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NORMING SETS AND RELATED REMEZ-TYPE INEQUALITIES

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Abstract

The classical Remez inequality ['Sur une propriété des polynomes de Tchebycheff', *Comm. Inst. Sci. Kharkov* **13** (1936), 9–95] bounds the maximum of the absolute value of a real polynomial P of degree d on $[-1, 1]$ through the maximum of its absolute value on any subset $Z \subset [-1, 1]$ of positive Lebesgue measure. Extensions to several variables and to certain sets of Lebesgue measure zero, massive in a much weaker sense, are available (see, for example, Brudnyi and Ganzburg ['On an extremal problem for polynomials of n variables', *Math. USSR Izv.* **37** (1973), 344–355], Yomdin ['Remez-type inequality for discrete sets', *Israel. J. Math.* **186** (2011), 45–60], Brudnyi ['On covering numbers of sublevel sets of analytic functions', *J. Approx. Theory* **162** (2010), 72–93]). Still, given a subset $Z \subset [-1, 1]^n \subset \mathbb{R}^n$, it is not easy to determine whether it is $\mathcal{P}_d(\mathbb{R}^n)$-norming (here $\mathcal{P}_d(\mathbb{R}^n)$ is the space of real polynomials of degree at most d on $\mathbb{R}^n$), that is, satisfies a Remez-type inequality: $\sup_{[-1,1]^n} |P| \le C \sup_Z |P|$ for all $P \in \mathcal{P}_d(\mathbb{R}^n)$ with C independent of P. (Although $\mathcal{P}_d(\mathbb{R}^n)$-norming sets are precisely those not contained in any algebraic hypersurface of degree d in $\mathbb{R}^n$, there are many apparently unrelated reasons for $Z \subset [-1, 1]^n$ to have this property.) In the present paper we study norming sets and related Remez-type inequalities in a general setting of finite-dimensional linear spaces V of continuous functions on $[-1, 1]^n$, remaining in most of the examples in the classical framework. First, we discuss some sufficient conditions for Z to be V-norming, partly known, partly new, restricting ourselves to the simplest nontrivial examples. Next, we extend the Turán–Nazarov inequality for exponential polynomials to several variables, and on this basis prove a new fewnomial Remez-type inequality. Finally, we study the family of optimal constants $N_V(Z)$ in the Remez-type inequalities for V, as the function of the set Z, showing that it is Lipschitz in the Hausdorff metric.

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1. Introduction

The classical Chebyshev inequality (see, for example, [36, pages 67–68]) bounds the maximum of the absolute value of a polynomial P of degree d on $[-1, 1]$ through the

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maximum of its absolute value on an interval $[a, b] \subset [-1, 1]$. In a little-known and hardly available paper [34], Remez generalized the Chebyshev inequality by replacing $[a, b]$ by an arbitrary measurable subset $Z \subset [-1, 1]$:

THEOREM 1.1. *Let $P \in \mathbb{R}[x]$ be a polynomial of degree d. Then, for any measurable $Z \subset [-1, 1]$,*

$$\sup_{[-1,1]} |P| \leq T_d\Big(\frac{4-\mu}{\mu}\Big) \sup_Z |P|, \tag{1.1}$$

where $\mu = \mu_1(Z)$ is the Lebesgue measure of Z and T_d is the Chebyshev polynomial of degree d.

This result has been rediscovered several times (see, for example, [17] and [14, Lemma 2]), but the Remez proof is still the simplest and most elegant.

A multidimensional inequality of this kind is due to Brudnyi and Ganzburg [14]:

THEOREM 1.2. *Let $B \subset \mathbb{R}^n$ be a convex body and $Z \subset B$ be a measurable subset. Then for every real polynomial P on $\mathbb{R}^n$ of degree d,*

$$\sup_B |P| \leq T_d\Big(\frac{1 + (1-\lambda)^{1/n}}{1 - (1-\lambda)^{1/n}}\Big) \sup_Z |P|. \tag{1.2}$$

Here $\lambda = (\mu_n(Z))/(\mu_n(B))$, with μ_n being the Lebesgue measure on $\mathbb{R}^n$.

This inequality is sharp and for $n = 1$ coincides with (1.1).

It is well known that inequalities of the form (1.2) may be true also for some sets Z of Lebesgue measure zero and even for certain finite sets Z; see, for example, [5, 11, 12, 16, 19, 22, 24, 33, 40–42].

There are numerous generalizations of the above inequalities (referred to as Remez-type inequalities) to wider classes of functions. Recently there has been considerable interest in such inequalities in connection with various problems of analysis; see, for example, the introduction to [10] and references therein, and results and references in [13, Ch. 2, 9, 10], [2, 6, 7, 18, 28]. Some of the results below can be extended in an appropriate manner to these classes.

In the present paper we study the problem of characterizing the objects subject to the following definition. Let $V \subset C(Q_1^n)$ be a finite-dimensional subspace of real continuous functions on the closed unit cube $Q_1^n := [0, 1]^n \subset \mathbb{R}^n$.

DEFINITION 1.3. A compact subset $Z \subset Q_1^n$ is said to be V-norming if there exists a constant $C > 0$ such that, for every $f \in V$,

$$\max_{Q_1^n} |f| \leq C \cdot \max_Z |f|. \tag{1.3}$$

The minimum of all such constants C is denoted by $N_V(Z)$ and is called the V-norming constant of Z.

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(The notion originates from the Banach space theory: the family $\{\delta_z\}_{z\in Z} \subset V^*$ of evaluation functionals at points of Z is a norming set for V, that is, $\|f\|_Z := \sup_{z\in Z} |\delta_z(f)|$, $f \in V$, is a norm equivalent to the supremum norm on V.)

Even if $V := \mathcal{P}_d(\mathbb{R}^n)$, the space of real polynomials of degree at most d on $\mathbb{R}^n$, the general problem of characterizing sets Z for which Remez-type inequality (1.3) is valid remains generally open. As we will see below, there is a wide variety of apparently unrelated geometric, algebraic, arithmetic, ... *sufficient conditions* on such Z, which are difficult to present in a coherent way.

In principle, there is a very simple description of V-norming sets:

Proposition 1.4. *A compact subset $Z \subset Q_1^n$ is V-norming (equivalently, $N_V(Z) < \infty$) if and only if the space $V|_Z$ of restrictions of functions in V to Z is of dimension* $\dim V$. *This is, in turn, equivalent to the condition that Z is not contained in any zero-level set $\{x \in Q_1^n : f(x) = 0\}$, $f \in V$.*

Proof. Indeed, (1.3) implies that Z is V-norming if and only if the restriction map $r : V \to V|_Z$, $r(f) := f|_Z$, is an isomorphism of Banach spaces equipped with the corresponding supremum norms. □

However, in general it is not easy to reformulate this condition in an 'effective' way and to provide explicit bounds on $N_V(Z)$ starting with an explicitly given Z.

In the present paper we first discuss (in Section 2) some sufficient conditions for Z to be $\mathcal{P}_d(\mathbb{R}^n)$-norming, partly known, partly new. This includes 'massiveness', algebraic, and topological properties of Z. We intend it as a survey of a variety of results in the area of Remez-type inequalities and a brief introduction to the subject for those who apply such inequalities in other fields of mathematics.

Next, in Section 3 we extend the Turán–Nazarov inequality for exponential polynomials to several variables, and on this basis prove a new fewnomial Remez-type inequality. It is worth mentioning that fewnomials appear in numerous fields of analysis, differential equations, diophantine geometry, probability, cryptography, complexity theory, to name but a few. In many of these fields (for example, probability, cryptography, complexity theory) Remez-type inequalities are in common use. So we expect that our fewnomial version of the Remez inequality (involving only the number of terms, but not the degree) will find important applications.

Finally, in Section 4 we study the behavior of the best constant $N_V(Z)$ in the Remez-type inequality, as a function of the set Z, showing, in particular, that it is Lipschitz in the Hausdorff metric. Lipschitz continuity of the Remez constant has an important corollary: the norming property of a compact set is an open condition in the Hausdorff metric, with an explicit bound on the norming constant of nearby sets. Once more, rather rough (and purely metric) information on sets implies a strong restriction on the behavior of functions (for example, polynomials) on these sets.

2. Examples of Remez sets

As mentioned in the Introduction, the problem of characterizing (in geometric terms) those sets Z for which Remez-type inequality is valid is generally open. In this section we provide a small selection from the wide variety of apparently unrelated geometric, algebraic, arithmetic, ... *sufficient conditions* for Z to be V-norming. We restrict ourselves to the simplest nontrivial examples of each kind, mostly for families V of real polynomials or analytic functions.

2.1. Interpolation system. For finite sets Z it is possible (in principle) to write an explicit answer through the determinants arising in the corresponding interpolation systems.

Indeed, suppose that $\mathcal{F} := \{f_1, \ldots, f_l\} \subset V$, $l := \dim V$, is a basis and that our set Z contains exactly l points $x^1, \ldots, x^l \in Q_1^n$. Assuming that the values of $f = \sum_{i=1}^{l} a_i f_i \in V$ on Z are given, $f(x^j) = v_j$, $j = 1, \ldots, l$, we get the following interpolation system:

$$\sum_{i=1}^{l} a_i f_i(x^j) = v_j, \qquad j = 1, \ldots, l.$$

Considered as a linear system with respect to the unknown variables a_i, this is a multidimensional Vandermonde-like system with the matrix $M_Z = (f_i(x^j))$. It is uniquely solvable if and only if its determinant $\Delta_{\mathcal{F}}(x^1, \ldots, x^l) = \det M_Z$ is nonzero. The V-norming constant $N_V(Z)$ is precisely the norm of the inverse matrix M_Z^{-1}, considered as the operator from the space of functions on Z to the space V, both equipped with the corresponding supremum norms. An easy application of Cramer's rule gives us a bound of the form

$$N_V(Z) \le \frac{(\max_{1\le i\le l}\{\max_{Q_1^n} |f_i|\})^l \cdot l \cdot l!}{|\Delta_{\mathcal{F}}(x^1, \ldots, x^l)|}. \tag{2.1}$$

Thus we have the following proposition.

Proposition 2.1. *A set $Z = \{x^1, \ldots, x^l\} \subset Q_1^n$ is V-norming if and only if $\Delta_{\mathcal{F}}(x^1, \ldots, x^l) \ne 0$. In this case, the upper bound for $N_V(Z)$ is given by (2.1).*

For specific families $\mathcal{F} \subset V$ and sets Z more accurate estimates of $N_V(Z)$ in terms of the characteristics of the matrix M can be produced.

In this extremal setting the Remez-type inequality is essentially equivalent to the stability estimate of the multidimensional interpolation problem; see, for example, [30] and references therein. As an immediate consequence we get the following corollary.

Corollary 2.2. *For each compact V-norming set $Z \subset Q_1^n$, the norming constant $N_V(Z)$ satisfies*

$$N_V(Z) \le \inf_{Z' \subset Z, \#Z' = l} N_V(Z') \le \inf_{\mathcal{F} \subset V} \frac{(\max_{1\le i\le l}\{\max_{Q_1^n} |f_i|\})^l \cdot l \cdot l!}{\sup_{x^1,\ldots,x^l \in Z} |\Delta_{\mathcal{F}}(x^1, \ldots, x^l)|}. \tag{2.2}$$

(Here $\#Z'$ stands for the cardinality of the subset Z' and $\mathcal{F}$ runs over all bases in V.)

The first inequality in (2.2) is almost optimal, as the following result shows.

Proposition 2.3. *For each compact V-norming set $Z \subset Q_1^n$, there exists a subset $Z' \subset Z$ of cardinality l such that*

$$\frac{1}{l} \cdot N_V(Z') \le N_V(Z).$$

Thus,

$$\frac{1}{l} \cdot \inf_{Z' \subset Z, \#Z'=l} N_V(Z') \le N_V(Z) \le \inf_{Z' \subset Z, \#Z'=l} N_V(Z').$$

Proof. For a fixed basis $\mathcal{F}$ in V, let $Z' = \{x_*^1, \ldots, x_*^l\} \subset Z$ be such that

$$|\Delta_{\mathcal{F}}(x_*^1, \ldots, x_*^l)| = \sup_{x^1,\ldots,x^l \in Z} |\Delta_{\mathcal{F}}(x^1, \ldots, x^l)|.$$

(Such points exist because Z is compact and $\Delta_{\mathcal{F}}$ is a continuous function on $(Q_1^n)^l \subset \mathbb{R}^{nl}$.) Since $\dim V|_Z = l$, and evaluations δ_z at points $z \in Z$ determine bounded linear functionals on $V|_Z$, the Hahn–Banach theorem implies easily that $\mathrm{span}\{\delta_z\}_{z \in Z} = (V|_Z)^*$ and, hence, $\Delta_{\mathcal{F}}(x_*^1, \ldots, x_*^l) \ne 0$. Next, we define functions $L_i \in V_{\mathcal{F}}$ by the formulas

$$L_i(x) := \frac{\Delta_{\mathcal{F}}(x_*^1, \ldots, x_*^{i-1}, x, x_*^{i+1}, \ldots, x_*^l)}{\Delta_{\mathcal{F}}(x_*^1, \ldots, x_*^l)}, \quad x \in Q_1^n, 1 \le i \le n.$$

Clearly, they satisfy the properties

$$L_i(x_*^j) = \delta_{ij} \text{ (the Kronecker delta)} \quad \text{and} \quad \max_Z |L_i| \le 1. \tag{2.3}$$

For a function h defined on Z', the Lagrange interpolation is given by the formula

$$(Lh)(x) := \sum_{i=1}^{l} h(x_*^i) L_i(x), \quad x \in Q_1^n. \tag{2.4}$$

From (2.3) and (2.4) we obtain

$$N_V(Z') \le \sum_{i=1}^{l} \max_{Q_1^n} |L_i| \le l \cdot N_V(Z),$$

as required. □

2.2. Sets with algebraically independent coordinates. Assume that $Z = \{x^1, \ldots, x^s\} \subset Q_1^n \subset \mathbb{R}^n$, where $s = \binom{n}{d} (= \dim \mathcal{P}_d(\mathbb{R}^n))$. As was shown above, Z is $\mathcal{P}_d(\mathbb{R}^n)$-norming (that is, satisfies Remez-type inequality (1.3) for real polynomials on $\mathbb{R}^n$ of degree at most d) if and only if the Vandermonde matrix M_Z determined with respect to the basis $\mathcal{M}_{d,n}$ of monomials in $\mathcal{P}_d(\mathbb{R}^n)$ is nondegenerate, that is, its determinant $\Delta_{\mathcal{M}_{d,n}}(x^1, \ldots, x^s) \ne 0$. But $\Delta_{\mathcal{M}_{d,n}}$ is a polynomial with integer coefficients in the coordinates of $x^1, \ldots, x^s$. Therefore, if they are algebraically independent over $\mathbb{Q}$, then $\Delta_{\mathcal{M}_{d,n}}(x^1, \ldots, x^s) \ne 0$. For instance, due to the classical Lindemann–Weierstrass theorem [39], the latter is true if all these coordinates are exponents of algebraic numbers linearly independent over $\mathbb{Q}$. Presumably, in some specific examples (for example, if the coordinates of $x^1, \ldots, x^s$ are Liouville numbers), the norming constant $N_{\mathcal{P}_d(\mathbb{R}^n)}(Z)$ can be estimated explicitly.

2.3. Hausdorff measure and metric entropy. By Theorem 1.2 above, each compact subset $Z \subset Q_1^n$ of positive Lebesgue n-measure is $\mathcal{P}_d(\mathbb{R}^n)$-norming for each natural d and

$$N_{\mathcal{P}_d(\mathbb{R}^n)}(Z) \le T_d\Big(\frac{1+(1-\lambda)^{1/n}}{1-(1-\lambda)^{1/n}}\Big) < \Big(\frac{4n}{\lambda}\Big)^d \tag{2.5}$$

with $\lambda = \mu_n(Z)$. Similarly, if V consists of real analytic functions defined in a neighborhood of Q_1^n and Z is as above, then it is V-norming and

$$N_V(Z) \le E\Big(\frac{1+(1-\lambda)^{1/n}}{1-(1-\lambda)^{1/n}}\Big)^C < \Big(\frac{4n}{\lambda}\Big)^C,$$

where $E(x) := x + \sqrt{x^2-1}$, $|x| \ge 1$, and C is a constant depending on V only. This follows from an inequality similar to (1.2) for real analytic functions; see [8, 9].

Next, it is shown in [12] that compact subsets $Z \subset Q_1^n$ of Hausdorff dimension greater than $n-1$ are $\mathcal{P}_d(\mathbb{R}^n)$-norming for each d.

In [11, 40] some $\mathcal{P}_d(\mathbb{R}^n)$-norming sets are characterized in terms of their metric entropy. Let us recall that the covering number $M(\epsilon, X)$ of a compact metric space X is the minimal number of closed ϵ-balls covering X (see [27]). Below $M(\epsilon, X)$ are defined for compact subsets $X \subset \mathbb{R}^n$ equipped with the induced l^∞ metric, that is, closed ϵ-balls in this metric are intersections with X of closed cubes of side length 2ϵ with centers at points of X.

Definition 2.4. Let Z be a compact subset of Q_1^n. The metric (d,n)-span $\omega_d(Z)$ of Z is defined as

$$\omega_{d,n}(Z) := \sup_{\epsilon>0} \epsilon^n [M(\epsilon, Z) - M_{n,d}(\epsilon)].$$

Here $M_{n,d}(\epsilon) := \sum_{i=0}^{n-1} C_i(n,d)(1/\epsilon)^i$ is a universal polynomial of degree $n-1$ in $1/\epsilon$ whose coefficients are positive numbers related to Vitushkin's bounds for covering numbers of polynomial sublevel sets of degree d; see [23, 38, 40]. The explicit formula for $M_{n,d}$ is given in [40]. In particular,

$$M_{1,d}(\epsilon) = d, \quad M_{2,d}(\epsilon) = (2d-1)^2 + 8d \cdot \Big(\frac{1}{\epsilon}\Big).$$

The following result is established in [40].

Theorem 2.5. *If* $\omega_{d,n}(Z) = \omega > 0$, *then* $N_{\mathcal{P}_d(\mathbb{R}^n)}(Z) < \infty$ *and satisfies*

$$N_{\mathcal{P}_d(\mathbb{R}^n)}(Z) \le T_d\Big(\frac{1+(1-\omega)^{1/n}}{1-(1-\omega)^{1/n}}\Big).$$

Thus, in some cases the Lebesgue measure $\mu_n(Z)$ in Theorem 1.2 can be replaced with $\omega_{d,n}(Z)$.

Note that the metric (d,n)-span $\omega_{d,n}(Z)$ may be positive even for some finite sets. Consider, for example, finite subsets of $\mathbb{R}$.

Corollary 2.6. *A set* $Z = \{x_1, \ldots, x_m\} \subset [-1, 1]$, $x_i \neq x_j$ *for all* $i \neq j$, *is* $\mathcal{P}_d(\mathbb{R})$*-norming if and only if* $m \geq d + 1$. *In this case* $\omega_{d,1}(Z) \geq \delta$ *where* δ *is the minimal distance between distinct* x_i *and* x_j *in* Z, *and* $N_{\mathcal{P}_d(\mathbb{R})}(Z) \leq T_d((2 - \delta)/\delta)$.

Proof. It is enough to take $\epsilon > 0$ tending to δ from the left in the definition of $\omega_{d,1}(Z)$. □

Some other examples are given in [40]. In particular, $\omega_{d,n}(Z) > 0$ for each Z with the entropy (or box) dimension $\dim_e Z$ greater than $n - 1$.

For similar results for spaces V of real analytic functions, see [11].

Still, subsets $Z \subset Q_1^n$ with $\omega_{d,n}(Z) > 0$ are (in a certain discrete sense) 'massive in dimension $n - 1$'. Below we give examples of $\mathcal{P}_d(\mathbb{R}^n)$-norming sets Z in Q_1^n which are contained in certain analytic curves in $\mathbb{R}^n$, and which have $\omega_{d,n}(Z) = 0$. So 'massiveness' is just one of the possible geometric reasons for a set to be $\mathcal{P}_d(\mathbb{R}^n)$-norming.

2.4. Capacity. Another class of 'massive' norming sets consists of the so-called nonpluripolar subsets of $\mathbb{R}^n$. Recall that a compact subset $Z \subset Q_1^n$ is *pluripolar* if there exists a nonidentically $-\infty$ plurisubharmonic function u on $\mathbb{C}^n$ such that $u|_Z \equiv -\infty$. It is known (see, for example, [26]) that a compact subset $Z \subset Q_1^n$ is nonpluripolar if and only if there exists a constant $C > 0$ depending on Z and n only such that Z is $\mathcal{P}_d(\mathbb{R}^n)$-norming for all d and the norming constants satisfy

$$N_{\mathcal{P}_d(\mathbb{R}^n)}(Z) \leq C^d.$$

For instance, inequality (2.5) shows that any compact subset $Z \subset Q_1^n$ of positive Lebesgue n-measure is nonpluripolar. However, nonpluripolar sets in $\mathbb{R}^n$ may be of arbitrary small positive Hausdorff dimension (for example, the n-fold direct product of Cantor sets in $[-1, 1]$ of sufficiently small positive Hausdorff dimensions is nonpluripolar in $\mathbb{R}^n$; see also [29]). If $Z \subset Q_1^n$ is nonpluripolar, then the upper bound for $N_{\mathcal{P}_d(\mathbb{R}^n)}(Z)$ can be expressed also in terms of the capacity $\mathrm{cap}(Z)$ of Z, a positive number defined in one of the following equivalent ways: in terms of the Monge–Ampère measure of Z, the Robin constant of Z, the Chebyshev constant of Z or the transfinite diameter of Z; see [1, 26, 35]. Then, for such Z and all $d \in \mathbb{N}$, one has

$$\ln(N_{\mathcal{P}_d(\mathbb{R}^n)}(Z)) \leq \frac{c \cdot d}{\mathrm{cap}(Z)},$$

where $c > 0$ depends on n only, and $\mathrm{cap}(Z)$ is defined in terms of the Monge–Ampère measure of Z; see [1].

Finally, observe that due to Proposition 1.4 each nonpluripolar compact subset $Z \subset Q_1^n$ is V-norming for every finite-dimensional space of real analytic functions defined in a neighborhood of Q_1^n.

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2.5. Nodal sets of elliptic partial differential equations. We consider a homogeneous elliptic differential equation of the form

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = 0 \tag{2.6}$$

defined in an open neighborhood of Q_1^n, where the coefficients a_{ij} satisfy

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda\|\xi\|_2^2, \quad \text{for any } \xi \in \mathbb{R}^n, x \in Q_1^n,$$

for some positive constant λ. We assume that a_{ij} are Lipschitz and b_i and c are at least bounded. The Lipschitz condition for the leading coefficients is essential. It implies the unique continuation for the operator $\mathcal{L}$. In other words, if a solution u vanishes to an infinite order at a point in Q_1^n, then u is identically zero; see [3].

For any C^2 nonidentically zero solution u in Q_1^n, we define the nodal set

$$\mathcal{N}(u) := \{x \in Q_1^n : u(x) = 0\}.$$

According to [25], the set $\mathcal{N}(u)$ has finite $(n-1)$-dimensional Hausdorff measure. Thus from here and Proposition 1.4 we obtain the following result. *Suppose that V is a finite-dimensional space of C^2 solutions of equation (2.6) and $Z \subset Q_1^n$ is compact of infinite $(n-1)$-dimensional Hausdorff measure. Then Z is V-norming.*

2.6. Algebraic curves of high degree. One apparent algebraic-geometric reason for a set to be $\mathcal{P}_d(\mathbb{R}^n)$-norming is that algebraic sets Z of degree higher than d 'generically' cannot be contained in a hypersurface of degree d. There are plenty of ways in which this general claim can be transformed into a Remez-type inequality. We give here only one simple example, in which the computations are fairly straightforward.

Consider a curve $S \subset Q_1^n \subset \mathbb{R}^n$ given in parametric form by $x = \Psi(t)$ where Ψ is defined by

$$x_1 = t^{d_1}, \quad x_2 = t^{d_2}, \dots, x_n = t^{d_n}, \quad t \in [-1, 1].$$

THEOREM 2.7. *If $d_1 \geq 1, d_2 > dd_1, \dots, d_n > dd_{n-1}$, then S is $\mathcal{P}_d(\mathbb{R}^n)$-norming, and $N_{\mathcal{P}_d(\mathbb{R}^n)}(S) \leq 2^{dd_n} \cdot \binom{d}{n}$.*

PROOF. Let $P(x_1, \dots, x_n) = \sum_{|\alpha|\leq d} a_\alpha x^\alpha$ be a real polynomial of degree d, $\alpha = (\alpha_1, \dots, \alpha_n)$ multi-indices, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. On the curve S we have $x^\alpha = t^{\beta(\alpha)}$, where $\beta(\alpha) = \alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n$.

LEMMA 2.8. *For $\alpha' \neq \alpha''$, we have $\beta(\alpha') \neq \beta(\alpha'')$.*

PROOF. Let $j \leq n$ be the largest index for which $\alpha'_j \neq \alpha''_j$, say, $\alpha'_j < \alpha''_j$. Then $\beta(\alpha') \leq \beta(\alpha'') - d_j + dd_{j-1} < \beta(\alpha'')$, since d_i increase, $|\alpha| \leq d$, and $d_j > dd_{j-1}$. □

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So the monomials of P remain 'separated' in the univariate polynomial $G(t) = P(\Psi(t))$ of degree $\beta((0,\dots,0,d)) = dd_n$, and their coefficients a_α remain the same. If P is bounded by 1 on S, then G is bounded by 1 on $[-1,1]$. We conclude via the Chebyshev inequality that all the coefficients of G do not exceed 2^{dd_n}, and hence the same is true for P. Finally, this implies that P is bounded by $2^{dd_n}\cdot\binom{d}{n}$ on Q_1^n. □

Applying to the univariate polynomial G the classical Remez inequality, or its discrete version given by Theorem 2.5 above, we immediately conclude that the corresponding subsets of the curve S are $\mathcal{P}_d(\mathbb{R}^n)$-norming in $\mathbb{R}^n$.

A more general class of real analytic curves S for which an analog of Theorem 2.7 is valid for certain spaces V of real analytic functions with explicit bounds of norming constant $N_V(S)$ is presented in [10, Theorem 2.3]. The role of the degree of a polynomial there is played by the 'valency' of an analytic function.

2.7. Transcendental surfaces. Each piece of an analytic curve

$$\Gamma := \left\{(x,\phi(x))\in\mathbb{R}^2 : x\in[a,b]\subset[-1,1], \sup_{[a,b]}|\phi|<1\right\}\subset Q_1^2$$

is $\mathcal{P}_d(\mathbb{R}^2)$-norming for all d if ϕ is a transcendental function. For instance, this is true for the curve $y=e^x$, $x\in[-1,0]$. However, it may be a delicate problem to bound explicitly the norming constants for such curves. In this section we formulate one of the results in this direction. Let us recall that an entire function f on $\mathbb{C}^n$ is of order $\rho\geq 0$ if

$$\rho = \limsup_{r\to\infty}\frac{\ln m_f(r)}{\ln r},\quad \text{where } m_f(r) = \ln\Big(\sup_{\|z\|_2\leq r}|f(z)|\Big).$$

If $\rho<\infty$, then f is called of finite order.

Suppose that f is a nonpolynomial entire function on $\mathbb{C}^n$ real on $\mathbb{R}^n$ and such that the hypersurface $\{(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1} : x_{n+1} = f(x_1,\dots,x_n)\}$ intersects Q_1^n by a subset Z of real dimension n. The next result follows from [10, Theorem 2.5] (see also [15, Theorem 1.1] for the particular case $n=1$ and f being of finite positive order).

Theorem 2.9. *There exist a sequence of natural numbers $\{d_j\}$ converging to ∞ (depending on f only) and a sequence of positive numbers $\{\epsilon_j\}$ converging to 0 (depending on f, n, and n-dimensional Hausdorff measure of Z only) such that*

$$\ln(N_{\mathcal{P}_{d_j}(\mathbb{R}^{n+1})}(Z)) \leq d_j^{2+\epsilon_j},\quad \textit{for all } d_j.$$

For $n=1$ this inequality is sharp in the sense that 2 in the exponent on the right-hand side cannot be replaced by a smaller number.

In [10, Theorem 2.8] some sufficient conditions for f are formulated under which the above inequality is valid for all polynomial degrees d. In particular, this is true if

$$f = \sum_{j=1}^m p_j\cdot e^{q_j},\quad \text{where all } p_j, q_j \text{ are real polynomials on } \mathbb{R}^n.$$

Also, [10, Theorem 2.5] deals with some other spaces V of analytic functions and gives upper bounds for their norming constants $N_V(Z)$.

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2.8. Topological conditions. Algebraic geometry provides a wide variety of specific topological properties of real algebraic sets of a given degree d in $\mathbb{R}^n$. Some of them are shared also by all *subsets* of these algebraic sets. So if a compact set $Z \subset Q_1^n$ violates one of these properties, it cannot be contained in an algebraic set of degree d, and hence Z is $\mathcal{P}_d(\mathbb{R}^n)$-norming. Below, we illustrate this with some simple examples.

We consider compact oriented hypersurfaces S in $\mathbb{R}^n$, bounding the corresponding domains D. A sequence $\Sigma = \{S_1, \ldots, S_m\}$ of such hypersurfaces is called 'nested' if $D_1 \subset D_2 \subset \cdots \subset D_m$. We define $\delta(\Sigma)$ as the minimum of the ℓ^∞ distances between the subsequent hypersurfaces in Σ.

Proposition 2.10. *Let $\Sigma = \{S_1, \ldots, S_{d+1}\}$ be a nested sequence of hypersurfaces in the unit cube $Q_1^n \subset \mathbb{R}^n$. Then $S = \bigcup_{j=1}^{d+1} S_j$ is $\mathcal{P}_{2d}(\mathbb{R}^n)$-norming, and $N_{\mathcal{P}_{2d}(\mathbb{R}^n)}(S) \leq T_{2d}((2-\delta)/\delta)$, where $\delta = \delta(\Sigma)$.*

Proof. Let $P \in \mathcal{P}_{2d}(\mathbb{R}^n)$ be a polynomial of degree at most $2d$ bounded by 1 on S. Fix a point $x_0 \in D_1$ and consider the straight line $l \subset \mathbb{R}^n$ passing through x_0 and a point $y \in Q_1^n$ such that $|P(y)| = \max_{Q_1^n} |P|$. We set $l \setminus \{x_0\} := l_1 \sqcup l_2$, where each l_i is an open ray with endpoint x_0. By assumptions, each l_i crosses S at not less than $d+1$ points, and we can fix exactly one point $x_{ij} \in l_i \cap S_j$, $j = 1, \ldots, d+1$, so that

$$\|x_{i1} - x_0\|_\infty < \|x_{i2} - x_0\|_\infty < \cdots < \|x_{id+1} - x_0\|_\infty.$$

Also, the points $x_{i1}, x_{i2}, \ldots, x_{id+1}$ on l_i are separated by ℓ^∞ distance at least $\delta = \delta(\Sigma)$ from one another. This implies that the points x_{11}, x_{ij}, $2 \leq j \leq d+1$, $i = 1, 2$, are separated by the ℓ^∞ distance at least $\delta = \delta(\Sigma)$ from one another as well. Applying Corollary 2.6 to the set Z consisting of these points and to the restriction of P to the interval $l \cap Q_1^n$ (which has ℓ^∞ length at most 2), we get the required bound. □

Let us give an example of more subtle topological restrictions.

Theorem 2.11. *Each compact set $Z \subset Q_1^2 \subset \mathbb{R}^2$ containing 11 ovals outside one another is $\mathcal{P}_6(\mathbb{R}^2)$-norming.*

Proof. If Z were contained in an algebraic curve Y of degree 6, then each oval of Z would be an oval of Y. But by the solution of the first part of Hilbert's 16th problem, Y cannot contain 11 ovals out from one another. Therefore Z cannot be contained in an algebraic curve of degree 6. □

Remark 2.12. It is not easy to give an explicit bound on $N_{\mathcal{P}_6(\mathbb{R}^2)}(Z)$ in geometric terms. Producing such bounds (in this and similar situations) is an important open problem. Its solution would clarify the interconnection of topological and analytical properties of polynomials. It would also clarify some 'rigidity' properties of smooth functions appearing in the framework of the approach developed in [41] (see also references therein). This approach transfers to several variables the classical Rolle lemma and some of its important consequences. In particular, combining the Taylor remainder formula with a bound for the norming constant in Theorem 2.5 would lead to the statement that any C^7 function f on the plane, vanishing on Z as above and satisfying $\max_{x\in Q_1^2}|f(x)| = 1$, must have its seventh derivative larger than a certain explicit constant c_Z (depending on Z only).

This last statement is directly related to the Whitney extension problem for smooth functions, especially in the form considered recently by Fefferman; see [21] and references therein. We expect that Remez-type inequalities can improve our understanding of the geometry of the Whitney extensions, and plan to present some results in this direction separately.

3. Remez-type inequalities for fewnomials

3.1. Turán–Nazarov inequality. This is an important *nonlinear* version of the Remez inequality having numerous applications in analysis, random functions theory, sampling theory, etc. (see [4, 31, 32, 37] and references therein), formulated as follows.

THEOREM 3.1 [31]. *Let $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$, $t \in \mathbb{R}$, be an exponential polynomial, where all $c_k, \lambda_k \in \mathbb{C}$. Let $I \subset \mathbb{R}$ be an interval and $Z \subset I$ be a measurable subset. Then*

$$\sup_I |p| \le e^{\mu_1(I)\cdot \max_{0\le k\le m}|\mathrm{Re}\,\lambda_k|}\cdot\left(\frac{c\mu_1(I)}{\mu_1(Z)}\right)^m\cdot \sup_Z |p|, \tag{3.1}$$

where μ_1 is the Lebesgue measure on $\mathbb{R}$ and $c > 0$ is an absolute constant.

This result was first established by Turán [37] for all λ_k being pure imaginary Gaussian integers and $Z \subset I$ being an interval, and in the general form by Nazarov [31]. Its discrete version, in the spirit of [40], was obtained in [22].

Let us prove the multidimensional version of Theorem 3.1.

THEOREM 3.2. *Let $A \subset \mathbb{R}^n$ be a d-dimensional affine subspace, $B \subset A$ be a convex body in A, and $Z \subset B$ be a Borel subset. Let $p(x) = \sum_{k=0}^m c_k e^{f_k(x)}$, $x \in \mathbb{R}^n$, be an exponential polynomial, where all $c_k \in \mathbb{C}$ and all f_k are complex-valued linear functionals on $\mathbb{R}^n$. Then*

$$\sup_B |p| \le e^{\max_{0\le k\le m}\{\sup_{x,y\in B}\mathrm{Re}\, f_k(x-y)\}}\cdot\left(\frac{cd\mathcal{H}_d(B)}{\mathcal{H}_d(Z)}\right)^m\cdot \sup_Z |p|, \tag{3.2}$$

where $\mathcal{H}_d$ is the Hausdorff d-measure on $\mathbb{R}^n$.

PROOF. Without loss of generality, we assume that Z is closed and $\mathcal{H}_d(Z) > 0$. Let $x_0 \in B$ be such that $\sup_B |p| = |p(x_0)|$. Due to [14, Lemma 3], there exists a ray $l := \{x_0 + t\cdot e \in \mathbb{R}^n : \|e\|_2 = 1, t \in \mathbb{R}_+\}$ with the endpoint x_0 such that

$$\frac{\mathcal{H}_1(B\cap l)}{\mathcal{H}_1(Z\cap l)} \le \frac{d\mathcal{H}_d(B)}{\mathcal{H}_d(Z)}. \tag{3.3}$$

The restriction of p to $l \cap B$ has a form

$$\sum_{k=0}^m (c_k\cdot e^{f_k(x_0)})e^{t\cdot f_k(e)}, \quad 0 \le t \le t_0,$$

where $t_0 > 0$ is such that $x_0 + t_0\cdot e$ belongs to the boundary of B in A.

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Thus according to inequality (3.1),

$$\sup_B |p| = \sup_{B\cap l} |p| \le e^{t_0\cdot \max_{0\le k\le m} |\mathrm{Re}\, f_k(e)|} \cdot \Big(\frac{c\mathcal{H}_1(B\cap l)}{\mathcal{H}_1(Z\cap l)}\Big)^m \cdot \sup_{Z\cap l} |p|.$$

It remains to use (3.3) and note that $t_0 \cdot |\mathrm{Re}\, f_k(e)| \le \sup_{x,y\in B} \mathrm{Re}\, f_k(x-y)$ and $\sup_{Z\cap l} |p| \le \sup_Z |p|$. □

3.2. Fewnomial Remez-type inequality. We deduce from (3.2) the following 'fewnomial' Remez-type inequality. Let $(\mathbb{R}^*_+)^n \subset \mathbb{R}^n$ be the set of points with positive coordinates. If $x_1, \ldots, x_n$ are coordinates on $\mathbb{R}^n$ we introduce a Riemannian metric on $(\mathbb{R}^*_+)^n$ by the formula

$$ds^2 = \frac{dx_1^2}{x_1^2} + \cdots + \frac{dx_n^2}{x_n^2}.$$

The map $e_n : \mathbb{R}^n \to (\mathbb{R}^*_+)^n$, $e_n((x_1, \ldots, x_n)) = (e^{x_1}, \ldots, e^{x_n})$ determines an isometry between $\mathbb{R}^n$ equipped with the Euclidean metric and $(\mathbb{R}^*_+)^n$ equipped with the Riemannian metric introduced above. Thus, the latter is a geodesically complete Riemannian manifold, and geodesics there are images by e_n of straight lines in $\mathbb{R}^n$. In particular, a geodesic segment joining points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $(\mathbb{R}^*_+)^n$ has the form

$$x^{\wedge t} \circ y^{\wedge 1-t} := (x_1^t \cdot y_1^{1-t}, \ldots, x_n^t \cdot y_n^{1-t}), \quad 0 \le t \le 1.$$

One can easily show that such a segment is the usual convex interval joining x and y if and only if there exist a partition of the set $\{1, \ldots, n\}$ into disjoint subsets I, J (one of which may be $\emptyset$) and a positive number λ such that $x_i = y_i$ for all $i \in I$ and $x_j = \lambda y_j$ for all $j \in J$.

A subset $S \subset (\mathbb{R}^*_+)^n$ is called *logarithmically convex* if for each pair of points in S the geodesic segment joining them belongs to S. In other words, S is logarithmically convex if it is the image under e_n of a convex subset of $\mathbb{R}^n$.

In general, logarithmically convex sets are not convex; the class of convex and logarithmically convex sets is relatively small (for instance, it contains d-dimensional rectangles in $(\mathbb{R}^*_+)^n$, $0 \le d \le n$, with edges parallel to coordinate axes).

A submanifold $M \subset (\mathbb{R}^*_+)^n$ is called *affine* if it is the image under e_n of an affine subspace of $\mathbb{R}^n$.

For a compact subset $S \subset (\mathbb{R}^*_+)^n$ and a natural number $d \le n$, we define

$$K_d(S) := \frac{\max_{\{i_1,\ldots,i_d\}\in I_d}\{\max_S x_{i_1}\cdots x_{i_d}\}}{\min_{\{i_1,\ldots,i_d\}\in I_d}\{\min_S x_{i_1}\cdots x_{i_d}\}},$$

where I_d is the family of all d-point subsets of the set $\{1, \ldots, n\}$. Clearly, $K_d(S) = K_d(t \cdot S)$ for each $t > 0$, and if L is a diagonal matrix with positive entries, then $K_n(L(S)) = K_n(S)$. Also,

$$K_d(S) \le K_1(S)^d \quad \text{for all } d. \tag{3.4}$$

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in (\mathbb{R}^*_+)^n$, we set $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $x/y := (x_1/y_1, \ldots, x_n/y_n)$.

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THEOREM 3.3. *Let $M \subset (\mathbb{R}_+^*)^n$ be a d-dimensional affine submanifold, $B \subset M$ be a d-dimensional logarithmically convex compact subset, and $Z \subset B$ be a Borel subset. Let $p(x) = \sum_{k=0}^m c_k x^{\alpha_k}$, $x \in (\mathbb{R}_+^*)^n$, where all $\alpha_k \in \mathbb{R}^n$ and all $c_k \in \mathbb{R}$. Then*

$$\sup_B |p| \leq \max_{0\leq k\leq m}\left\{\sup_{x,y\in B}\left(\frac{x}{y}\right)^{\alpha_k}\right\}\cdot\left(\frac{cdK_d(B)\cdot\mathcal{H}_d(B)}{\mathcal{H}_d(Z)}\right)^m\cdot\sup_Z|p|. \tag{3.5}$$

PROOF. The substitution $x = e_n(u)$, $u \in \mathbb{R}^n$, reduces (3.5) to a particular case of (3.2) for the exponential polynomial $p \circ e_n$. To estimate the ratio of measures $(\mathcal{H}_d(e_n^{-1}(B)))/(\mathcal{H}_d(e_n^{-1}(Z)))$ in the inequality obtained, we express the Hausdorff d-measure of a subset $S \subset M$ as

$$\mathcal{H}_d(S) = \int_{e_n^{-1}(S)} J_d(e_n|_A)(x)\, d\mathcal{H}_d(x),$$

where $A = e_n^{-1}(M) \subset \mathbb{R}^n$ is a d-dimensional affine subspace and $J_d(e_n|_A)$ is the d-Jacobian of the map $e_n|_A$; see [20, Theorem 3.2.3]. Under a suitable affine parameterization $\phi = (\phi_1, \ldots, \phi_d) : \mathbb{R}^d \to A$ of A we obtain

$$\mathcal{H}_d(S) = \int_{(\phi^{-1}\circ e_n^{-1})(S)}\left(\sum_{\{i_1,\ldots,i_d\}\in I_d} a_{i_1,\ldots,i_d}^2\cdot e^{2\phi_{i_1}(x)+\cdots+2\phi_{i_d}(x)}\right)^{1/2} d\mu_d(x),$$

where $a_{i_1,\ldots,i_d}$ is the determinant of the $d\times d$ matrix defined by the $i_1,\ldots,i_d$ rows of the linear part of ϕ. The latter implies, for $S' := (e_n\circ\phi)^{-1}(S)$,

$$\left(\sum_{\{i_1,\ldots,i_d\}\in I_d} a_{i_1,\ldots,i_d}^2\right)^{1/2}\cdot\min_{\{i_1,\ldots,i_d\}\in I_d}\{\min_S x_{i_1}\cdots x_{i_d}\}\cdot\mu_d(S') \leq \mathcal{H}_d(S)$$
$$\leq \left(\sum_{\{i_1,\ldots,i_d\}\in I_d} a_{i_1,\ldots,i_d}^2\right)^{1/2}\cdot\max_{\{i_1,\ldots,i_d\}\in I_d}\{\max_S x_{i_1}\cdots x_{i_d}\}\cdot\mu_d(S').$$

In turn, this yields

$$\frac{\mathcal{H}_d(e_n^{-1}(B))}{\mathcal{H}_d(e_n^{-1}(Z))} = \frac{\mu_d((e_n\circ\phi)^{-1}(B))}{\mu_d((e_n\circ\phi)^{-1}(Z))} \leq \frac{K_d(B)\cdot\mathcal{H}_d(B)}{\mathcal{H}_d(Z)}. \qquad \square$$

$\square$

If $B = \{x = (x_1, \ldots, x_n) \in (\mathbb{R}_+^*)^n : a_i < x_i < b_i, 1 \leq i \leq n\}$ is an n-dimensional rectangle, then since $\mathcal{H}_n(e_n^{-1}(B)) = \ln(b_1/a_1)\cdots\ln(b_n/a_n)$, one obtains easily from the proof that the constant in (3.5) can be replaced by a smaller one, that is, for $Z \subset B$ and p as above, and $a := (a_1, \ldots, a_n)$, $b := (b_1, \ldots, b_n)$,

$$\sup_B|p| \leq \left(\max_{0\leq k\leq m}\left(\frac{b}{a}\right)^{\alpha_k}\right)\cdot\left(\frac{cn\cdot\prod_{i=1}^n\left(b_i\ln\left(\frac{b_i}{a_i}\right)\right)}{\mu_n(Z)}\right)^m\cdot\sup_Z|p|.$$

Also, using inequality (3.4), we obtain the following corollary.

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Corollary 3.4. *Under the assumptions and in the notation of Theorem 3.3,*

$$\sup_B |p| \le K_1(B)^{m+\max_{0\le k\le m}|\alpha_k|} \cdot \left(\frac{cd \cdot \mathcal{H}_d(B)}{\mathcal{H}_d(Z)}\right)^m \cdot \sup_Z |p|. \tag{3.6}$$

An important feature of inequality (3.6) is that while the degree $\deg p := \max_{0\le k\le m}|\alpha_k|$ of p enters the constant of the inequality as the exponent of a certain geometric characteristic of B, the exponent of $(\mathcal{H}_d(B))/(\mathcal{H}_d(Z))$ is m, that is, it depends only on the number of terms of p, and not on its degree.

If $B \subset (\mathbb{R}_+^*)^n$ in Theorem 3.3 is of the form $x_0 + Q_1^n$, $x_0 \in (\mathbb{R}_+^*)^n$, then one easily deduces from inequality (3.5) its discrete version by replacing the Hausdorff n-measure of Z in (3.5) by a fewnomial version of the metric span of Z. It is defined as in Definition 2.4 above, with the polynomial $M_{n,d}$ replaced by its 'fewnomial' version; see [23]. Since the expression is rather cumbersome, we do not state this result explicitly in full generality, restricting ourselves to the particular case of univariate polynomials.

Theorem 3.5. *Let $[a,b] \Subset \mathbb{R}_+^*$ and $Z \subset [a,b]$ be a measurable subset. Suppose that $p(x) = \sum_{k=0}^m c_k x^{n_k}$, $x \in \mathbb{R}$, where $0 \le n_0 < \cdots < n_m$ are nonnegative integers, is a real polynomial of degree at most n_m. Then*

$$\sup_{[a,b]} |p| \le \left(\frac{b}{a}\right)^{n_m} \cdot \left(\frac{c\left(b\ln\left(\frac{b}{a}\right)\right)}{\omega_{m,1}(Z)}\right)^m \sup_Z |p|,$$

where $\omega_{m,1}(Z)$ is the metric span of Z.

Proof. The proof repeats verbatim the proof of Theorem 2.5 above presented in [40]. The only property required in the proof is that, according to the Descartes rule, for every $C \in \mathbb{R}$, the polynomial $p - C$ has at most m positive roots. □

As a simple example of sets Z satisfying fewnomial Remez-type inequalities, we consider a nested sequence of hypersurfaces $\Sigma = \{S_1, \ldots, S_{m+1}\}$, as in Section 2.8 above. Assume that $\Sigma \subset Q_R^n \setminus Q_\rho^n \subset Q_n^1$, where $Q_s^n \subset \mathbb{R}^n$ stands for the closed ℓ^∞ ball (-'cube') of radius s centered at $0 \in \mathbb{R}^n$. As above, let $\delta = \delta(\Sigma)$ denote the minimal ℓ^∞ distance between S_j.

Consider the family of monomials $\mathcal{F} = \{x^{\alpha_k}\}_{0\le k\le m}$, where all $\alpha_k \in \mathbb{Z}_+^n$ and $N := \max_{0\le k\le m}|\alpha_k|$. The linear space $V_{\mathcal{F}}$ generated by $\mathcal{F}$ consists of multivariate polynomials of the form

$$P(x) = \sum_{k=0}^m c_k x^{\alpha_k}, \quad c_k \in \mathbb{R}, x \in \mathbb{R}^n.$$

Theorem 3.6. *The set $S = \bigcup_{j=1}^{m+1} S_j$ is a $V_{\mathcal{F}}$-norming and, for each $P \in V_{\mathcal{F}}$,*

$$\sup_{Q_R^n} |P| \le \left(\frac{R}{\rho}\right)^N \cdot \left(\frac{cR\ln(R/\rho)}{\delta}\right)^m \cdot \sup_S |P|.$$

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PROOF. Consider a ray $l = \{tv, v \in \mathbb{R}^n, \|v\|_\infty = 1, t \geq 0\} \subset \mathbb{R}^n$ with endpoint 0 passing through a point $y \in Q_R^n$ such that $|P(y)| = \max_{Q_R^n} |P|$, and apply Theorem 3.5 to the interval $B_l = \{tv \in l : \rho \leq t \leq R\}$ and the subset $Z_l = S \cap l$. By assumptions, B_l crosses S at not less than $m + 1$ points, and we can fix exactly one point $x_j \in l \cap S_j$, $j = 1, \ldots, m + 1$, so that the points $x_1, x_2, \ldots, x_{m+1}$ on l are ordered and separated by ℓ^∞ distance at least $\delta = \delta(\Sigma)$ from one another. Applying the estimate of the metric span in Corollary 2.6 above, we conclude that $\omega_{m,1}(Z_l) \geq \delta$. This gives us the required bound. □

4. Lipschitz continuity of the norming constant

Let (X, d) be a metric space. A real function f on X is said to belong to the space $\mathrm{Lip}(X)$ if

$$L_f := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty.$$

In this case, L_f is called the *Lipschitz constant of* f.

Let ω be an increasing concave function on $\mathbb{R}_+$, equal to 0 at 0, and such that $\lim_{t\to\infty} \omega(t) = \infty$. One can easily check that $d_\omega(x, y) := \omega(\|x - y\|_\infty)$, $x, y \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$ compatible with the standard topology. Note that $C(Q_1^n) = \bigcup_\omega \mathrm{Lip}_{d_\omega}(Q_1^n)$, where the union is taken over all possible ω. (Indeed, for $f \in C(Q_1^n)$, set $\varphi(t) := t + \omega(t; f), \psi(t) := \int_0^t \varphi^{-1}(s)\, ds$, $t > 0$, where $\omega(\cdot; f) \in C(\mathbb{R}_+)$ is the modulus of continuity of f defined with respect to the metric $\|\cdot\|_\infty$. Then the inequality $\psi(t) \leq \varphi^{-1}(t)$, $t \in [0, 1]$, implies that $f \in \mathrm{Lip}_{d_\omega}(Q_1^n)$ for $\omega := \psi^{-1}$.)

Let $V \subset \mathrm{Lip}_{d_\omega}(Q_1^n)$, for some ω as above, be a finite-dimensional space.

PROPOSITION 4.1. *There exists a constant $M > 0$ such that, for each $f \in V$,*

$$L_f \leq M \sup_{Q_1^n} |f|. \tag{4.1}$$

PROOF. Let $\mathcal{F} = \{f_1, \ldots, f_l\}$, $l := \dim V$, be a basis in V. For $f = \sum_{i=1}^l a_i f_i \in V$, we set $\|f\|_1 := \sum_{i=1}^l |a_i|$. Since V is l-dimensional, $\|\cdot\|_1$ is a norm on V equivalent to the norm induced from $C(Q_1^n)$. In particular, for some $\tilde{c} > 0$ and all $x, y \in Q_1^n$, $x \neq y$, we have

$$\frac{|f(x) - f(y)|}{d_\omega(x, y)} \leq \Big(\max_{1\leq i\leq l} L_{f_i}\Big) \cdot \|f\|_1 \leq \Big(\max_{1\leq i\leq l} L_{f_i}\Big) \cdot \tilde{c} \cdot \sup_{Q_1^n} |f|. \qquad \square$$

The optimal constant M_V in (4.1) is called the *Markov constant of* V. For instance, the classical A. Markov polynomial inequality implies that $M_V = d^2 n$ if $V = \mathcal{P}_d(\mathbb{R}^n)$ and $\omega(t) := t$. In turn, the classical Bernstein inequality implies that $M_V = \pi d n$ if V is the space of trigonometric polynomials of degree d on $\mathbb{R}^n$ of period 2 in each coordinate and $\omega(t) = t$. The constant M_V can be effectively estimated applying the Gram–Schmidt process to a basis $\mathcal{F} = \{f_1, \ldots, f_l\}$ in V considered in a suitable space $L^2(\mu)$ on Q_1^n:

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Proposition 4.2. *Suppose that all* $f_i \in \mathrm{Lip}_{d_\omega}(Q_1^n)$ *and the family* $\mathcal{F}$ *is orthonormal with respect to a regular Borel measure* μ *on* Q_1^n. *Then*

$$M_V \le \Big(\max_{1\le i\le l} L_{f_i}\Big)\cdot \sqrt{l}\cdot \sqrt{\mu(Q_n^1)}.$$

Proof. For $f = \sum_{i=1}^{l} a_i f_i \in V$, we have

$$\|f\|_1 \le \sqrt{l}\cdot\Big(\sum_{i=1}^{l}|a_i|^2\Big)^{1/2} = \sqrt{l}\cdot\Big(\int_{Q_1^n} f^2\,d\mu\Big)^{1/2} \le \sqrt{l}\cdot\sqrt{\mu(Q_n^1)}\cdot\sup_{Q_1^n}|f|.$$

This and the argument of the proof of Proposition 4.1 give the required inequality. □

Let $\mathcal{K}_n$ be the set of all closed subsets of Q_1^n equipped with the Hausdorff metric d_H: if $K_0, K_1 \in \mathcal{K}_n$, then

$$d_H(K_0,K_1) := \max_{i=0,1}\Big\{\sup_{y\in K_i}\inf_{x\in K_{1-i}}\|x-y\|_\infty\Big\}.$$

It is well known that $(\mathcal{K}_n, d_H)$ is a compact metric space. Let us consider $\mathcal{K}_n$ with the metric $d_{\omega H} := \omega\circ d_H$. Then one can check easily that $(\mathcal{K}_n, d_{\omega H})$ is compact as well and that the metrics d_H and $d_{\omega H}$ determine the same topology on $\mathcal{K}_n$.

The main result of this section is the following Lipschitz continuity property of norming constants $N_V(Z)$, $Z\in\mathcal{K}_n$.

Theorem 4.3. *The function* $1/N_V \in \mathrm{Lip}_{d_{\omega H}}(\mathcal{K}_n)$ *and its Lipschitz constant* $L_{1/N_V} \le M_V$. *(Here we define* $1/(N_V(Z)) = 0$ *for* Z *not* V*-norming.)*

Proof. Let $Z_1, Z_2 \in \mathcal{K}_n$ be V-norming sets. Assume without loss of generality that $N_V(Z_1) \ge N_V(Z_2)$. Suppose that $f\in V$ is such that $\sup_{Z_1}|f| = 1$ and $\sup_{Q_1^n}|f| = N_V(Z_1)$. For each $z_2\in Z_2$, we choose $z_1 \in Z_1$ so that $\|z_2 - z_1\|_\infty \le d_H(Z_2, Z_1)$. Then

$$|f(z_2)| \le |f(z_2) - f(z_1)| + |f(z_1)| \le M_V\cdot d_{\omega H}(Z_1,Z_2)\cdot N_V(Z_1) + 1.$$

Thus, by the definition of $N_V(Z_2)$,

$$N_V(Z_1) = \sup_{Q_1^n}|f| \le (M_V\cdot d_{\omega H}(Z_1,Z_2)\cdot N_V(Z_1) + 1)\cdot N_V(Z_2).$$

This implies the required statement:

$$N_V(Z_1) - N_V(Z_2) \le M_V\cdot d_{\omega H}(Z_1,Z_2)\cdot N_V(Z_1)\cdot N_V(Z_2).$$

Further, assume that Z_1 is not V-norming, while Z_2 is. Then due to Proposition 1.4 there exists a function $f\in V$ such that $f|_{Z_1} = 0$ and $\sup_{Q_1^n}|f| = 1$. Arguing as above (with $N_V(Z_1)$ replaced by 1), we obtain

$$1 = \sup_{Q_1^n}|f| \le M_V\cdot d_{\omega H}(Z_1,Z_2)\cdot N_V(Z_2),$$

that is,

$$\frac{1}{N_V(Z_2)} - \frac{1}{N_V(Z_1)} \le M_V\cdot d_{\omega H}(Z_1,Z_2).$$ □

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Corollary 4.4. *If $Z \in \mathcal{K}_n$ is V-norming, then each Y in the open ball of radius $1/(M_V N_V(Z))$ with center at Z is V-norming and*

$$N_V(Y) \leq \frac{N_V(Z)}{1 - M_V \cdot N_V(Z) \cdot d_{\omega H}(Z, Y)}.$$

Proof. From the previous theorem we obtain

$$\frac{1}{N_V(Y)} = \frac{1}{N_V(Z)} - \left(\frac{1}{N_V(Z)} - \frac{1}{N_V(Y)}\right) \geq \frac{1}{N_V(Z)} - M_V \cdot d_{\omega H}(Z, Y) > 0.$$

Passing here to reciprocals, we get the required statement. □

In particular, the set of non-V-norming sets is a closed subset of $\mathcal{K}_n$. In many cases (for example, if V consists of analytic functions) this set is meagre.

Proposition 4.5. *The set of non-V-norming sets is a meagre subset of $\mathcal{K}_n$ if and only if zero loci of functions in $V\backslash\{0\}$ are meagre subsets of Q_1^n.*

Proof. Suppose that zero loci of functions in $V\backslash\{0\}$ are meagre subsets of Q_1^n but the set of non-V-norming sets is not meagre in $\mathcal{K}_n$. Then there exists a non-V-norming set $Z \subset Q_1^n$ and $\varepsilon > 0$ such that each $Y \in \mathcal{K}_n$ with $d_H(Z; Y) \leq \varepsilon$ is non-V-norming as well. In particular, this is valid for $Y = [Z]_\varepsilon$, the closed ε-neighborhood of Z in Q_1^n. Then Proposition 1.4 produces a function $f \in V\backslash\{0\}$ such that $f|_{[Z]_\varepsilon} = 0$. Since $[Z]_\varepsilon$ contains interior points, the zero locus of f is not meagre, a contradiction.

Conversely, suppose that the set of non-V-norming sets is a meagre subset of $\mathcal{K}_n$ but there exists $f \in V\backslash\{0\}$ whose zero locus contains an open ball $Q_\varepsilon^n := \{x \in \mathbb{R}^n : \|x - y\|_\infty < \varepsilon\} \subset Q_1^n$ for some $z \in Q_1^n$, $\varepsilon > 0$. Each $Y \in \mathcal{K}_n$ such that $d_H(Y, \{z\}) < \varepsilon$ is a subset of $Q_\varepsilon^n(z)$ and so is non-V-norming, that is, the set of non-V-norming sets contains an interior point, a contradiction completing the proof. □

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