o, not in S, such that $x + \mathrm{o} = \mathrm{o} + x = x$ for each x in S. Thus o plays the role of an additive ˙dentity and (2) becomes valid. Then if the associative law is to hold

$$a + a' = (a + a) + a' = a + (a + a') \supset a + \mathrm{o} = a.$$

Similarly we get $a + a' \supset a'$. Conversely if we require $a + a'$ to consist of a,a',o the associative law holds; we formalize and complete the discussion in the following definition and theorem.

Definition 5. Let S' be the set formed by adjoining to S an "ideal" element o, which is not in S. We extend Definition 2 on sum of *elements* to S' as follows:

$$a + a' = a \cup a' \cup \mathrm{o}, \qquad a \subset S;$$
$$b + \mathrm{o} = \mathrm{o} + b = b, \qquad b \subset S'.$$

Sum of *sets* is determined in S' as in S by Definition 3.[11]

Theorem 10. *In S' we have* (a) $a+b=b+a$; (b) $(a+b)+c=a+(b+c)$.

Proof. (a) This follows easily from O2, the corollary to Theorem 1 and Definition 5.

(b) The degenerate cases in which one of a,b,c is o or one is the opposite of another can be disposed of using Definition 5, Theorem 7 and O5. Suppose $a,b,c \neq \mathrm{o}$ and neither is the opposite of another. Then $a + b$, $b + c$ are significant in the sense of Definition 2, and the result holds by O5 unless one of the following is true:

(1) $$c' \subset a + b,$$

(2) $$a' \subset b + c.$$

But (1) and (2) are equivalent. For (1) implies $a \neq b$ (otherwise $c' = a$) and so $(ac'b)$. Thus $(ac'b) \rightarrow (c'ba') \rightarrow (ba'c) \rightarrow (2)$. Similarly we can show (2) implies (1). Thus we have only to consider the case in which both (1) and (2) hold. In this case we may apply Theorem 8 to $a + b$ and c', and using the associative law for cases already mentioned we have

[10] We have implicitly required that $a + a'$ be non-void since the sum of $a + a'$ and p must be significant by Definition 3, which excludes the void set from consideration. However the theorem is also valid if we allow $a + a'$ to be void.

[11] Note that the converse of Theorem 2 still holds for a,b,c in S with the added proviso $c \neq a'$.

$$\begin{aligned}(a + b) + c &= (a + c' \cup c' \cup b + c') + c \\ &= a + c' + c \cup c' + c \cup b + c' + c \\ &= a + (c' \cup c \cup o) \cup c' \cup c \cup o \cup b + (c' \cup c \cup o) \\ &= a + c' \cup a + c \cup a \cup c' \cup c \cup o \cup b + c' \cup b + c \cup b \\ &= a + c' \cup c' \cup b + c' \cup b + c \cup a + c \cup a \cup b \cup c \cup o \\ &= a + b \cup b + c \cup c + a \cup a \cup b \cup c \cup o.\end{aligned}$$

Since the last expression is symmetrical in a,b,c, when we apply the same argument to $a + (b + c) = (b + c) + a$, as we may in view of (2), we get the same result. Thus (b) is verified.

4. Spherical geometries as multigroups. Continuing the discussion of the last section we show that S', with $+$ as defined, is a multigroup with strong regularity properties and that the theory of spherical geometries is in a sense equivalent to that of a certain class of abelian multigroups.

We begin with the following definition.

DEFINITION 6. A *regular multigroup* is a set G of elements $a,b,c, \ldots$ in which is defined a 2-term operation $+$ satisfying postulates[12] M1, . . . , M5:

M1. *$a + b$ is a uniquely determined non-void subset of G.*

M2. $(a + b) + c = a + (b + c)$.

M3. $a + b = b + a$.

M4. *There exists in G an element* o, *called an identity element, such that $a + o = a$ for each a in G.*

In G we define $a - b$ to be the set of all x satisfying $b + x \supset a$.

M5. For each b in G there exists b^* in G satisfying[13]

$$a - b = a + b^*. \tag{1}$$

The *order* of regular multigroup G is its cardinal number.

It is easily seen that G has a unique identity element, which may then be represented unambiguously by o. Observe that in view of M5, M1 $a - b \neq O$.[14] M5, M4, M3 imply

$$o - b = b^*. \tag{2}$$

Thus $o - b$ is a single element and b^* in M5 is uniquely determined. In view of (2) we naturally call b^* the *negative* or *inverse* of b and denote it $-b$. Thus $-b = o - b$ and in view of the definition of $o - b$, we may characterize $-b$ as the unique solution x of the relation $b + x \supset o$. It easily follows that $-(-b) = b$. Replacing b^* in (1) by $-b$, (1) assumes the form

$$a - b = a + (-b)$$

which is the familiar relation between subtraction and addition of abelian group theory. Thus an abelian group is seen to be a regular multigroup.

[12]We maintain the agreements on identification of elements and unit sets and the use of $\supset$, $\subset$ adopted above (Definition 2) and we extend $+$ from elements to non-void sets by Definition 3.

[13]We are using the term regularity in a much more restricted sense than Dresher and Ore [6, p. 708]; in our sense it implies self-reversibility and complete regularity [6, pp. 717, 723].

[14]O denotes the void set.

We now discuss the relation between spherical geometries and regular multigroups.

THEOREM 11. *If S is a spherical geometry then S', with $+$ as defined, is a regular multigroup in which the negative of point a is its opposite a'.*

Proof. M1, ..., M4 hold in S' in view of Definitions 2,5 and Theorem 10. If $b = o$ M5 holds with $b^* = o$ since $a - o = a$. If $b \neq o$ then $b \subset S$ and we choose $b^* = b'$, the *opposite* of b. Then M5 is easily verified if $a = o,b$ or b'. Suppose $a \neq o,b,b'$. Supopse $x \subset a - b$. Then $b + x \supset a$ and $x \neq b,b',o$. Thus $b + x$ is defined by Definition 2 so that by Theorem 2, $b + x \supset a$ implies (bax) and so (axb'). Thus $x \subset a + b'$. Conversely $x \subset a + b' \rightarrow (axb') \rightarrow (b'xa) \rightarrow (xab) \rightarrow (bax) \rightarrow a \subset b + x \rightarrow x \subset a - b$. Thus M5 is completely verified and the theorem is proved.

This result suggests

DEFINITION 7. Let S be a spherical geometry. Then S' with $+$ as defined, is called the *associated multigroup of S.*[15]

The last result does not distinguish associated multigroups of spherical geometries from abelian groups or other regular multigroups. Thus we must find special properties to characterize these multigroups. First we introduce

DEFINITION 8. Let G be a regular multigroup. A *submultigroup* of G is a non-void subset of G which contains with a,b also $- a$ and $a + b$.[16] The *order* of element a of G is the cardinal number of the submultigroup of G generated by a, that is the least submultigroup of G which contains a.

Now we can state and easily derive the characteristic properties of multigroup S'.

THEOREM 12. *The associated multigroup of a spherical geometry is regular, satisfies the idempotent law and each of its elements, with the exception of* o, *has order* 3.

Proof. In view of the last theorem and Definitions 2,5 we have only to show that if S is a spherical geometry and $S' \supset a \neq o$ then the order of a is 3. Any submultigroup of S' which contains a must contain $A = a \cup a' \cup o$, since $- a = a'$ and $a + a' \supset o$. Moreover the negatives of the elements of A are a', $-(a') = a$,o; and A is closed under $+$ in view of the idempotent law and Definition 5. Thus A is the least submultigroup of S' which contains a. The cardinal number of A is 3, since $a,a' \neq o$ and by O3, $a \neq a'$. Thus a has order 3 and the theorem is proved.

Now we prove a sort of converse of this result and characterize the multigroups associated with spherical geometries.

THEOREM 13. *Let G be a regular multigroup which satisfies the idempotent*

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[15]Strictly speaking S' is not uniquely determined, since o is not, but we naturally consider the various S' to be identical.

[16]Observe that a submultigroup of G is a regular multigroup with respect to the composition of G. The term submultigroup is often used in a weaker sense than that of Definition 8 to denote a subset which is a multigroup with respect to the composition of the given multigroup [6, p. 714].

law and each element of which, with the exception of o, *has order* 3. *Then* G *is the associated multigroup of a spherical geometry.*

Proof. Let S be the set obtained by deleting from G its identity element o. In S we define (abc) to mean $c \neq a, -a$ and $b \subset a + c$ and we show that S, with order so defined, is a spherical geometry and that S' its associated multigroup coincides with G.

First we show for $a \neq \text{o}$

(1) $$a + (-a) = a \cup (-a) \cup \text{o}.$$

Adding a to both members of the relation $\text{o} \subset a + (-a)$ we have

$$a \subset a + (a + (-a)) = (a + a) + (-a) = a + (-a).$$

Similarly $-a \subset a + (-a)$; $a, -a, \text{o}$ are distinct, for $-a \neq \text{o}$ and $a = -a$ implies $a + (-a) = a + a = a$ so that the set $a \cup \text{o}$ is the least submultigroup of G containing a, and a has order 2 contrary to hypothesis. Thus since $a + (-a) \supset a, -a, \text{o}$ and a has order 3, (1) is verified.

To show S a spherical geometry we observe O1 is a consequence of M5 and (1); O2 follows from M3 and $-(-a) = a$; O3 can be verified by taking p (the opposite of a) to be $-a$, for a in S; O4 follows from M1. To verify O5 consider the operation $\oplus$ defined in S by Definition 2: if $b \neq a, -a$ then $a \oplus b$ is the set of x for which (axb); $a \oplus a = a$. We see immediately that $a \oplus b = a + b$ for $a,b \subset S$ provided $b \neq -a$. Thus since $+$ is associative, the associative law for $\oplus$ certainly holds for those triples a,b,c in S for which it is significant. Hence O5 is verified and S is a spherical geometry.

Now to construct S', the associated multigroup of S, we adjoin o to S to form set S' so that as a *set* $S' = G$. Then we extend $\oplus$ to S' by the agreements (Definition 5) $a \oplus (-a) = a \cup (-a) \cup \text{o}$ for $a \subset S$ and $b \oplus \text{o} = \text{o} \oplus b = b$ for $b \subset S'$. Thus in view of (1) $a \oplus b = a + b$ for all $a,b \subset S'$ and as a *multigroup* $S' = G$.

5. Regular multigroups. In this section we sketch the theory of regular multigroups. The results are analogues of familiar theorems of group theory and are given without proof to avoid duplication of methods in the literature.[17] There is implicit in the discussion, in view of sec. **4**, a corresponding theory for arbitrary spherical geometries, which we explicitly derive for Euclidean spherical geometries. The theory of course also applies to abelian groups. In later sections we add restrictions when necessary and obtain finally the multigroups associated with spherical geometries.

In this section G denotes an arbitrary regular multigroup with elements $a,b,c,\ldots$ and operation $+$; $A,B,C,\ldots$ denote subsets of G which are non-void unless the contrary is stated. For simplicity of expression we shall refer to G as a *group* and to its submultigroups as *subgroups*; and we shall call the usual type of group with *single*-valued composition a *classical* group. The operations

[17]See in particular Dresher and Ore [6]; observe however that many of our definitions differ from theirs.

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of subtraction and taking inverses are defined for sets in the natural way: $A - B = \sum_{a \subset A, b \subset B}(a - b)$; $-A$ denotes the set of all $-a$ for $a \subset A$. Familiar formal laws of additive algebra hold for sets: $A \subset A'$, $B \subset B'$ imply $A + B \subset A' + B'$; $(A + B) + C = A + (B + C)$; $A + B = B + A$; $A - B = A + (-B)$; $-(-A) = A$; $-(A + B) = (-A) + (-B)$. Subgroups can be characterized formally. *A is a subgroup of G if and only if* (a) $A + A = A = -A$ *or* (b) $A - A = A$. *Generation* of subgroups is defined in the usual way:

DEFINITION 9. Let M be an arbitrary (not necessarily non-void) subset of G. By the *subgroup of G generated by M*, denoted $\{M\}$, we mean the least subgroup of G which contains M. If $\{M\} = A$ we say *M generates A* or is a *set of generators* of A. In general if M_i, $i \epsilon I$, is a system of arbitrary subsets of G we define $\{M_i; i \epsilon I\}$, the *subgroup of G generated by* M_i, $i \epsilon I$, to be the least subgroup of G which contains each M_i. If I is the set $1, \ldots, n$ we use the notation $\{M_1, \ldots, M_n\}$ for $\{M_i; i \epsilon I\}$. Note that $\{O\} = o$ for any G; if G is the associated multigroup of spherical geometry S then $\{a\} = a \cup (-a) \cup o$, and if S is Euclidean and $a,b \subset S$, $(b \neq a,a')$ then $\{a,b\}$ is the great circle containing a,b to which is adjoined o.

In classical group theory $\{M\}$, where $M \neq O$, consists of all "polynomial" combinations of elements of M which can be formed using the group operation and taking inverses. Here we have an analogous result.

THEOREM 14. *$\{M\}$, if $M \neq O$, is the set union of all expressions $a_1 + \ldots + a_n$ where $a_i \subset M$ or $a_i \subset -M$, $1 \leqq i \leqq n$.*

COROLLARY. (Finiteness of dependence). Suppose $M \neq O$. Then $x \subset \{M\}$ if and only if $x \subset \{a_1, \ldots, a_n\}$ where $a_i \subset M$, $1 \leqq i \leqq n$.

Exactly as in classical abelian group theory we have

THEOREM 15. *If A,B are subgroups of G then* $\{A,B\} = A + B$.

From this Dedekind's famous modular law [4, p. 34, L5] follows, essentially by Dedekind's proof [4, p. 35, Theorem 3.2].

THEOREM 16. (Modularity). *If A,B,C are subgroups of G and $A \subset C$ then*[18] $\{A,B\}.C = \{A,B.C\}$.

Now we point out the geometrical significance of the ideas presented thus far. Let G be the associated multigroup of a Euclidean spherical geometry S and let A be a subgroup of G. By Definition 8, $A \supset a,b$ implies $A \supset -a$, $a + b$. Hence A contains with each point a, its opposite a' and with each pair of points a,b $(b \neq a,a')$ the minor arc of a great circle which joins a and b. An arbitrary (not necessarily non-void) subset of S which enjoys these properties we call a *spherical subspace* or simply a *linear subspace* of S. (Examples are: O, a pair of opposite points, a great circle, etc.) Observe that a linear subspace of S contains with a,b $(b \neq a,a')$ the great circle passing through a,b and so is an analogue in spherical geometry S of a linear subspace of a projective or affine geometry. Let B be the set obtained by deleting o, the

[18] We use the symbol . to denote set theoretic multiplication.

identity element of G, from subgroup A of G. Then B is a linear subspace of S. Furthermore if we adjoin o to B any linear subspace of S we obtain, in view of Definition 5, a corresponding subgroup A of G which we call the subgroup *associated* to B. Thus the trivial operation of adjoining o effects a $(1 - 1)$ correspondence between the set of linear subspaces of S and the set of subgroups of G, which we call the *natural* correspondence between these sets. In view of this we may consider the concept *linear subspace* of S as essentially identical with *subgroup* of G.[19]

To obtain geometrical significance for $\{M\}$, we suppose $M \not\supset$ o, which is not essentially restrictive. If we delete o from $\{M\}$ we obtain a linear subspace $\overline{M}$ of S which, in view of the natural correspondence between linear subspaces of S and subgroups of G, is the *least* linear subspace of S containing M. Thus $\overline{M}$ is called the linear subspace of S *determined* or *spanned* by M. For example the linear subspace of S determined by point a is $a \cup a'$, by $a \cup b$ $(b \neq a,a')$ is the great circle containing a and b. Thus the geometrical notion *determination of linear subspaces* is subsumed under the familiar algebraic concept *generation of subgroups*.[20] Furthermore we note that the natural correspondence associates $\{M\}$ to $\overline{M}$, in particular it associates o to O,$\{a\}$ to the linear space $a \cup a'$, and $\{a,b\}$ to the great circle containing a,b where $b \neq a,a'$.

We continue with *coset* and associated ideas.

DEFINITION 10. Let H be a subgroup of G. Then $a + H$ is called the *coset of H determined by a* and is denoted $(a)_H$. The set of all cosets $(a)_H$ where $a \subset A$ is denoted $(A)_H$. Let G/H denote $(G)_H$. In G/H we define addition thus: $(a)_H \oplus (b)_H = (a + b)_H$. We call G/H with addition so defined the *factor group of G with respect to H.*

As in classical group theory the cosets of H form a decomposition of G. Furthermore the sum of two elements of G/H (cosets) is independent of their representation and G/H, like G, is a *group* (regular multigroup). The correspondence $x \to (x)_H$ maps G on G/H in such a way as to preserve addition. This suggests

DEFINITION 11. Let K_1, K_2 be arbitrary systems (not necessarily groups) consisting of a set of elements and a 2-term operation (not necessarily single-valued) the composition in each being denoted $+$. Let there exist a single-valued mapping f of K_1 on K_2 which satisfies $f(x + y) = f(x) + f(y)$. Then we call f a *homomorphism* of K_1 on K_2 and say K_1 is *homomorphic* to K_2. If f is $(1 - 1)$ we use the terms *isomorphism*, *isomorphic* and write $K_1 \cong K_2$.[21]

If H is a subgroup of G then G is homomorphic to G/H. Furthermore if G is homomorphic to K, then K also is a group (regular multigroup) and is isomorphic to G/H, where H is the set of elements of G mapped by the homomorphism on the identity of K. If A,B are subgroups of G the mapping

[19] Compare [14, §4], [15, §5].

[20] Compare [14, §5], [15, p. 350, Definition 2].

[21] Congruence relations in groups can be introduced by the definition of [15] and have the familiar relations to homomorphisms [4, pp. 2,3].

$(b)_A \rightarrow (b)_{A.B}$ effects an isomorphism of $\{A,B\}/A$ into $B/A.B$ and we may assert

THEOREM 17 (Isomorphism Theorem). *If A,B are subgroups of G then $\{A,B\}/A$ is isomorphic to $B/A.B$.*[22]

From this the Jordan Hölder theorem can be deduced as in classical group theory.

We conclude this section with the geometrical significance of coset and factor group. First let G for simplicity of illustration be the associated multigroup of S, the spherical geometry of a Euclidean 2-sphere, and let H be the subgroup of G formed by adjoining o, the identity element of G, to T a great circle of S. Suppose $a \not\subset H$. Then $(a)_H = a + (\text{o} \cup T) = a \cup a + T$. That is $(a)_H$ consists of a and all interior points of minor arcs of great circles which join a to points of T. This is of course the *hemisphere* of S, bounded by T, which contains a. On the other hand $(a)_H = H$ if $a \subset H$. Similarly if we replace T by a pair of opposite points p,p' and let H be the subgroup of G composed of p,p',o we find that the cosets of H are the (open) semicircles with endpoints p,p', and H itself. In general let S be any Euclidean spherical geometry, T be a linear subspace of S, G be the associated multigroup of S and $H = \text{o} \cup T$. Then $(a)_H$ is the "hemisphere" bounded by T which contains a provided $a \subset H$, otherwise $(a)_H = H$. Thus the coset concept subsumes the idea *half-space* (point, semicircle, 2-hemisphere, etc.). Furthermore the coset decomposition of G determined by H yields, by exclusion of o from consideration, a decomposition of S into the set of half-spaces (or hemispheres) bounded by T, together with T. Examples are the decomposition of a 2-sphere (1) into a great circle and the hemispheres which it bounds and (2) into a pair of opposite points and the semicircles joining them.

To illustrate the notion factor group consider the second example of the preceding paragraph in which T consists of a pair of opposite points p,p'. Then G/H is the set composed of H and the semicircles joining p and p' in which the "join" or "sum" of two non-opposite semicircles consists of all the semicircles in the *lune* bounded by the given semicircles. G/H is easily seen geometrically to be isomorphic to the multigroup associated with a great circle of S. We prove this formally as a simple application of the Isomorphism Theorem. Let K be the subgroup of G formed by adjoining o to a great circle U which contains neither p nor p'. Then $G = \{H,K\}$ and $H.K = \text{o}$ so that by Theorem 17

$$G/H = \{H,K\}/H \cong K/H.K = K/\text{o} = K.$$

6. Linear independence and dimension. We continue with the theory of linear independence and dimension or rank which are of importance both in classical group theory and spherical geometry. We consider the assignment of dimension to subgroups of G and its relation to generation and intersection

[22]Compare [6, p. 726, Theorem 6], also see [7, p. 68]. For classical groups see [1, p. 134, Theorem 15], [18, p. 136, the first Isomorphism Theorem].

properties of subgroups. The theory covers the corresponding topics for linear subspaces of a Euclidean spherical geometry and is applicable to spherical geometries in general. The theory requires a restriction on the regular multigroups G we have been studying which relates them in an interesting way to projective geometries (Theorem 19).

In developing the familiar theory of dimension for a Euclidean spherical geometry S we assign to the linear subspaces in order of increasing complexity: O, pair of opposite points, great circle, 2-sphere, . . . , the "dimensions": -1, 0, 1, 2, ... We may take this to signalize that a linear subspace of S of each type is a maximal proper subspace of one of the succeeding type. Thus a necessary condition for validation of the familiar theory of dimension of S is that O be a maximal proper linear subspace of each pair of opposite points, in other words that there be no linear subspace "between" O and a pair of opposite points a,a'. Translating this into the corresponding restriction on G, the associated multigroup of S, we get since $\{a\}$ is the subgroup of G associated to the linear space composed of a,a': *there is no subgroup of* G *"between"* o *and* $\{a\}$ *if* $a \neq$ o. This property is sufficient to yield the desired dimension theory. In order to phrase it more carefully and conveniently we introduce

DEFINITION 12. Let A, B be distinct subgroups of a regular multigroup G such that $A \supset X \supset B$ (where X is a subgroup of G) implies $X = A$ or $X = B$. Then we say *A covers B.*

We state the desired property of a regular multigroup G which we assume throughout this section as the

COVERING POSTULATE. If $a \neq$ o, $\{a\}$ covers o.

We continue with consequences of this postulate, postponing to the end of the section interpretations of the theory. Suppose $\{a\} \supset b \neq$ o. Then $\{a\} \supset \{b\} \supset$ o, and $\{b\} \neq$ o. Hence by the Covering Postulate $\{b\} = \{a\}$. Thus we may assert the

COROLLARY. If $a \neq$ o, $\{a\}$ is generated by each of its elements other than o.

We generalize the Covering Postulate in

THEOREM 18. *If H is a subgroup of G and $a \not\subset H$ then $\{a,H\}$ covers H.*

Proof. Suppose $H \subset X \subset \{a,H\}$ where $X \neq H$ and is a subgroup of G. Suppose $x \subset X$, $x \not\subset H$. Then $x \subset \{\{a\},H\} = \{a\} + H$ so that $x \subset b + h$ where $b \subset \{a\}$ and $h \subset H$. If $b =$ o then $x = h$ contrary to $x \not\subset H$. Thus $b \neq$ o. We have $b \subset x - h \subset X$. Thus using the last corollary $X \supset \{b\} = \{a\}$. Hence $X \supset \{a,H\}$ and $X = \{a,H\}$. Since $\{a,H\} \neq H$, by definition $\{a,H\}$ covers H.

It is well known that a Euclidean sphere is convertible into a projective geometry by defining "point" as a pair of opposite points of the sphere, and "line" as the set of "points" contained in a great circle. The following theorem which we shall not prove is, in essence, a generalization of this and implies that spherical geometries are related to projective geometries in essentially the same way.

THEOREM 19. *Let P be the set of subgroups of G of the form* $\{a\}$, $a \neq$ o. *Then P becomes a projective geometry if we define "point" to be element of P and "line" to be the set of "points" contained in a subgroup of G of the form* $\{a,b\}$, *where* $\{a\} \neq \{b\}$.[23]

The Covering Postulate implies that G has marked homogeneity of structure.

THEOREM 20. *If* $a,b \neq$ o *then* $\{a\} \cong \{b\}$.

Proof. Suppose $\{a\} \neq \{b\}$, since otherwise the result is trivial. Suppose o $\neq c \subset a + b$. Then $\{c\} \neq \{a\}$ for otherwise $\{a\} \supset c - a \supset b$ and by the corollary to the Covering Postulate $\{a\} = \{b\}$. We have $\{a\} \subset \{a,c\} \subset \{a,b\}$ so that $\{a,c\} = \{a,b\}$ by Theorem 18. By symmetry $\{b,c\} = \{a,b\}$ so that $\{a,c\} = \{b,c\}$. We have, using the Isomorphism Theorem

$$\{a,c\}/\{c\} = \{\,\{a\},\{c\}\,\}/\{c\} \cong \{a\}/(\{a\}.\{c\}) = \{a\}/\text{o} = \{a\}.$$

By symmetry $\{b,c\}/\{c\} \cong \{b\}$. Thus $\{a,c\} = \{b,c\}$ implies $\{a\} \cong \{b\}$.

COROLLARY. If A covers B, A' covers B' then $A/B \cong A'/B'$.

In choosing a set of generators for a group we naturally want to exclude redundant elements. This suggests the following definition of *linear independence.*

DEFINITION 13. M an arbitrary (not necessarily non-void) subset of G is *linearly independent* or *independent* if[24] $\{M \dot{-} x\} \not\supset x$ for each $x \subset M$. In the contrary case we say M is *dependent.* Observe that O is independent but o is dependent.

If M is dependent then the corollary to Theorem 14 implies that some finite subset of M is likewise dependent. The converse is obvious. Thus we may state

THEOREM 21. *M is independent if and only if its finite subsets are independent.*[25]

We now derive a criterion for independence of a finite set very similar to the familiar algebraic one for independence of elements of a linear vector space or an abelian group.

THEOREM 22. *Suppose* $a_1, \ldots, a_n$ *are distinct and* $a_i \neq$ o, $1 \leqq i \leqq n$. Then *they constitute an independent set if and only if*

$$p_1 + \ldots + p_n \supset \text{o}, \qquad p_i \subset \{a_i\}, \qquad (1 \leqq i \leqq n)$$

always implies $p_1 = \ldots = p_n =$ o.

Proof. Suppose $a_1, \ldots, a_n$ distinct and form an independent set, $p_i \subset \{a_i\}$, $1 \leqq i \leqq n$, $p_1 + \ldots + p_n \supset$ o, but one of the p's, say $p_n \neq$ o. Then $p_1 + \ldots + p_{n-1} \supset - p_n \neq$ o and using the corollary to the Covering Postulate we have

$$\{a_1, \ldots, a_{n-1}\} \supset \{-p_n\} = \{a_n\} \supset a_n,$$

[23]Compare Carmichael [5, Chap. XI] where finite projective geometries are represented by systems of subgroups of a certain type of finite abelian group.

[24]We use the symbol $\dot{-}$ to denote set theoretic subtraction.

[25]Compare [12, Theorem 2], [13, Theorem 2.3].

contrary to supposition. Conversely suppose $a_1, \ldots, a_n$ satisfy the given condition, each distinct from o, but they do not form an independent set. It is not restrictive to assume $\{a_1, \ldots, a_{n-1}\} \supset a_n$. Then by Theorem 15, $a_n \subset \{a_1\} + \ldots + \{a_{n-1}\}$ so that there exist $p_i \subset \{a_i\}$, $1 \leqq i \leqq n-1$, satisfying $a_n \subset p_1 + \ldots + p_{n-1}$. Adding $-a_n$ to both members of this relation we obtain $o \subset p_1 + \ldots + p_n$, where $p_n = -a_n \neq o$, contrary to supposition and the proof is complete.

In view of the corollary to the Covering Postulate, if $M \not\supset o$ it is independent if and only if $\{M \dot{-} x\} . \{x\} = o$ for $x \subset M$. This property is now strengthened.

THEOREM 23. *Suppose $M \not\supset o$. Then M is independent if and only if M_1, $M_2 \subset M$ and $M_1.M_2 = O$ always imply $\{M_1\}.\{M_2\} = o$.*[26]

Proof. Supose M independent, $M_1, M_2 \subset M$ and $M_1.M_2 = O$ but $\{M_1\}.\{M_2\} \neq o$. Then $o \neq p \subset \{M_1\}.\{M_2\}$ for some p. Thus $M_1 \neq O$ and by the corollary to Theorem 14 there exist $a_i \subset M_1$, $1 \leqq i \leqq n$, such that $p \subset \{a_1, \ldots, a_n\}$. We may assume that in this relation redundant a's have been deleted. Thus $a_n, p \not\subset \{a_1, \ldots, a_{n-1}\}$.[27] Hence by Theorem 18,

$$\{a_1, \ldots, a_n\} \text{ covers } \{a_1, \ldots, a_{n-1}\} \text{ and }^{28}$$

$$a_n \subset \{a_1, \ldots, a_n\} = \{a_1, \ldots, a_{n-1}, p\} \subset \{a_1, \ldots, a_{n-1}, M_2\} \subset \{M \dot{-} a_n\},$$

contrary to supposition and the necessity of the condition is proved. Its sufficiency is immediate if $M \not\supset o$, since it implies $\{M \dot{-} x\}.\{x\} = o$ for $x \subset M$.

The theory of dimension for subgroups of G is covered by the theory of exchange lattices of MacLane [12] since Theorems 16, 18 imply the subgroups of G form a modular exchange lattice. We have the following results. Each subgroup A of G has a *basis*, that is an independent set of generators. Any two bases of A have the same cardinal number, which we call the *dimension* or *rank* of subgroup A, denoted functionally $d(A)$.[29] If B also is a subgroup of G, $A \supset B$ implies $d(A) \geqq d(B)$. For each subgroup A there exists a *complement*, A', that is a subgroup of G such that $\{A,A'\} = G$, $A.A' = o$. For subgroups A,B of finite dimension we have the *dimension formula*

$$d(\{A,B\}) + d(A.B) = d(A) + d(B).$$

Furthermore if $A \supset B$, the relation A covers B is equivalent to $d(A) = d(B) + 1$. For finite n, a set of n independent elements is contained in a unique subgroup of G of dimension n. Finally, if $d(A) = n$ is finite, any independent set of n elements of A is a basis of A.

Now we consider applications of the theory developed in this section. It certainly applies to the associated multigroup of a spherical geometry since in

[26] Compare [13, Definition 2.1].

[27] If $n = 1$ this expression stands for $\{a_i; i \epsilon O\} = o$.

[28] If $n = 1$ we naturally take the expression $\{a_1, \ldots, a_{n-1}, p\}$ to be $\{p\}$.

[29] In applying MacLane's theory [12, Theorem 6] $d(A)$ would be defined as the cardinal number of a basis of A in the lattice of subgroups of G. Using Theorem 23 this can be shown equivalent to our definition of $d(A)$.

this case $\{a\}$ consists of $a, -a$, o and the Covering Postulate obviously applies. Thus the results of the last paragraph yield a theory of linear independence and dimension for spherical geometries in general and, in view of the discussion in sec. **5** of the geometrical significance of subgroups, cover the theory of alignment and intersection of linear subspaces of a Euclidean spherical geometry.[30]

Next we naturally enquire which classical abelian groups G satisfy the theory, that is to which does the Covering Postulate apply. Suppose then that G is a classical abelian group satisfying the Covering Postulate. Clearly the cyclic subgroups $\{a\}$ of G, where $a \neq$ o, must have prime order, and by Theorem 20 all must have the same order p. The existence of a basis M of G implies that G is a direct sum of cyclic groups of order p. In fact G is the direct sum [2] of the system of cyclic subgroups $\{x\}$, $x \subset M$. For in the first place $G = \{M\} = \{\{x\}; x \subset M\}$, that is G is generated by this system of groups. Secondly the intersection of each group of the system with the group generated by the remaining groups of the system is o, since $\{a\}.\{\{x\}; x \subset M \dot{-} a\} = \{a\}.\{M \dot{-} a\} =$ o by Theorem 23. Conversely any direct sum of (classical) cyclic groups of prime order p satisfies the Covering Postulate since each of its cyclic subgroups, other than the identity, has order p. Thus the theory of multigroups developed so far relates Euclidean spheres and direct sums of (classical) cyclic groups of fixed prime order, which agree in the more general group theoretic properties of the preceding section as well as in the dimensional properties of this section.[31]

7. Separation and factor groups. In this final section we derive conditions that the familiar type of separation theory, which holds for the linear spaces of a Euclidean spherical geometry, be valid in a regular multigroup, and show in effect (Theorem 24, Corollary 4) that this theory holds for any spherical geometry.

In this section G will denote an arbitrary regular multigroup, all other restrictions on G will be stated explicitly. We begin with the precise sense in which the term *separation* will be used in G.

Definition 14. Let A,B be subgroups of G and let X,Y exist such that (1) $A = B \cup X \cup Y$, $B.X = X.Y = B.Y = O$, $X,Y \neq O$ and (2) $x_1,x_2 \subset X$, $y_1,y_2 \subset Y$ imply $x_1 + x_2 \subset X$, $y_1 + y_2 \subset Y$, $(x_1 + y_1).B \neq O$. Then we say *B separates A*.

The theory of separation based on this definition is independent of the dimension of the subgroups involved, which may be finite or infinite. We begin with a basic criterion for separation in terms of factor group.

[30] If G is the associated multigroup of a Euclidean spherical geometry S and A is the subgroup of G associated to linear subspace T of S, then $d(A)$ exceeds by unity the dimension of T as ordinarily defined.

[31] This is related in view of Theorem 19 to results of Carmichael [5] on the representation of finite projective geometries by systems of subgroups of finite groups of the type mentioned. For deep analogies between projective geometries and classical abelian groups see Baer [3].

THEOREM 24. *B separates A if and only if A/B is isomorphic to the group G_3 of order 3 whose addition table is the following:*[32]

	o	p	$-p$
o	o	p	$-p$
p	p	p	o,p,$-p$
$-p$	$-p$	o,p,$-p$	$-p$

Proof. Suppose B separates A, and that X,Y satisfy the conditions of Definition 14. We show

$$A = B \cup X \cup Y \tag{1}$$

is the coset decomposition of A determined by its subgroup B. First we suppose $a \subset X$ and show $X = a + B$. We have $-a \subset Y$. For $-a \subset X$ implies $o \subset a + (-a) \subset X$ contrary to $o \subset B$; and $-a \subset B$ implies $a \subset B$ contrary to $a \subset X$. Hence for arbitrary $x \subset X$ we have $x + (-a) \supset b$ for some $b \subset B$. Thus $x \subset a + b \subset a + B$. Conversely for arbitrary $x \subset a + B$ we have $x \subset a + b$ for some $b \subset B$. If $x \subset Y$ then $b \subset x - a = x + (-a) \subset Y$ contrary to $b \subset B$; if $x \subset B$ then $a \subset x - b \subset B$ contrary to $a \subset X$. Hence $x \subset X$ and $X = a + B$. By symmetry since $-a \subset Y$ we have $Y = (-a)+B$. Thus (1) becomes

$$A = B \cup a + B \cup (-a) + B,$$

and A/B is composed of the cosets $(a)_B$, $(-a)_B$, $(o)_B$. We determine the addition table of A/B. We have $(a)_B \oplus (a)_B = (a + a)_B$. Since $a + a \subset X = (a)_B$, we have $(a + a)_B = (a)_B$ so that $(a)_B \oplus (a)_B = (a)_B$. Likewise $(-a)_B \oplus (-a)_B = (-a)_B$. Since $(a)_B \oplus (-a)_B \supset (o)_B$, we have adding $(a)_B$ to both members, $(a)_B \oplus (-a)_B \supset (a)_B$. Similarly $(a)_B \oplus (-a)_B \supset (-a)_B$. Thus since $(a)_B$, $(-a)_B$, $(o)_B$ are distinct, A/B is easily seen to be isomorphic to G_3.

Conversely suppose A/B isomorphic to G_3. Let $A = B \cup X \cup Y$ be the coset decomposition of A determined by B. Then $B \, . \, X = X \, . \, Y = B \, . \, Y = O$ and $X,Y \neq O$. Since $A/B \cong G_3$ we have in A/B, $X \oplus X = X$, $Y \oplus Y = Y$, $X \oplus Y \supset B$. Thus if $x_1,x_2 \subset X$ we have $X \oplus X = (x_1)_B \oplus (x_2)_B = (x_1+x_2)_B = X$, so that $x_1 + x_2 \subset X$. Similarly $y_1,y_2 \subset Y$ imply $y_1 + y_2 \subset Y$. Finally $X \oplus Y = (x_1)_B \oplus (y_1)_B = (x_1 + y_1)_B \supset B$ so that $x_1 + y_1 \supset b$ for some $b \subset B$. Thus B separates A by definition.

COROLLARY 1. B separates A implies A covers B.

Proof. The hypothesis implies $A/B \cong G_3$. Since G_3 covers its identity, A/B has the same property and the conclusion follows easily.[33]

The typical separation property of linear subspaces of a Euclidean spherical geometry (or of a Euclidean *space* for that matter) suggests the converse of the corollary, namely: *A covers B implies B separates A.*[34] We seek conditions

[32] Observe that G_3 is $\{p\}$ if $p \neq o$ is an element of the associated multigroup of any spherical geometry.

[33] For classical groups compare [1, p. 134, Theorem 14].

[34] A similar property holds in descriptive geometries [15, p. 372, Theorem 6].

that this hold in G. In view of Theorem 24, the desired property is equivalent to: *A covers B implies* $A/B \cong G_3$. Suppose G satisfies the Covering Postulate and to exclude trivial cases suppose $G \neq o$. Then by the corollary to Theorem 20 all factor groups A/B, where A covers B, are isomorphic; thus all are isomorphic to G_3 if (and only if) one is. This one may be chosen arbitrarily. Taking it to be $\{a\}/o = \{a\}$, where $a \neq o$, we have

COROLLARY 2. Let $G \neq o$ satisfy the Covering Postulate. Then A covers B implies B separates A, and if and only if G has a subgroup isomorphic to G_3.

Suppose G satisfies the Covering Postulate and has a subgroup isomorphic to G_3. This subgroup must be of the form $\{a\}$, $a \neq o$, so that by Theorem 20 all subgroups of this form are isomorphic to G_3. But the latter condition implies the Covering Postulate. Thus we may reformulate the sufficiency in Corollary 2 as

COROLLARY 3. Suppose all subgroups of G of the form $\{a\}, a \neq o$, are isomorphic to G_3. Then A covers B implies B separates A.

Finally we observe that $\{a\}$ is isomorphic to G_3 if and only if a has order 3 and $a + a = a$ (see the derivation of (1) in the proof of Theorem 13). Thus we have

COROLLARY 4. Suppose G satisfies the idempotent law and each of its elements, with the exception of o, has order 3. Then A covers B implies B separates A.

In view of Theorem 12 this result yields a separation theory for spherical geometries.

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