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Geometric and Potential Theoretic Results on Lie Groups

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Abstract. The main new results in this paper are contained in the geometric Theorems 1 and 2 of Section 0.1 below and they are related to previous results of M. Gromov and of myself (*cf.* [11], [29]). These results are used to prove some general potential theoretic estimates on Lie groups (*cf.* Section 0.3) that are related to my previous work in the area (*cf.* [28], [34]) and to some deep recent work of G. Alexopoulos (*cf.* [3], [4]).

The subject has unfortunately by now become very technical. But an effort has been made to make the introduction at least, of this paper, readable by a non-specialist. My advice to a non-specialist who wants to read this paper is not to be intimidated by unknown words and to read on.

I use throughout the convention that, in a formula, the letters C or c, possibly with suffixes, indicate, possibly different, positive constants that are independent of the important parameters of the formula.

0 Introduction

0.1 Geometric Results

Let G be some real connected Lie group and let Q be its radical (*cf.* [24]), *i.e.*, the largest connected soluble subgroup. Let us recall that G is amenable if and only if G/Q is compact (*cf.* [23]).

Let $e \in \Omega \subset G$ be some compact neighbourhood of the neutral element e of G and let us denote:

$$|g| = |g|_G = \inf\{n\,;\, g \in \Omega^n = \Omega \cdot \Omega \cdots \Omega\}, \quad g \in Q,$$

we can define then a left invariant distance on G by:

$$d(g,h) = |g^{-1}h|.$$

Both $|\quad|$ and $d(\cdot,\cdot)$ depend of course on Ω, but they do so in an inessential way (*cf.* [11], [29]). Let $H \subset G$ be some closed connected subgroup of G then it is known (*cf.* [29]) that there exist constants C such that

$$|h|_G \le C|h|_H \le \exp(C|h|_G + C)\,; \quad h \in H.$$

We shall say that the closed connected subgroup $H \subset G$ is of *strict exponential distortion* if there exist constants s.t.

$$\exp(c|h|_G) \le C|h|_H + C, \quad h \in H.$$

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We shall say that G is an R-group if for every closed connected subgroup $H \subset G$ there exists C such that

$$|h|_H \leq C|h|_G^C + C\ ; \quad h \in H.$$

This is equivalent to the fact that the volume growth of G is polynomial (*cf.* [29], only one of the two implications has been proved there. The other implication is essentially trivial and it will not be used here), *i.e.*, that for every $\Omega \subset G$, compact Nhd of the identity of G, we have:

$$\text{Haar measure } (\Omega^n) \leq Cn^C\ ; \quad n \geq 1.$$

Let us assume now that G is an amenable simply connected Lie group and let $\mathcal{A}$ be the class of all closed normal connected subgroups of G that are of strict exponential distortion in G. We have:

Theorem 1 *Let G be some connected amenable Lie group then $A_0 = \bigcup_{H\in\mathcal{A}} H$, the union of all closed connected normal subgroups of strict exponential distortion, is a closed connected normal subgroup of G of strict exponential distortion in G.*

We can speak thus of the maximal normal closed connected subgroup of strict exponential distortion.

Theorem 2 *Let G be some simply connected amenable Lie group and let $A_0 \subset G$ be the maximal normal closed connected subgroup of strict exponential distortion in G. Then there exists $G_R \subset G$ some closed simply connected subgroup of G such that:*

(i) *G_R is an R-group*
(ii) *$G = A_0 \cdot G_R$ (: set product in the group)*
(iii) *$A_0 \subset (A_0 \cap N) \cdot S$ where N is the nilradical of G and S is a (compact) Levi subgroup (cf. [24]).*

The fact that G is amenable is essential for the above two theorems to hold. Indeed we have:

Theorem 3 *Let G be some simply connected Lie group, let $A \subset G$ be some normal closed connected subgroup of strict exponential distortion in G and let $G_R \subset G$ some connected closed subgroup that is an R-group. Let us assume that $G = A \cdot G_R$. Then G is amenable.*

In Theorems 2 and 3 we have restricted ourselves to simply connected groups for simplicity. Observe, however, that if $\pi\colon \tilde{G} \to G$ is the simply connected cover of G and if $H \subset G$ is of strict exponential distortion, $\pi^{-1}(H) \subset \tilde{G}$ is not necessarily of strict exponential distortion (*e.g.* $\tilde{G} = \mathbb{R}$, $G = \mathbb{T} = H$), *cf.* Section 1.5.

0.2 Random Walks and Diffusion on Lie Groups. The Basic Definitions

Let G be some locally compact group we shall denote by $[G, G]$ its "closed commutator", which can be defined to be the smallest normal closed subgroup $H \subset G$ such that G/H is abelian. When G is a connected Lie group we have

$$G/[G, G] \cong \mathbb{R}^n \times \mathbb{T}^m,$$

(*cf.* [22]) where the compact torus $\mathbb{T}^m$ is uniquely determined. Furthermore, in this case $[G, G]$ is connected (for otherwise we can consider the component of the identity of $[G, G]$). We can consider then

$$\varphi = p \circ \pi \colon G \underset{\pi}{\longrightarrow} G/[G, G] \underset{p}{\longrightarrow} \mathbb{R}^n,$$

and we shall say that $\mu \in \mathbb{P}(G)$, a probability measure on G, is centered if the measure $\check{\mu} = \check{\varphi}(\mu) \in \mathbb{P}(\mathbb{R}^n)$ is centered, *i.e.*, if

$$\int_{\mathbb{R}^n} x \, d\check{\mu}(x) = 0,$$

where the integral is assumed to be absolutely convergent.

Let $\mu \in \mathbb{P}(G)$ be some probability measure on the locally compact group G. I shall consider in this paper the random walk controlled by that measure. This, by definition, is the Markov chain $\{Z_n \in G \,; n \geq 1\}$:

$$\int f(y) \mathbb{P}[Z_n \in dy // Z_{n-1} = x] = f * \mu(x) = \int f(xy^{-1}) \, d\mu(y).$$

We shall say that the above random walk is centered if $\mu \in \mathbb{P}(G)$ is centered. For the above notion to make sense G has to be a connected Lie group. In fact the notion of a centered measure also makes sense for any *connected* locally compact group G for it is then known that we can find $K \subset G$ some compact normal subgroup such that G/K is a Lie group (*cf.* [20]).

Similarly, if G is a connected Lie group we can consider

$$\Delta = \Delta_0 + Y \,; \quad \Delta_0 = -\sum_{j=1}^{n} X_j^2, \quad X_1, \ldots, X_n, Y \in \mathfrak{g} \,; \tag{0.1}$$

some left invariant second order differential operator on G that generates a continuous diffusion $\{z(t) \in G \,; t > 0\}$. The notations and the definitions in this paper will be as [32], [28], [34], [27]. $\mathfrak{g}$ denotes the Lie algebra of G so that $X_1, \ldots, X_n, Y$ are left invariant vector fields on G. We shall assume that the fields $X_1, \ldots, X_n$ satisfy the Hörmander condition, *i.e.*, that they are generators of the Lie algebra $\mathfrak{g}$. We shall denote by

$$dg = d^{\ell} g \,; \quad d^r g = dg^{-1} = m(g) \, dg$$

the left and the right invariant Haar measure on G, where $m(g)$ is the modular function, and we shall denote by $\phi_t(g)$ the convolution kernel of the semigroup $T_t = e^{-t\Delta}$, *i.e.*,

$$T_t f(x) = \int p_t(x, y) f(y) \, dy = \int \phi_t(y^{-1}x) f(y) \, dy \,; \quad f \in C_0^\infty(G).$$

The measure:

$$d\mu_t(x) = \phi_t(x) d^r x \tag{0.2}$$

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is a probability measure on G such that

$$T_t f = f * \mu_t \; ; \quad t > 0, \tag{0.3}$$

and

$$\mathbb{P}_e[z(t) \in dg] = d\mu_t(g). \tag{0.4}$$

If we discretize the time $t = 1, 2, \ldots$ we obtain therefore a random walk controlled by the measure $\mu_1 \in \mathbb{P}(G)$. We shall say that the Laplacian Δ in (0.1) is centered if and only if the above random walk is centered. We say then that the diffusion $\{z(t) \in G \; ; \; t > 0\}$ is centered. It is easy to see (but this fact is not essential here), that if G is amenable, then Δ in (0.1) is centered if and only if $Y \in [\mathfrak{g}, \mathfrak{q}] + \mathfrak{g}_0$ where $\mathfrak{q}$ is the radical of $\mathfrak{g}$ and $\mathfrak{g}_0$ is the Lie algebra of some maximal compact subgroup of G (*cf.* Appendix).

Gaussian Estimates and the Reduction Let G be some connected Lie group, in this paper we shall consider random walks that are controlled by the probability measure

$$d\mu = \varphi(x)dx,$$

where the function $\varphi \in L^1(G \; ; \; dx)$ is assumed to satisfy one of the following additional conditions:

(i) $\varphi(x) \in C_0(G)$, *i.e.*, is continuous and compactly supported. We shall then say that the corresponding random walk is compactly supported.
(ii) $\varphi(x)$ is C^∞ and satisfies the following Gaussian estimates

$$C_- \exp(-c_-|x|^2) \leq \varphi(x) \leq C_+ \exp(-c_+|x|^2) \; ; \quad x \in G. \tag{0.5}$$

For any choice of left invariant fields $X_1, \ldots$ there exists a constant C such that

$$X_1 X_2 \cdots X_k \varphi \leq C \exp(-C|x|^2) \; ; \quad x \in G. \tag{0.6}$$

We then say that μ is a Gaussian measure (*cf.* [27, Ch. 3]).

The importance of this definition lies in the fact that the measure μ_1 attached to a left invariant diffusion as in Section 0.2.(0.4) is Gaussian. The proof of this fact in full generality is not trivial (*cf.* [27, Appendix A.4], [33]). If in (0.1) Δ satisfies $Y = 0$ this fact can be found in [32]. The main difficulty is the proof of the lower Gaussian estimate is (0.5) when $Y \neq 0$. The proof is easy when $Y = \sum X_i + \sum [X_j, X_k]$ where the X_i's are the fields that appear in Δ_0. If we work harder (*cf.* [34]), we can even prove a strict Gaussian estimate for ϕ_t. More precisely for all $\varepsilon > 0$ we can set

$$c_\pm = 1/(4 \pm \varepsilon)$$

in (0.5); but then, of course, $C_\pm$ depend on ε.

The reason why centered Laplacians are important is, that for any Laplacian Δ as in (0.1), we can find

$$\chi \colon G \to \mathbb{R}_+^*,$$

a multiplicative character on G, and a constant $\kappa_0 \geq 0$, such that

$$\chi\Delta\chi^{-1} = \Delta_c + \kappa_0,$$

where Δ_c is centered. A similar reduction to a centered probability measure can be done by replacing $\mu \in \mathbb{P}(G)$ by

$$\nu = \frac{\chi \cdot \mu}{\int_G \chi d\mu}$$

(*cf.* [18]). This reduction requires that $e \in (\operatorname{supp} \mu)^0$). For the rest of this paper all the random walks and all the diffusions that will be considered will be centered, and either compactly supported or Gaussian (*i.e.*, the measure that controls the random walk will be assumed Gaussian).

What corresponds to the convolution kernel in the case of a random walk is the function $\tilde{\phi}_n$ defined by

$$d\mu^{*n} = d(\mu * \mu * \cdots * \mu) = \tilde{\phi}_n(x) d^r x\,; \quad n \geq 1, \quad x \in G \tag{0.7}$$

We shall finally say that the diffusion generated by Δ in (0.1) is symmetric if $Y = 0$. We shall say that a probability measure $\mu \in G$ is symmetric if

$$d\mu(x) = d\mu(x^{-1})$$

we shall then say that the corresponding random walk is symmetric.

The Local Harnack Estimate Let $K \subset G$ be some compact subset. Then there exists $C > 0$ such that

$$\phi_t(k_1 x k_2) \leq C\phi_{t+1}(x)\,; \quad x \in G\,; \quad k_1, k_2 \in K, \quad t \geq 1. \tag{0.8}$$

This for $k_1 = e$ is the standard Harnack estimate for the Heat diffusion kernel (*cf.* [32], [34]). If the diffusion is symmetric we have

$$\phi_t(x^{-1}) = \phi_t(x)m(x),$$

and in general we have

$$\phi_t(x^{-1}) = \phi_t^*(x)m(x),$$

where ϕ_t^* is the heat diffusion kernel that corresponds to the Laplacian

$$\Delta^* = \Delta_0 - Y.$$

From this (0.8) follows in full generality. The analogous "discrete Harnack" estimate that holds for $\tilde{\phi}_n$ is elementary and very easy to prove (*cf.* [34], [26]).

The above definitions and facts are useful because they clarify the general picture to the reader. Many of the above facts however (and in particular the difficult lower estimate (0.5) and the estimate (0.6), (0.8) for ϕ_t) are inessential for the proofs of the main results of this paper. My advice to the reader who is not an expert in the subject, is not to worry unduly about the proofs of the above facts and simply to *read on*.

0.3 Random Walks and Diffusion on Lie Groups. Statement of the Theorems

In this section G will denote some connected amenable real Lie group and

$$(\mathcal{Z})\colon z(t) \in G\,; \quad t > 0 \text{ (resp.: } t = 1, 2, \dots)$$

will denote some centered diffusion (resp: centered random walk that is either compactly supported or Gaussian). We have then:

Upper Gaussian Estimate

$$\mathbb{P}_e[\sup_{0<s<t} |z(s)| \geq m] \leq C \exp\left(-\frac{m^2}{ct}\right); \quad m, t \geq 1. \tag{0.9}$$

This combined with the local Harnack estimate (0.8) yields the estimate

$$\phi_t(g) \leq \exp\left(-\frac{|g|^2}{ct}\right); \quad g \in G, \quad t > 1. \tag{0.10}$$

We have an analogous result for $\tilde{\phi}_n(g)$ (*cf.* [32] for estimates of this kind when the group is unimodular, and *cf.* [21] for an interesting recent development in a slightly different direction).

Lower Gaussian Estimate

$$\mathbb{P}_e[\sup_{0<s<t} |z(s)| \leq m] \geq C \exp\left(-\frac{t}{cm^2}\right); \quad t, m \geq 1. \tag{0.11}$$

Surprisingly enough, the lower estimate (0.11) is essentially an *automatic* consequence of the upper estimate (0.9) and the Makovian property of $z(t)$ $(t > 0)$.

Corollary 1 *Let us assume that G is unimodular and that the diffusion $(\mathcal{Z})$ is symmetric (i.e., $Y = 0$ in (0.1)). Then the heat diffusion kernel $\phi_t(g)$ satisfies:*

$$\phi_{2t}(g) \leq C\phi_t(e) \exp\left(-\frac{|g|^2}{ct}\right); \quad t \geq 1, \quad g \in G.$$

Corollary 2 *The convolution kernels ϕ_t $(t > 0)$ or $\tilde{\phi}_n$ $(n = 1, 2, \dots)$ [cf. (0.2), (0.7)] satisfy:*

$$\phi_t(e) \geq C \exp(-ct^{1/3})\,; \quad \tilde{\phi}_n(e) \geq C \exp(-cn^{1/3})\,; \quad t \geq 1, \quad n = 1, 2, \dots .$$

The Corollary 2 is an immediate consequence of the lower Gaussian estimate. Indeed if we set $m \sim t^{1/3}$ in (0.11) and use the Harnack estimate (0.8) and (0.2), (0.4) we deduce that there exists $x_t \in G$ $(t \geq 1)$ such that:

$$|x_t| \leq ct^{1/3} \quad \phi_t(x_t) \geq C \exp(-ct^{1/3}).$$

By a repeated use ($ct^{1/3}-$ times) of the local Harnack estimate (0.8) we deduce the corollary. If the diffusion or the random walk in the Corollary 2 is assumed to be symmetric, the Corollary 2 is already known [1].

Further Results and Comments

(A) If G is an R-group both the upper and the lower Gaussian estimate are easy consequences of the recent work of G. Alexopoulos [3], [4].

(B) If G is an amenable C-group (*cf.* [34]) then the diffusion kernel satisfies

$$C^{-1}\exp(-ct^{1/3}) \leq \phi_t(e) \leq C\exp(-ct^{1/3})\ ; \quad t \geq 1. \tag{0.12}$$

(C) If G is an amenable NC-group (*cf.* [34]) then the diffusion kernel satisfies

$$C^{-1}t^{-\nu} \leq \phi_t(e) \leq Ct^{-\nu}\ ; \quad t \geq 1,$$

where $\nu = \nu(G\ ;\ \Delta)$ depends on G and Δ and satisfies $\nu(G\ ;\ \Delta) = \nu(G\ ;\ \Delta_0)$. This means that ν is determined as in [34] by G and the driftless (symmetric) Laplacian Δ_0, *i.e.*, by the "quadratic component" of $\Delta = \Delta_0 + Y$ (*cf.* (0.1)). In other words, the presence of a drift term Y does not change the behaviour of $\phi_t(e)$ for an NC-group, as long as Δ is centered. This fact when G is an R-group is contained in the recent results of G. Alexopoulos (*cf.* [3], [4]).

The results in (B) and (C) generalize to centered random walks also. When the diffusion is symmetric (B) and (C) is the content of [34], [35]. With the exception of the lower estimate (0.12) the details of the above generalizations will not be given in this paper. The reason is that once we have the results of [3], [4] the above generalizations are, if lengthy and tedious, essentially routine, and it is only the lower estimate (0.12) that needs new ideas.

0.4 Various Generalizations

General Connected Locally Compact Groups Let us place ourselves now in the more general setting of a general connected amenable locally compact group. If we use the *main approximation theorem* (we have already invoked that theorem to define a centered measure for a general connected locally compact group *cf.* [20]) we can generalize, in a natural and obvious way, the $C - NC$ classification in this more general setting. Furthermore, all the above results generalize automatically (*i.e.*, they can be deduced at once from the corresponding Lie group results) in this setting.

What is perhaps more interesting is that the connectedness is essential for the above circle of ideas to work. Indeed let W be the wreath product [12]

$$W = \Big(\sum_{\gamma \in X} Z_\gamma\Big) \rtimes \Gamma \quad \text{(: semidirect product)},$$

where $X = \Gamma \cong \mathbb{Z}^k$ (for some $k \geq 1$), $Z_\gamma \cong \mathbb{Z}$ and where the action of Γ on X is given by translation. W is of course soluble and finitely generated but, as the interested reader can readily verify for himself, the above estimates break down for W.

Generalizations to Non-Amenable Groups Let now G be a general connected real Lie group, *i.e.*, we no longer assume that G is amenable. Let also $\Delta = \Delta_0 + Y$ be some centered

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Laplacian as in (0.1). To estimate the corresponding diffusion kernel $\phi_t(g)$ (*cf.* (0.2), (0.3)), we shall need two new parameters.

First we shall need to use λ the spectral gap of Δ which can be defined by (*cf.* [27])

$$\lambda = \inf\{(\Delta f, f) \ ; \ f \in C_0^\infty(G), \|f\|_2 = 1\}$$

where both $\| \quad \|_2$ and the scalar product is taken in $L^2(G, d^r g)$.

We shall also need to define the fundamental spherical function φ_0 on G. Let Q be the radical of G and let $S = G/Q$ which is a semisimple real Lie group which we shall assume to be non-compact. Let us factor out the center $S/Z = S_1$ and let $K \subset S_1$ be some maximal compact subgroup. The Harish-Chandra spherical functions $\varphi_\lambda(\lambda \in \mathrm{Hom}_{\mathbb{R}}[\mathfrak{a} \ ; \mathbb{C}]$; where $\mathfrak{n} + \mathfrak{a} + \mathfrak{k}$ is the Iwasawa decompostion of the Lie algebra of S) can then be defined (*cf.* [13], [10]). The fundamental spherical function φ_0 (*i.e.*, $\lambda = 0$, φ_0 is sometimes denoted by Ξ *cf.* [10]) is particularly important. φ_λ define (: unique up to inner automorphisms) functions on G (indeed a different choice of the maximal compact subgroup $K \subset S_1$ will change $\varphi_\lambda(x)$ into $\varphi_\lambda(gxg^{-1})$ for some fixed $g \in G$). In particular, the fundamental spherical function $\varphi_0(g)$ $(g \in G)$ can be uniquely defined (up to inner automorphism) on G. We have then the following generalization of (0.10) and of Corollary 2.

$$\phi_t(g) \leq Ce^{-\lambda t}\varphi_0(g) \exp\left(-\frac{|g|^2}{ct}\right) ; \quad t \geq 1. \tag{0.13}$$

$$\phi_t(e) \geq Ce^{-\lambda t - ct^{1/3}} ; \quad t \geq 1. \tag{0.14}$$

The estimate (0.14) for a driftless Laplacian (*i.e.*, $Y = 0$) is contained in [27] and is clearly unimprovable.

The upper estimate (1.13) is also in some sense unimprovable. At least this is the case when G is semisimple. Indeed the behaviour of $\varphi_0(g)$ is very well understood and is exponential at infinity (*cf.* [13], [10]). In the special cases when an explicit formula is known (*cf.* [9]) the estimate (0.13) is unimprovable up to a polynomial factor of the form

$$t^A(1 + |g|)^B ; \quad A, B \in \mathbb{R}.$$

The methods of the proofs of (0.13), (0.14) are an easy generalization of the methods in this paper. But what is used in addition is the general technology that has been developed in [27]. The proofs of (0.13) and (0.14) will therefore not be given here since this would force us to introduce too much additional background material. These proofs will be given in [31], *cf.* also [5].

The Role of the Amenability Condition Let G be some locally compact group that is *not* amenable then no upper Gaussian estimate such as (0.9) can possibly hold in any form whatsoever. The reason is that when G is not amenable, the random walks considered in (0.9) satisfy almost surely:

$$\lim_{n\to\infty} \frac{|z(n)|_G}{n} > 0$$

(The existence of the limit is guaranteed by the subadditive ergotic theorem). This fact is very well known to the experts and is easy to prove (*cf.* [17], [25]).

1 Proof of the Geometric Results

1.1 Algebraic Considerations

We shall follow very closely the ideas and even the notation of [27, Sections 1.1–1.4].

Let $\mathfrak{g}$ be some real Lie algebra, let $\mathfrak{n} \subset \mathfrak{q} \subset \mathfrak{g}$ be the radical and the nilradical and let us assume that $\mathfrak{g}$ is amenable. This implies that there exists $\mathfrak{s} \subset \mathfrak{g}$ a semisimple subalgebra of compact type such that

$$\mathfrak{g} = \mathfrak{q} + \mathfrak{s}, \quad \mathfrak{q} \cap \mathfrak{s} = \{0\}.$$

This is but the standard Levi decomposition together with the definition of amenability.

As shown in [6], [27], [2], we can then find $\mathfrak{h} \subset \mathfrak{q}$ some nilpotent subalgebra such that:

$$\mathfrak{q} = \mathfrak{n} + \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{s}] = \{0\}.$$

The key to our construction is to consider the real root space decomposition of $\mathfrak{n}$ under the action of $\mathfrak{h}$ (*cf.* [27])

$$\mathfrak{n} = \mathfrak{n}_0 \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k,$$

where $\mathfrak{n}_j \neq \{0\}$ $(1 \leq j \leq k)$ is the sum of all the root space of $\mathfrak{n}$ under the action of $\mathfrak{h}$ with roots that have the same (fixed) non-zero real part. $\mathfrak{n}_0$ is the sum of all the root spaces with pure imaginary roots and $\mathfrak{n}_0$ could be $\{0\}$. We clearly have

$$[\mathfrak{n}_j, \mathfrak{s}] \subset \mathfrak{n}_j, \quad 0 \leq j \leq k. \tag{1.1}$$

It follows that:

$$\mathfrak{g}_R = \mathfrak{n}_0 + \mathfrak{h} + \mathfrak{s}$$

is a subalgebra and since all the roots of the action of $\mathfrak{h}$ on $\mathfrak{n}_0$ are pure imaginary, $\mathfrak{g}_R$ is an R-algebra and every Lie group that corresponds to $\mathfrak{g}_R$ is an R-group (*cf.* [29]). We shall denote also by

$$\mathfrak{a} = \operatorname{Alg}(\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k) \subset \mathfrak{n}$$

the algebra generated by $\mathfrak{n}_j$ $(1 \leq j \leq k)$, so that:

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{g}_R.$$

It follows from (1.1) that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$

$$\mathfrak{a} \lhd \mathfrak{g}\ ; \quad \mathfrak{g}/\mathfrak{a} = \mathfrak{g}_R/\mathfrak{g}_R \cap \mathfrak{a} = \mathfrak{g}^*, \tag{1.2}$$

where $\mathfrak{g}^*$ is an R-algebra.

The algebraic considerations that follow are interesting because they "complete the picture", but they are not essential for the proof of our theorems. We have

$$\mathfrak{a} = (\mathfrak{n}_0 \cap \mathfrak{a}) \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k.$$

This is because

$$\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k \subset \mathfrak{a} \subset \mathfrak{n}.$$

We have:

$$\mathfrak{a} \cap \mathfrak{n}_0 \subset [\mathfrak{a}, \mathfrak{a}].$$

This holds because $\mathfrak{h}$ acts on $\mathfrak{a} \cap \mathfrak{n}_0$ with pure imaginary roots and therefore in the projection

$$\pi\colon \mathfrak{a} \to \mathfrak{a}/[\mathfrak{a}, \mathfrak{a}] = \pi(\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k)$$

$\mathfrak{a} \cap \mathfrak{n}_0$ has to go to zero since the roots of the action of $\mathfrak{h}$ on $\pi(\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k)$ are never pure imaginary. It follows that

$$\mathfrak{a} = (\mathfrak{n}_0 \cap [\mathfrak{a}, \mathfrak{a}]) \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_k,$$

and since

$$[\mathfrak{h}, \mathfrak{n}_j] = \mathfrak{n}_j\,; \quad 1 \leq j \leq k$$

we have

$$\mathfrak{a} \subset [\mathfrak{g}, \mathfrak{g}].$$

Since on the other hand (trivially):

$$[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}\,; \quad [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{a} + [\mathfrak{g}_R, \mathfrak{g}_R],$$

we conclude that

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{a} + [\mathfrak{g}_R, \mathfrak{g}_R].$$

When $\mathfrak{g}$ is an NC-group, we have

$$\mathfrak{n}_0 \cap \mathfrak{a} = \mathfrak{g}_R \cap \mathfrak{a} = \{0\}.$$

A close analysis of the algebra $\mathfrak{n}_0 \cap \mathfrak{a}$ is essential for the good understanding of the geometry of the Lie group (*cf.* [30]) but we shall not go into the details any further.

1.2 A Reduction of the Geometric Theorems

All the notation of the previous section will be preserved, and we shall consider G some simply connected Lie group whose Lie algebra is $\mathfrak{g}$. The analytic subgroup $A \subset G$ that corresponds to $\mathfrak{a}$ is a subgroup of $N \subset G$, the nilradical of G, and is therefore a normal closed simply connected subgroup. The analytic subgroup $G_R \subset G$ that corresponds to $\mathfrak{g}_R$ is also closed and simply connected. Indeed G_R is the semidirect product:

$$G_R = Q_R \bowtie S \subset Q \bowtie S = G,$$

where Q is the radical of G and S is the Levi subgroup of G that corresponds to $\mathfrak{s}$. And Q_R is the subgroup of Q that corresponds to $\mathfrak{n}_0 + \mathfrak{h} \subset \mathfrak{q}$ and is therefore closed and simply connected (*cf.* [24]). It is clear that

$$G = A \cdot G_R. \tag{1.3}$$

We also have:

Basic Proposition *A has strict exponential distortion in G.*

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Both Theorems 1 and 2 follow from the above proposition because it implies that $A \cdot S$ has strict exponential distortion and that $A \cdot S$ contains every closed normal connected subgroup of G of strict exponential distortion. Indeed let $A_1 \subset G$ be such a subgroup and let $\xi \in \mathfrak{q} \cap \mathfrak{a}_1$ where $\mathfrak{a}_1$ is the Lie algebra of A_1. We have then:

$$|\operatorname{Exp}(t\xi)|_G \leq C \log(|t| + C) \;; \quad t \in \mathbb{R}.$$

This implies that $\xi \in \mathfrak{a}$ for otherwise $H = \pi\big(\operatorname{Exp}(t\xi)\big)$ is a nontrivial one parameter subgroup of the radical of the simply connected group G^* that corresponds to $\mathfrak{g}^*$ (*cf.* (1.2); π denotes here the canonical projection $G \to G^*$). Therefore, H is closed and $\cong \mathbb{R}$ and the fact that G^* is an R-group implies that

$$\big|\pi\big(\operatorname{Exp}(t\xi)\big)\big|_{G^*} \geq |t|^c - C \;; \quad t \in \mathbb{R}.$$

This contradiction shows that $\mathfrak{a}_1 \cap \mathfrak{q} \subset \mathfrak{a}$.

Since, on the other hand, A_1 is normal (this is the first point that we use the normality) and since S is compact $A_1 \cdot S = \tilde{A}_1$ is also a closed connected subgroup of strict exponential distortion. From the above considerations applied to $\tilde{A}_1$ it follows therefore that:

$$\text{Connected component } (\tilde{A}_1 \cap Q) \subset A,$$

and since $\tilde{A}_1 = (\tilde{A}_1 \cap Q) \cdot S$ it follows that

$$A_1 \subset \tilde{A}_1 \subset A \cdot S.$$

Observe also that if A_0 is as in Theorem 1, then the above shows that $A_0 \cap N = A_0 \cap Q = A$, and that A_0 is the largest normal connected subgroup of G that lies inside $A \cdot S$. ($A \cdot S$ is not necessarily a normal subgroup.)

1.3 Distances on Nilpotent Groups

Let N be some connected nilpotent group let $\mathfrak{n}$ be the corresponding Lie algebra and let $\xi_1, \dots, \xi_q \in \mathfrak{n}$ be a fixed set of generators of that algebra. We shall denote then:

$$g_i(t) = \operatorname{Exp}(t\xi_i) \in N \;; \quad i = 1, \dots, q, \quad t \in \mathbb{R},$$

and we have:

Proposition *The elements $g_i(t)$ ($i = 1, \dots, q$, $t \in \mathbb{R}$) are generators of N.*

This fact is not trivial but it is a standard consequence of the Backer-Campbell-Hausdorff formula (*cf.* [24, Section 2.15]), and its inverse, the Zassenhaus formula (*cf.* [19, Th. 5.21]). The reader could also consult [15], [32] where an explicit use of the above fact is made.

By an obvious compactness argument or directly by going through the proof we can refine the above proposition by putting "bounds" to the "generation of the element $g \in N$". We have:

Lemma 1.1 *Let N, $\xi_j \in \mathfrak{n}$, $g_j(t)$, $j = 1, \dots, q$ ($t \in \mathbb{R}$) be as above and let $K \subset\subset N$ be some compact subset of N. Then there exists $C \geq 1$ such that for all $g \in K$ we have*

$$g = g_{i_1}(t_1) \cdots g_{i_p}(t_p)$$

for some $0 \leq p \leq C$, $i_1, \dots, i_p = 1, \dots, q$ and $t_j \in \mathbb{R}$, $|t_j| \leq C$, $1 \leq j \leq p$.

The above proposition can be improved even further and we have

Lemma 1.2 *Let N, $\xi_j \in \mathfrak{n}$, $j = 1, \dots, q$ ($t \in \mathbb{R}$) be as above then there exists $C > 0$ such that every element $g \in N$ can be written*

$$g = g_{i_1}(t_1) \cdots g_{i_p}(t_p)$$

for some $0 \leq p \leq C$, $i_1, \dots, i_p = 1, \dots, q$ and

$$|t_i| \leq C|g|_N \; ; \quad 1 \leq i \leq p.$$

It is of course possible to procede directly with the proof of Lemma 1.2, but it is preferable to deduce it from Lemma 1.1 with the use of a simple algebraic device. Indeed let us assume that $\mathfrak{n}$ is nilpotent of order s, *i.e.*, that $[\mathfrak{n}, [\mathfrak{n}, [\mathfrak{n} \cdots] \cdots] \cdots]$ (s-times) $\equiv 0$, and let us consider $\tilde{\mathfrak{n}}$ the free nilpotent algebra of order s generated by the free generator $\zeta_1, \dots, \zeta_q$ (the same number as the ξ's in $\mathfrak{n}$) *cf.* [16]. A unique homomorphism can then be defined by

$$\varphi \colon \tilde{\mathfrak{n}} \to \mathfrak{n} \; ; \quad \varphi(\zeta_j) = \xi_j, \quad 1 \leq j \leq q.$$

This induces a homomorphism

$$\varphi \colon \tilde{N} \to N$$

from $\tilde{N}$, the simply connected Lie group that corresponds to $\tilde{\mathfrak{n}}$, onto the original group N. If we recall the obvious fact that for every $g \in N$ we can find $\tilde{g} \in \tilde{N}$ such that

$$\varphi(\tilde{g}) = g \; ; \quad |\tilde{g}|_{\tilde{N}} \leq C|g|_N,$$

we deduce that it suffices to prove the Lemma 1.2 for the group $\tilde{N}$ and the generators $\zeta_1, \dots, \zeta_q$ of the Lie algebra $\tilde{\mathfrak{n}}$. But for the group $\tilde{N}$ and the generators $\zeta_1, \dots, \zeta_q$, the Lemma 1.2 is an automatic consequence of Lemma 1.1. The reason is that $\tilde{\mathfrak{n}}$ and $\tilde{N}$ have a natural dilation structure. More explicitly, the mapping

$$\tilde{\mathfrak{n}} \ni \zeta_j \to \lambda\zeta_j \in \tilde{\mathfrak{n}} \; ; \quad \lambda > 0, \quad 1 \leq j \leq q$$

extends to a unique algebraic homomorphism

$$\delta_\lambda \colon \tilde{\mathfrak{n}} \to \tilde{\mathfrak{n}} \; ; \quad \delta_\lambda \colon \tilde{N} \to \tilde{N}$$

that has the additional property

$$|\delta_\lambda(g)|_{\tilde{N}} = \lambda|g|_{\tilde{N}},$$

for an appropriate definition of $|\ \ |_{\tilde{N}}$ (*cf.* [8], [32]). An appropriate dilation will therefore bring any $g \in \tilde{N}$ to the compact unit Nhd of $\tilde{N}$

$$K = \{g \in \tilde{N} \; ; |g|_{\tilde{N}} \leq 1\},$$

and the Lemma 1.1 applies. This completes the proof.

If we use the Lemma 1.2, we see that to prove the Basic Proposition it suffices to prove the following

Reduction With all the notations as before there exists some C such that for all $1 \le j \le k$ and for all $\xi \in \mathfrak{n}_j$ with $|\xi| \le 1$ (here we have fixed in any way whatsoever some norm $|\ \ |$ on the vector space $\mathfrak{g}$) we have

$$|\operatorname{Exp}(t\xi)|_Q \le C \log(|t| + C)\ ; \quad t \in \mathbb{R}. \tag{1.4}$$

Towards that we can push the reduction even further: We can fix the $1 \le j \le k$ in (1.4) and reduce the group Q to $Q_1 \subset Q$, which is the group that corresponds to the Lie algebra

$$\mathfrak{q} = \{\mathfrak{n}, \tau\}\ ;$$

where $\tau \in \mathfrak{h}$ is such that $L \in \mathfrak{h}^*$, the real part of the roots of $\mathfrak{n}_j$, satisfy

$$L(\tau) < 0. \tag{1.5}$$

The intuitive reason why (1.4) holds should be quite obvious to any reader that has some experience with geometric considerations:

Indeed what happens is that the vectors ξ and τ determine some "hyperbolic section". In abusive, but intuitively correct terms, if we pretend that we are working in the "two dimensional subspace" of Q that is "determined" by ξ and τ, then, by the negative curvature of that subspace, the length of the geodesic between e and $\operatorname{Exp}(T\xi)$ is $\sim \log T$. This idea will be made formally correct in the next two sections.

1.4 An Elementary Formula

The considerations in this section should be *entirely trivial* to the expert in the subject. Such a reader should refer directly to the formula (1.7) below and move on.

To prove (1.4) we have to start from a general formula on the differential of the product in a Lie group G. Let M be some manifold and let

$$\phi_i \colon M \to G\ ; \quad i = 1, 2$$

be two C^∞ mappings. Let us also denote

$$L_g, R_g \colon G \to G\ ; \quad L_g(x) = gx, \quad R_g(x) = xg, \quad x, g \in G,$$

the left and right translation operator on G. We shall consider the product mapping

$$M \ni m \to F(m) = \phi_1(m) \cdot \phi_2(m) \in G,$$

where $\cdot$ indicates the group product.

First of all if $\phi_i(m) = e$, $i = 1, 2$ it is clear (by the Backer-Campbell-Hausdorff formula among other things) that:

$$d\phi_i \colon T_m M \to \mathfrak{g}\ ; \quad dF = d\phi_1 + d\phi_2.$$

For every fixed $m_0 \in M$ we have therefore

$$\begin{aligned} dF_0 &= d\big(\phi_1^{-1}(m_0)\phi_1(m)\phi_2(m)\phi_2^{-1}(m_0)\big) \\ &= d\big(\phi_1^{-1}(m_0)\phi_1(m)\big) + d\big(\phi_2(m)\phi_2^{-1}(m_0)\big) \\ &= dL_{\phi_1^{-1}} \circ d\phi_1 + dR_{\phi_2^{-1}} \circ d\phi_2 \ ; \quad \forall \xi \in T_{m_0}M. \end{aligned}$$

Where both the left and the right hand side of the above formula are evaluated on $\xi \in T_{m_0}M$. On the other hand:

$$F = \phi_1(m_0)F_0\phi_2(m_0)$$

so that:

$$dF = dL_{\phi_1} \circ dR_{\phi_2} \circ dF_0 = dR_{\phi_2} \circ dL_{\phi_1} \circ dF_0.$$

We finally obtain the formula

$$dF = dR_{\phi_2} \circ d\phi_1 + dL_{\phi_1} \circ d\phi_2 \tag{1.6}$$

where both sides of (1.6) are linear mappings:

$$T_{m_0}M \to T_{\phi_1\phi_2(m_0)}G.$$

Let us perform now:

$$L_{\phi_2^{-1}} \circ L_{\phi_1^{-1}} = L_{(\phi_1\phi_2)^{-1}} \colon G \to G$$

and let us recall that:

$$\operatorname{Ad} g = dR_g \circ dL_{g^{-1}} = dL_{g^{-1}} \circ dR_g \colon \mathfrak{g} \to \mathfrak{g}.$$

These give:

$$dL_{(\phi_1\phi_2)^{-1}} \circ dF = \operatorname{Ad}(\phi_2)[dL_{\phi_1^{-1}} \circ d\phi_1] + dL_{\phi_2^{-1}} \circ d\phi_2.$$

What we have therefore is

Formula for the Differential of the Product If we identify T_gG with $\mathfrak{g}$ by left translation, we have:

$$d(\phi_1 \cdot \phi_2) = \operatorname{Ad}(\phi_2) \circ d\phi_1 + d\phi_2 \tag{1.7}$$

We shall make use of the above formula in the special case when $M \subset \mathbb{R}$ and

$$\phi_1(t) = \operatorname{Exp}\big(\varphi_1(t)\xi\big), \quad \phi_2(t) = \operatorname{Exp}\big(\varphi_2(t)\zeta\big) \ ; \quad \varphi_i(t) \in \mathbb{R},\ i = 1, 2,$$

where $\xi, \zeta \in \mathfrak{g}$. We obtain then:

$$d(\phi_1 \cdot \phi_2) = \varphi_2'(t)\zeta + \varphi_1'(t)\exp\big[\operatorname{ad}\big(\varphi_2(t)\zeta\big)\big]\xi \tag{1.8}$$

because:

$$\operatorname{Ad}(\operatorname{Exp} x)y = \exp[\operatorname{ad} x]y.$$

1.5 Proof of (1.4)

Let all the notations be as in 1.2 (especially (1.4), (1.5)). We shall prove the estimate (1.4) by considering an appropriate mapping

$$f\colon [0,T] \to Q\,; \qquad f(s) = \operatorname{Exp}\bigl(\varphi_1(s)\xi\bigr)\operatorname{Exp}\bigl(\varphi_2(s)\tau\bigr), \tag{1.9}$$

for an appropriate choice of $\varphi_1, \varphi_2 \in C^\infty[0,T]$ that satisfy:

$$\varphi_1(s) = 0, \quad 0 \le s \le 10\,; \qquad \varphi_1(s) \equiv T, \quad T-10 \le s \le T,$$

$$\varphi_2(0) = \varphi_2(T) = 0\,; \qquad f(0) = e, \quad f(T) = \operatorname{Exp}(T\xi) \in Q, \tag{1.10}$$

$$|\varphi_2'| \le C\log(|T| + C). \tag{1.11}$$

The only additional condition that we have to impose on φ_1, φ_2 is that φ_1 is "almost linear" in $[10, T-10]$ and that

$$\varphi_2(s) = C\log(|T| + C)\,; \qquad s \in [1, T-1]. \tag{1.12}$$

This is clearly compatible with (1.10), (1.11).

If we apply (1.8) and take into account (1.5) we see that for an appropriate choice of the constants in (1.12) we have:

$$\left| df\left(\frac{\partial}{\partial s}\right)\right|_{TQ} \le CT^{-10}\,; \qquad s \in [1, T-1], \tag{1.13}$$

where TQ is as usual the tangent bundle of Q. From this it follows that the length (in the appropriate Riemannian structure on Q) of the path (1.9) is less than $C\log(|T| + C)$ and (1.4) follows.

The proof of (1.13) is an easy exercise in the computation of the norm of a positive power of a matrix of the form

$$T = \begin{pmatrix} t_{11} & & & \\ & t_{22} & & t_{ij} \\ & & \ddots & \\ 0 & & & t_{nn} \end{pmatrix}$$

where $|t_{ii}| = \lambda > 0$, $|t_{ij}| \le M$ $(i, j = 1, 2, \dots, n)$. Indeed we have

$$\|T^m\| \le \|\lambda^m (I + T^*)^m\| \le CM^C\lambda^{m-C}\,; \qquad m = 1, 2, \dots$$

where C only depends on the dimension of the space and where $T^* = (t^*_{ij})$ with

$$t^*_{ij} = 0, i \ge j \quad t^*_{ij} = \lambda^{-1}|t_{ij}|, \quad i < j.$$

(The details are obvious *cf.* [27, Section 1.5].)

The Case when G is Non-Simply Connected Let us denote by $\bar{A}$, $\overline{G}_R \subset G$ the closure of the analytic subgroups A and G_R of Section 1.2. Then $\bar{A} \subset N$ and therefore $\bar{A} = A \cdot M$ where M is a compact central subgroup of N. By adapting the above argument we can easily show that $\bar{A}$ has strict exponential distortion in G. The group $\overline{G}_R$ on the other hand is clearly an R-group (*cf.* [14], [29]). One can also adapt the argument in Section 1.2 (*cf.* [14] and the Appendix) and deduce that any normal closed connected subgroup of strict exponential distortion is contained in $\bar{A}G_0$, where G_0 is a compact subgroup. This easily implies Theorem 1 for general connected Lie groups and even for connected locally compact groups (*cf.* [20]).

1.6 The Section

The results of this section are geometrically interesting because they complete the general picture and because they can also be used to give an alternative approach to the potential theoretic results of the next section. But since no direct use of these results will be made in this paper, the proofs, which are essentially routine, will not be given.

Let G be some connected R-Lie group and let $N \subset G$ be its nilradical and let $K \subset N$ be some normal (in G) closed connected subgroup. We say that $\Sigma \subset G$, some Borel subset, is a section of K if $G = K \cdot \Sigma$ in a (1-1) way. More precisely, we have

$$\sigma\colon G/K \to \Sigma \subset G \xrightarrow[\pi]{} G/K, \tag{1.14}$$

where π is the canonical projection and σ is a Borel mapping such that

$$\pi \circ \sigma = \mathrm{Id}(G/K). \tag{1.15}$$

For any $x_1, \dots, x_n \in G/K$ we clearly have

$$\sigma(x_1) \cdot \sigma(x_2) \cdots \sigma(x_n) = \varphi(x_1, \dots, x_n) \cdot \sigma(x_1 \cdots x_n)\ ; \quad n \geq 1, \tag{1.16}$$

where the "$\cdot$" indicate the group product in G/K or in G as the case might be, and $\varphi = \varphi_n : C^\infty(G/K \times \cdots \times G/K\ ; K)$, *i.e.*, $\varphi \in K$.

It is not entirely trivial but easy to show that Σ can be chosen so that for all $c > 0$ (or equivalently for all $c > 0$ small enough) there exists $C > 0$ such that

$$|\varphi(x_1, \dots, x_n)|_K \leq Cn^C + C\ ; \quad n \geq 1, \quad |x_j| \leq c, \quad 1 \leq j \leq n.$$

One can easily deduce that Σ is a "*polynomial section*" in the following sense:

Definition of a Polynomial Section Let G be some connected Lie group and let $K \subset G$ be some normal closed subgroup and $\Sigma \subset G$ some section of K in G in the sense of (1.14), (1.15) and let $\varphi = \varphi_n$ be defined as in (1.16). We say that Σ is a polynomial section if there exist constants $C > 0$ such that:

$$|\sigma(x)|_G \leq C|x|_{G/K} + C\ ; \quad x \in G/K,$$
$$|\varphi(x_1, \dots, x_n)|_K \leq (\Sigma|x_j|_{G/K} + n)^C + C\ ; \quad n \geq 1, \quad x_1, \dots, x_n \in G/K.$$

When G is not an R-group, the existance of polynomial sections to one of its normal subgroups is a rare event indeed. But by Theorem 2 of Section 0.1 we can deduce from the R-group case the following:

Proposition *Let G be some simply connected amenable Lie group and let $A \subset G$ be its maximal normal subgroup of strict exponential distortion. Then A admits a polynomial section in G.*

Similarly, when G is not simply connected, then the subgroup $\bar{A}$ of Section 1.5 admits a polynomial section.

1.7 The Basic Geometric Transformation

This section is a preparation to the potential theoretic results that will be given in the second part of this paper. Let G be some connected amenable Lie group (simply connected or not). Let $\tilde{G} \to G$ be the simply connected cover of G and let

$$A \subset \tilde{G}\,; \quad G_R \subset \tilde{G}$$

the two subgroups that correspond to $\tilde{G}$ as in the Basic Proposition of Section 1.2. G_R acts by inner automorphism and we can therefore consider the semidirect product and the homomorphisms

$$\overline{G} = A \bowtie G_R \to \tilde{G} \to G. \tag{1.17}$$

By the argument of Sections 1.2–1.4 (repeated in $\overline{G}$) it follows that A is of strict exponential distortion in $\overline{G}$. In this context the important thing to observe is that both for the upper and for the lower Gaussian estimates (0.9), (0.11) it suffices to give the proof for the group $\overline{G}$ and that these estimates "go through the factorization $\pi\colon \overline{G} \to G$ of (1.17)".

Indeed if $d\mu = \varphi(x)dx$, $\varphi(x) \in C_0^\infty$, is some compactly supported symmetric probability measure and if Δ is some symmetric Laplacian on G, we can lift them to a centered measure $\overline{\mu} \in \mathbb{P}(\overline{G})$ and to a centered Laplacian $\overline{\Delta}$ such that

$$\check{\pi}(\overline{\mu}) = \mu\,; \quad d\pi(\overline{\Delta}) = \Delta.$$

When G is already simply connected one can, just as easily, lift a general centered Laplacian (*cf.* Appendix). It is a trifle less obvious to lift general Gaussian measures but we shall not worry about this difficulty here.

To work on the group $\overline{G}$ we shall use the following:

Basic Transformations Let

$$x_j = a_j g_j \in \overline{G}\,; \quad a_j \in A, \quad g_j \in G_R, \quad j = 1, 2, \dots .$$

The "partial products" can then be transformed as follows:

$$s_n = x_1 \cdots x_n = (a_1 a_2^{g_1} a_3^{g_1 g_2} \cdots a_n^{g_1 g_2 \cdots g_{n-1}}) g_1 \cdots g_n = a_n^* g_1 \cdots g_n, \tag{1.18}$$

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where we use the standard notation for any Lie group G:

$$x^y = yxy^{-1}\,; \quad x, y \in G.$$

In this context it is very easy to verify that if $x \in N$ lies in some normal closed subgroup of G we have:

$$|x^y|_N \le C|x|_N \exp(C|y|_G + C),$$

with constants that are independent of x and y but depend of course on N and G. [: it suffices to write $x = x_1 \cdots x_p$, $(x_j \in N)$ $y = y_1 \cdots y_q$, with $|x_j|_N \le 1$, $|y_j|_G \le 1$, $p \sim |x|_N$, $q \sim |y|_G$].

What makes the transformation (1.18) important is that if we start from the information

$$|x_j|_{\overline{G}} \le K\,; \quad |g_1 \cdots g_j|_{G_R} \le M\,; \quad 1 \le j \le n, \tag{1.19}$$

for some $K, M \ge 1$, then we can conclude that:

$$|a_j|_A \le \exp(CK)\,; \quad |a_j^{g_1\cdots g_{j-1}}|_A \le \exp(CK + CM)\,; \quad |a_n^*|_A \le \exp(CK + CM + \log n)$$

with constants that are independent of K and M. By the strict exponential distortion of A in $\overline{G}$, we conclude therefore that the hypothesis (1.19) implies that

$$|s_n|_{\overline{G}} \le C(M + K + \log n). \tag{1.20}$$

The estimate (1.20) will be basic for our potential theoretic applications.

The Proof of Theorem 3 Let $G = Q \bowtie (S_1 \oplus \cdots \oplus S_k)$; $S_j =$ simple group, $(1 \le j \le k)$ be the Levi decomposition of G. We shall show that all the factors S_j $(1 \le j \le k)$ are compact and this will prove the Theorem. Indeed for each $1 \le j \le k$, S_j is not distorted in G, *i.e.*, $|x|_G \approx |x|_{S_j}$, $x \in S_j$. If, on the other hand, S_1, say, is not compact, we must have $S_1 \subset A$ because G/A is an R-group. This gives a contradiction and completes the proof.

2 The Potential Theoretic Results

2.1 The Gaussian Estimates for the Kernel

Let X be some metric space assigned with the distance $d(\cdot,\cdot)$. We shall consider time homogeneous Markov chains on X

$$\mathcal{Z} : z(n) \in X\,; \quad n = 1, 2, \ldots,$$

and we shall say that $\mathcal{Z}$ satisfies the Gaussian estimate if:

$$\mathbb{P}_x\big[d\big(z(n), x\big) \ge m\big] \le C \exp\left(-\frac{m^2}{Cn}\right); \quad x \in X, \quad m \ge 1, \quad n = 1, 2, \ldots. \tag{2.1}$$

For constants $C > 0$ that are independent of x, m and n.

Let now dx be some measure on X we can then define the kernel of the chain with respect to dx by

$$p_n(x,y)dy = \mathbb{P}_x[z(n) \in dy]\ ; \quad x, y \in X, \quad n = 1, 2, \dots .$$

We shall say that the chain $\mathcal{Z}$ is reversible with respect to dx if the kernel $p_n(x,y)$ is bi-markovian, *i.e.*, if

$$\int p_n(x,y)\,dx \le 1\ ; \quad n \ge 1, \quad y \in X.$$

We can define then the reversed chain

$$(\mathcal{Z}^*) \qquad z^*(n) \in X\ ; \quad n = 1, 2, \dots,$$

by

$$\mathbb{P}_x[z^*(n) \in dy] = p_n^*(x,y)dy\ ; \quad n = 1, 2, \dots, x, \quad y \in X,$$

where by definition:

$$p_n^*(x,y) = p_n(y,x)\ ; \quad x, y \in X, \quad n = 1, 2, \dots .$$

The first thing to observe is that the Gaussian estimate (2.1) implies (in fact is equivalent to) a functional estimate that we shall now describe:

Let $\varphi \in L^\infty(X)$ be some bounded function that satisfies the following Lipschitz condition

$$\text{(Lip)} \qquad |\varphi(x) - \varphi(y)| \le d(x,y)\ ; \quad x, y \in X.$$

Let us also consider the submarkovian operator

$$T_n f(x) = \int p_n(x,y) f(y)\,dy\ ; \quad x \in X, \quad n = 1, 2, \dots\ ; \quad f \in L^\infty(X).$$

Then the chain $\mathcal{Z}$ satisfies the upper Gaussian estimate (2.1) if and only if

$$\text{(2.2)} \qquad \|e^{-\lambda\varphi} T_n e^{\lambda\varphi}\|_{\infty\to\infty} \le \exp(cn\lambda^2 + c)\ ; \quad \lambda \in \mathbb{R}\ ; \quad n = 1, 2, \dots$$

with constants $c > 0$ that are independent of λ, n or φ.

The operator above is the composition of the following three operators (and in that order):

1) Multiplication by the L^∞ function $e^{\lambda\varphi}$
2) Action by the submarkovian operator T_n
3) Multiplication by the L^∞ function $e^{-\lambda\varphi}$.

$\| \quad \|_{p\to q}$ indicates, of course, as usual the $L^p(X\ ; dx) \to L^q(X, dx)$ norm. Observe also that the kernel of $e^{-\lambda\varphi} T_n e^{\lambda\varphi}$ with respect to dx is

$$p_n(x,y) \exp\bigl[\lambda\bigl(\varphi(y) - \varphi(x)\bigr)\bigr].$$

By choosing some bounded regularization of the distance

$$\varphi(y) = \min[m, d(x,y)], \quad y \in X$$

(with fixed x), we see that (2.2) implies that

$$e^{\lambda m} \int_{d(x,y)\geq m} p_n(x,y)\,dy \leq C\exp(cn\lambda^2)\,; \quad m \geq 1, \quad n = 1,2,\ldots, \quad \lambda \in \mathbb{R}, \quad x \in X.$$

Optimizing over λ we see that the Gaussian estimate (2.1) follows.

Conversely, let us assume that the Gaussian estimate (2.1) holds for the chain $\mathcal{Z}$, let us fix $x \in X$ and let

$$\text{(2.3)} \quad F_n(\xi) = \mathbb{P}_x\big[d\big(z(n),x\big) \geq \xi\big] \leq c\exp\left(-\frac{\xi^2}{cn}\right); \quad x \in X, \quad n = 1,2,\ldots, \quad \xi \geq 0.$$

We clearly have:

$$|(e^{-\lambda\varphi}T_n e^{\lambda\varphi})f(x)| \leq \int_X \exp\big[\lambda\big(\varphi(y)-\varphi(x)\big)\big]\mathbb{P}_x[z(n)\in dy]$$

for any $f \in L^\infty$ with $\|f\|_\infty \leq 1$. We can therefore bound:

$$\|e^{-\lambda\varphi}T_n e^{\lambda\varphi}\|_{\infty\to\infty} \leq -\int_0^\infty \exp(\lambda\xi)dF_n(\xi) \leq C\lambda\int_0^\infty \exp\left(\lambda\xi - \frac{\xi^2}{cn}\right)d\xi\,; \quad \lambda > 0,$$

where the second estimate follows from (2.3) and a simple integration by parts. The estimate (2.2) follows upon performing the above integration. We also have the following simple but important:

Proposition *Let us assume that the chain $\mathcal{Z}$ is reversible with respect to the measure dx on X and that both $\mathcal{Z}$ and $\mathcal{Z}^*$ satisfy the Gaussian estimate (2.1). The kernel $p_n(x,y)$ satisfies then:*

(2.4)
$$p_{2n}(x,y) \leq C\sup_z\{p_n(x,z)p_n(z,y)\}^{1/2}\exp\left(-\frac{d^2(x,y)}{cn}\right); \quad x,y \in X, \quad n = 1,2,\ldots.$$

The proof is very easy. Set

$$P_n(x,y) = \sup_z\{p_n(x,z)p_n(z,y)\}^{1/2}\,; \quad q_n^{(\lambda)}(x,y) = p_n(x,y)\exp\lambda\big(\varphi(x)-\varphi(y)\big)$$

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where $n = 1, 2, \ldots, \lambda \in \mathbb{R}$ and φ satisfies (Lip). We have

$$p_{2n}(x, y) = \int p_n(x, z) p_n(z, y)\, dz \le P_n(x, y) \int p_n^{1/2}(x, z) p_n^{1/2}(z, y)\, dz\ ;$$
$$x, y \in X, \quad n = 1, 2, \ldots .$$

$$\begin{aligned} q_{2n}^{(\lambda/2)}(x, y) &\le P_n(x, y) \int \big(q_n^{(\lambda)}(x, z) q_n^{(\lambda)}(z, y)\big)^{1/2}\, dz \\ &\le P_n(x, y) \left(\int q_n^{(\lambda)}(x, z)\, dz\right)^{1/2} \left(\int q_n^{(\lambda)}(z, x)\, dz\right)^{1/2} \\ &\le P_n(x, y) \|e^{\lambda\varphi} T_n e^{-\lambda\varphi}\|_{\infty\to\infty}^{1/2} \|e^{\lambda\varphi} T_n^* e^{-\lambda\varphi}\|_{\infty\to\infty}^{1/2} \\ &\le C P_n(x, y) e^{c\lambda^2 n}\ ; \quad x, y \in X, \quad n = 1, 2, \ldots, \quad \lambda \in \mathbb{R}. \end{aligned}$$

This gives of course

$$p_{2n}(x, y) \le C P_n(x, y) e^{c\lambda^2 n - c\lambda(\varphi(x) - \varphi(y))}$$

and upon optimizing over φ and λ we obtain (2.4).

The above proposition has a partial converse. Indeed if we assume that for some $D > 0$ we have:

$$V_x(n) = \int_{d(x,z)\le n} dz \le cn^D\ ; \quad p_n(x, y) \le Cn^{-D/2} \exp\left(-\frac{d^2(x, y)}{cn}\right)\ ; \tag{2.5}$$
$$n = 1, 2, \ldots, \quad x, y \in X.$$

Then an easy integration argument shows that the estimate (2.1) holds for $\mathcal{Z}$. In fact, the conditions (2.5) implies the following stronger estimate

$$\mathbb{P}_x\big[\sup_{0\le j\le n} d\big(z(j), x\big) \ge m\big] \le C \exp\left(-\frac{m^2}{cn}\right). \tag{2.6}$$

The argument to prove this is well known (*cf.* [34, III, Section 1.3]).

The estimates (2.5) for a centered compactly supported random walk on a Lie group G of polynomial volume growth are contained in the recent work of G. Alexopoulos [3], [4]. Similarly, in [3], [4] one also finds the continuous time version of these estimates for centered diffusions on such a group G:

$$p_t(x, y) \le ct^{-D/2} \exp\left(-\frac{d^2(x, y)}{ct}\right)\ ; \quad t \ge 1, \quad x, y \in G. \tag{2.7}$$

If the diffusion is symmetric the estimate (2.7) is well known (*cf.* [32]).

For a centered diffusion on a Lie group of polynomial volume growth G we also have the continuous time analogue of (2.6):

$$\mathbb{P}_x\big[\sup_{0<s<t} d\big(z(t), x\big) \ge m\big] \le C \exp\left(-\frac{m^2}{ct}\right)\ ; \quad x \in G\ ; \quad m, t \ge 1. \tag{2.8}$$

This continuous time variant (2.8), if the diffusion is symmetric, can be proved exactly as (2.6) (because the small time behaviour of $p_t(x, y)$ is known *cf.* [34]). For a general centered diffusion the small time behaviour creates an extra difficulty that has to be dealt with (*cf.* [27, Appendix A.14]).

Let us now quite generally make the assumption that the Markov chain $(\mathcal{Z})$ is (strictly) Markovian, *i.e.*, that

$$T_n 1 = 1, \quad n \geq 1.$$

Then clearly if $(\mathcal{Z})$ satisfies the Gaussian estimate (2.1), we can find some $c_0 > 0$ such that

(2.9)
$$\mathbb{P}_x\big[d\big(z(t), x\big) \leq c_0 t^{1/2}\big] = 1 - \mathbb{P}_x\big[d\big(z(t), x\big) > c_0 t^{1/2}\big] \geq c_0^{-1}\,; \quad x \in X, \quad t = 1, 2, \ldots.$$

And if the stronger estimate (2.6) holds, we can even find $c_1 \geq c_0$ and $\varepsilon_0 > 0$ such that

(2.10)
$$\mathbb{P}_x\big[d\big(z(t), x\big) \leq c_0 t^{1/2}\,; \quad \sup_{0 \leq s \leq t} d\big(z(s), x\big) \leq c_1 t^{1/2}\big] \geq \varepsilon_0\,; \quad x \in X, \quad t = 1, 2, \ldots.$$

In other words, we can bound from below the probability that at time t we are in the ball of radius $c_0 t^{1/2}$ without having exited from the (say much larger : $c_1 \gg c_0$) ball of radius $c_1 t^{1/2}$. The analogous facts clearly also hold for a continuous time parameter.

If we are given now $t, m \geq 1$ with $t/m^2 \gg 1$, we can iterate the above operation $[t/m^2]$-times. [More explicitly, the operation goes as follows: at time $t \sim m^2$ we return to the ball of radius $R_1 \sim m$ without having exited from the much larger ball $R_2 \gg R_1$ (but still with $R_2 \sim m$).] The Markov property then applies and we obtain automatically the lower Gaussian estimate:

$$\mathbb{P}_x\big[\sup_{0<s<t} d\big(z(s), x\big) \leq m\big] \geq C \exp\Big(-\frac{t}{m^2}\Big)\,; \quad m \geq 1, \quad t = 1, 2, \ldots.$$

The analogous result for a continuous time parameter clearly also holds.

Observe, finally, that the above considerations easily adapt to time inhomogeneous Markov chains and to time inhomogeneous random walks on groups. But we shall not elaborate.

2.2 The Proof of the Upper and the Lower Gaussian Estimate on an Amenable Lie Group

As already pointed out, in the special case when the Markov chains are symmetric or when G is simply connected, for the proofs of the upper and lower Gaussian estimates (0.9), (0.11) it suffices to restrict our attention to a Lie group G of the form:

(2.11)
$$G = A \bowtie G_R,$$

where G_R is an R-group and where A is strictly exponentially distorted in G. We shall not, *a priori*, make the assumption that G is amenable but this *a posteriori* (as already pointed out at the end of Section 0) will turn out to be the case.

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The Compactly Supported Random Walks on G Let G be as in (2.11) and let $\pi\colon G \to G_R$ be the canonical projection. Let $\{z(n) \in G\,;\, n = 1, 2, \dots\}$ be the random walk generated by the centered measure $\mu \in \mathbb{P}(G)$ that satisfies:

$$\operatorname{supp}\mu \subset B_e(K) = \{g \in G\,;\, |g|_G \le K\}.$$

We shall then consider the projected random walk

$$\{\dot z(n) = \pi\big(z(n)\big) \in G_R\,;\, n \ge 1\}.$$

The basic transformations in Section 1.6 (esp. (1.20)) shows then that:

$$\mathbb{P}_e[\sup_{0\le j\le n} |z(j)|_G \le C(m + K + \log n) // \sup_{0\le j\le n} |\dot z(j)|_{G_R} \le m] = 1. \tag{2.12}$$

The upper Gaussian estimate for $z(n)$ follows therefore from the corresponding result for $\dot z(n)$. The term $\log n$ is irrelevant here since in the upper Gaussian estimate we may assume that $m \ge c\sqrt{n}$. This completes the proof of (0.9) and (0.11) for compactly supported random walks.

The Gaussian Random Walks Let us now assume that the random walk $\{z(n) \in G\,;\, n \ge 1\}$ is centered and Gaussian, *e.g.* generated by some centered Laplacian (*cf.* Section 0.2). We shall then modify the conditional probability (2.12) of the basic transformation of Section 1.6 as follows:

(2.13)
$$\begin{aligned}\mathbb{P}_e[\sup_{0< j\le n} |z(j)|_G \le C(m + K + \log n) // \sup_{0< j\le n} |\dot z(j)|_{G_R} \le m, \sup_{0< j\le n} |z^{-1}(j) z(j-1)|_G \le K]\\ = 1\,;\quad m, K \ge 1, \quad n = 1, 2, \dots .\end{aligned}$$

By the upper Gaussian estimate (0.5) we have on the other hand:

$$\begin{aligned}&\mathbb{P}_e[\sup_{0< j\le n} |z^{-1}(j) z(j-1)|_G \ge K]\\ &\qquad \le \exp(-cK^2 + \log n)\,;\quad K \ge 1, \quad n = 1, 2, \dots .\end{aligned} \tag{2.14}$$

Now in the estimate (0.9) we may assume without loss of generality that $m \ge c\sqrt{n}$. Let us then set $K = \varepsilon_0 m$ in (2.14), (2.13) for an appropriate $\varepsilon_0 > 0$. We see then that the estimate (0.9) for the random walk $\{z(n) \in G\}$ follows by the corresponding result for $\{\dot z(n) \in G_R\}$ (*cf.* (2.8)).

Having proved the upper Gaussian estimate (0.9), the estimate (0.11) follows automatically by the considerations of the previous section because the left invariant diffusion semigroup $e^{-t\Delta}$ is clearly Markovian [The function $u(t, g) = e^{-t\Delta}1(g)$ is independent of $g \in G$]. The Corollary 1 follows from the Proposition in Section 2.1.

The General Case The proof in the general case, when G is not simply connected, is almost identical but we have to argue on the product $G = \bar A \cdot \Sigma$ where Σ is a polynomial Borel section of the subgroup $\bar{A}$ (: notations of Sections 1.5, 1.6) rather than with the semidirect product (2.11). The complications caused by the fact that Σ is not a subgroup are easily dealt with by the properties of the polynomial section. The details will be left to the interested reader who should start by adapting the Basic Transformation of Section 1.7.

Final Remark For the proof of our Gaussian estimates we have made use of the results of G. Alexopoulos [3], [4] for groups of polynomial growth. It is worth pointing out however that only the upper Gaussian estimates of [3] and [4] were used and that these upper Gaussian estimates can be obtained directly without the deep and difficult Harnack estimate that Alexopoulos proves in [3], [4].

The reason why these upper Gaussian estimates are easier to prove is that in a group of polynomial volume growth G, for fixed $x, y \in G$, $d(x,y) = \delta \gg 1$, one can find an appropriate φ that satisfies (Lip) as above, and which has the additional properties

$$\|e^{-\lambda\varphi}e^{-t\Delta}e^{\lambda\varphi}\|_{2\to 2} \leq Ce^{ct\left(\lambda^2+\frac{\lambda}{\delta}\right)}\ ; \quad \lambda > 0$$
$$|\varphi(x)-\varphi(y)| \sim \delta = d(x,y)$$

where Δ is some fixed *centered* Laplacian. From this, by the uniform estimates of the kernel (*cf.* [33, Th. 3]) and standard methods (*cf.* [7]) one deduces (2.6) and (2.7) for G and Δ.

The construction of the above function φ is easy if G is nilpotent. For a general G it takes some doing. But still it is easier, I feel, than the global Harnack estimate of Alexopoulos.

Appendix

Let G be some connected Lie group and let $\mathcal{H}$ be the class of all closed normal subgroups H such that $G/H \cong \mathbb{R}^a$ for some $a \geq 0$. It is clear that the intersection of two subgroups in $\mathcal{H}$ also lies in $\mathcal{H}$ and that the component of the identity of a subgroup in $\mathcal{H}$ also lies in $\mathcal{H}$. There exists therefore a minimal element in $\mathcal{H}$ that is connected and which we shall denote by H_0. Let $\mathfrak{h}_0 \lhd \mathfrak{g}$ be the Lie algebra of H_0 which is an ideal in $\mathfrak{g}$, the Lie algebra of G. It is clear that the Laplacian $\Delta = \Delta_0 + Y$ in (0.1) is centered if and only if $Y \in \mathfrak{h}_0$.

Let us denote by $\mathfrak{q} \lhd \mathfrak{g}$ the radical of $\mathfrak{g}$ and let $\mathfrak{s}$ be some Levi subgroup. The obvious fact that

$$\mathfrak{h}_0 \supset [\mathfrak{g},\mathfrak{g}] + \mathfrak{s} = [\mathfrak{g},\mathfrak{q}] + \mathfrak{s}$$

clearly implies that when G is simply connected then

$$\mathfrak{h}_0 = [\mathfrak{g},\mathfrak{q}] + \mathfrak{s}.$$

When G is not simply connected the situation is more complicated. Let $C \subset Q$ be some maximal connected compact analytic subgroup of the radical of G (*cf.* [14]), then $H_0 \supset C$ and therefore, if we denote by $\mathfrak{c}$ the Lie algebra of C we have:

$$\mathfrak{h}_0 \supset [\mathfrak{q},\mathfrak{g}] + \mathfrak{c} + \mathfrak{s} = \mathfrak{h}_1.$$

$\mathfrak{h}_1$ is an ideal in $\mathfrak{g}$. Let us denote by H_1 the analytic subgroup generated by $\mathfrak{h}_1$ and let $\overline{\mathfrak{h}}_1$ be the Lie algebra of $\overline{H}_1$ the closure of H_1. We then have:

$$(*) \qquad \mathfrak{h}_0 = \overline{\mathfrak{h}}_1.$$

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Indeed, first of all, if we quotient by the maximal central torus, we can easily reduce the proof of (*) to the case when the nilradical is simply connected. Let then S be some Levi subgroup of G and let

$$\theta\colon \tilde{G} = Q \bowtie S \to G$$

be the canonical mapping. Then $\tilde{H}_1$ the analytic subgroup generated by $\mathfrak{h}_1$ in $\tilde{G}$ *is* closed because $[\mathfrak{q}, \mathfrak{g}]$ lies in the nilradical; furthermore, $\tilde{G}/\tilde{H}_1 \cong \mathbb{R}^a$ $(a \geq 0)$ (*cf.* [14, XV, 3.7]). It follows that $G/\overline{H}_1 \cong \tilde{G}/\theta^{-1}(\overline{H}_1) \cong \mathbb{R}^b$ $(0 \leq b \leq a)$ therefore $\overline{H}_1 \supset H_0$. This proves (*) because the opposite inclusion is obvious. It follows in particular that when G is amenable and S is compact (*i.e.*, in the context of this paper) we have $\mathfrak{h}_0 = \mathfrak{h}_1$.

The choice of the Levi subalgebra $\mathfrak{s}$ and of the maximal compact subalgebra $\mathfrak{c} \subset \mathfrak{q}$ in the definition of $\mathfrak{h}_1$ was arbitrary. When G is amenable, by taking conjugates, it follows that we can choose both S and C to lie inside some maximal compact subgroup $G_0 \subset G$. Since on the other hand clearly the Lie algebra $\mathfrak{g}_0$ of G_0 satisfies $\mathfrak{g}_0 \subset \mathfrak{h}_0$ we finally obtain:

$$\mathfrak{h}_0 = [\mathfrak{g}, \mathfrak{q}] + \mathfrak{g}_0.$$

Another geometric fact that we used (in Section 1.5) is this. Let G be some R-group and let $\mathfrak{h} \subset \mathrm{Lie}(G)$ be some ideal that has the following property: There exists $C > 0$ s.t. for all $n \geq 1$ and all $\gamma_1, \dots, \gamma_n \in \mathfrak{h}$, $|\gamma_j| \leq 1$, $1 \leq j \leq n$, we have:

$$(**) \qquad |\operatorname{Exp}(\gamma_1) \cdots \operatorname{Exp}(\gamma_n)|_G \leq C \log(n + C).$$

Then $\operatorname{Exp}(\mathfrak{h})$ is precompact in G. The first step towards the proof is to show that $\mathfrak{h} + \mathfrak{s}$ satisfies (**) for any Levi subalgebra $\mathfrak{s}$. This reduces the problem to the case when G is soluble and $\mathfrak{h}$ is a subalgebra. The details will be left to the reader.

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