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Abstract

Let K be a number field of degree n, and let d_K be its discriminant. Then, under the Artin conjecture, the generalized Riemann hypothesis and a certain zero-density hypothesis, we show that the upper and lower bounds of the logarithmic derivatives of Artin L-functions attached to K at $s=1$ are $\log\log|d_K|$ and $-(n-1)\log\log|d_K|$, respectively. Unconditionally, we show that there are infinitely many number fields with the extreme logarithmic derivatives; they are families of number fields whose Galois closures have the Galois group C_n for $n=2,3,4,6$, D_n for $n=3,4,5$, S_4 or A_5.

1. Introduction

Let K be a number field of degree n with discriminant d_K, and let $\zeta_K(s)$ be the Dedekind zeta function of K, with the following Laurent expansion at $s=1$:

$$\zeta_K(s)=c_{-1}(s-1)^{-1}+c_0+c_1(s-1)+c_2(s-1)^2+\cdots.$$

Then $\gamma_K=c_0/c_{-1}$ is called the Euler–Kronecker constant of K. If $K=\mathbb{Q}$, then $\gamma_{\mathbb{Q}}$ is just the Euler constant $\gamma=0.57721566\ldots$. When K is an imaginary quadratic field, the Kronecker limit formula expresses γ_K in terms of special values of the Dedekind η-function. It was Ihara who began a systematic study of the Euler–Kronecker constant; we refer to [Iha06] for details.

We can see that $\zeta_K'/\zeta_K(s)=-1/(s-1)+\gamma_K+(s-1)h(s)$, for some holomorphic function $h(s)$ at $s=1$. Let $\widehat{K}$ be the Galois closure of K. Then we have $\zeta_K(s)=\zeta(s)L(s,\rho)$ for some $(n-1)$-dimensional complex representation ρ of the Galois group $\mathrm{Gal}(\widehat{K}/\mathbb{Q})$. So

$$\gamma_K=\gamma+\frac{L'}{L}(1,\rho). \qquad (1.1)$$

This leads to a study of the logarithmic derivative of $L(s,\rho)$ at $s=1$. In [Iha06], Ihara found an upper bound and a lower bound for γ_K under the generalized Riemann hypothesis (GRH). The main terms in his upper and lower bounds under the GRH are

$$2\log\log\sqrt{|d_K|}, \quad -2(n-1)\log\left(\frac{\log\sqrt{|d_K|}}{n-1}\right).$$

In [IMS09, p. 260], the authors remarked that in the case of Dirichlet characters, the coefficient 2 can be replaced by $1+o(1)$. In § 10 we prove that under the Artin conjecture, the GRH and a certain zero-density hypothesis (Conjecture 10.4), the upper and lower

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bounds are

$$\log\log|d_K| + O(\log\log\log|d_K|), \quad -(n-1)\log\log|d_K| + O(\log\log\log|d_K|),$$

respectively.

When K is a quadratic field $\mathbb{Q}(\sqrt{d})$, the value $\frac{L'}{L}(1,\chi_d)$ determines $\gamma_{\mathbb{Q}(\sqrt{d})}$, where χ_d is the Dirichlet character attached to the quadratic field $\mathbb{Q}(\sqrt{d})$. Recently, Mourtada and Murty [MM] showed unconditionally that there are infinitely many Dirichlet L-functions of quadratic character whose logarithmic derivatives at $s=1$ have large values. Specifically, there are infinitely many fundamental discriminants d such that $|-\frac{L'}{L}(1,\chi_d)| \gg \log\log|d|$. This implies that $|\gamma_{\mathbb{Q}(\sqrt{d})}| \gg \log\log|d|$ for infinitely many quadratic fields $\mathbb{Q}(\sqrt{d})$. We realized that the techniques we used to obtain extreme values of $L(1,\rho)$ in [Cho, CKa, CKb] can be applied to generalize Mourtada and Murty's result to arbitrary Artin L-functions.

Let $f(x,t)$ be an irreducible polynomial of degree n, giving rise to a regular Galois extension over $\mathbb{Q}(t)$ with Galois group G. Let K_t be the number field obtained by adjoining a root of $f(x,t)$ with a specialization $t\in\mathbb{Z}$. We study the Artin L-functions $L(s,\rho,t)=\zeta_{K_t}(s)/\zeta(s)$. Under several assumptions (the strong Artin conjecture and Assumptions 4.1 and 4.2), we show in § 4 that there are infinitely many number fields such that

$$\frac{L'}{L}(1,\rho,t) \geqslant \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|) \tag{1.2}$$

and infinitely many number fields such that

$$\frac{L'}{L}(1,\rho,t) \leqslant -(n-1)\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|). \tag{1.3}$$

The idea is to calculate the average of the logarithmic derivatives over certain sets. Here the estimate in Proposition 4.5 is crucial. It follows from Weil's theorem on the number of rational points of algebraic curves over finite fields. In a continuation paper [CKc], we prove a refinement of Weil's theorem (Theorem 5.4).

In §§ 6–8, we exhibit several examples from [Cho, CKa, CKb] for which the strong Artin conjecture and Assumptions 4.1 and 4.2 hold. These are number fields whose Galois closures have Galois groups C_n for $n=2,3,4,6$, D_n for $n=3,4,5$, S_4 or A_5. So in these cases, (1.2) and (1.3) are true unconditionally. In the case of A_4, we were not able to verify Assumption 4.2; so in this case (1.2) and (1.3) are true modulo Assumption 4.2. See Remark 6.3 for the case of C_5.

2. Regular extensions and their Galois representations

A finite extension E of the rational function field $\mathbb{Q}(t)$ is said to be regular if $\overline{\mathbb{Q}}\cap E=\mathbb{Q}$. Suppose that $f(x,t)$ is an irreducible polynomial of degree n and gives rise to a regular Galois extension over $\mathbb{Q}(t)$ with Galois group G. Let K_t be a field obtained by adjoining to $\mathbb{Q}$ a root of $f(x,t)$ with a specialization $t\in\mathbb{Z}$, and let $\widehat{K_t}$ be the Galois closure of K_t. Let C be any conjugacy class of G. Serre made the following important observation regarding the distribution of Frobenius elements in a regular Galois extension [Ser08, p. 45].

Theorem 2.1. *There is a constant $c_f>0$, depending on f, such that for any prime $p\geqslant c_f$ there is a $t_C\in\mathbb{Z}$ such that for any $t\equiv t_C \pmod p$ with $\mathrm{Gal}(\widehat{K_t}/\mathbb{Q})\simeq G$, p is unramified in $\widehat{K_t}/\mathbb{Q}$ and $\mathrm{Frob}_p\in C$.*

Recall the following regular inverse Galois problem.

Conjecture 2.2. Given a finite group G, there exists a polynomial $f(x,t) \in \mathbb{Z}[t][x]$ such that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ has the Galois group G and is a regular extension.

In [Ser08, p. 35], Serre called this the Gal_T property. It is known that the Gal_T property is satisfied for abelian groups, dihedral groups, A_n and S_n.

With the specialization $t \in \mathbb{Z}$, let $n = [K_t : \mathbb{Q}]$. We can consider the following refinement. Let $\mathfrak{K}(n, G, r_1, r_2)$ be the set of number fields of degree n with signature (r_1, r_2) whose normal closures have G as their Galois group (if they exist).

Conjecture 2.3. Given a finite group G, there exists a polynomial $f(x,t) \in \mathbb{Z}[t][x]$ such that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ has the Galois group G and is a regular extension, and there exists a certain infinite subset $S \subset \mathbb{Z}$ such that $K_t \in \mathfrak{K}(n, G, r_1, r_2)$ for $t \in S$.

In the explicit examples of §§ 6–9, we specify S.

3. Approximation of $\frac{L'}{L}(1,\rho)$ and the zero-free region

Proposition 3.1 [Dai06]. *Let $F/\mathbb{Q}$ be a finite Galois extension and let ρ be an n-dimensional complex representation of* $\mathrm{Gal}(F/\mathbb{Q})$ *with conductor N. Let $6/7 < \alpha < 1$. If $L(s,\rho)$ is entire and free from zeros in the rectangle $[\alpha, 1] \times [-(\log N)^2, (\log N)^2]$, and if N is sufficiently large, then*

$$\frac{1}{2\pi i}\int_{(2)} \frac{L'}{L}(s+u,\rho)\Gamma(s)x^s\,ds - \frac{L'}{L}(u,\rho) \ll_n \frac{x^2}{(1-\alpha)^2\sqrt{N}} + \frac{(\log N)^2}{(1-\alpha)^3 x^{(1-\alpha)/8}}$$

for $1 \leqslant u \leqslant 3/2$ and $x \geqslant 1$.

Set $u = 1$ in Proposition 3.1. Then

$$-\frac{L'}{L}(1,\rho) + \frac{1}{2\pi i}\int_{(2)} \frac{L'}{L}(s+1,\rho)x^s\Gamma(s)\,ds \ll_n \frac{x^2}{(1-\alpha)^2\sqrt{N}} + \frac{(\log N)^2}{(1-\alpha)^3 x^{(1-\alpha)/8}}. \tag{3.2}$$

Let

$$L(s,\rho) = \prod_p L(s,\rho)_p = \sum_{n=1}^{\infty} \lambda(n)n^{-s}, \quad L(s,\rho)_p = \prod_{i=1}^{n}(1-\alpha_i(p)p^{-s})^{-1}.$$

Then $\lambda(p) = \sum_{i=1}^n \alpha_i(p)$ and $|\lambda(p)| \leqslant n$. By taking the logarithmic derivative of $L(s,\rho)$ and the Mellin inversion of $\Gamma(s)$, we obtain

$$-\frac{1}{2\pi i}\int_{(2)} \frac{L'}{L}(s+1,\rho)\Gamma(s)x^s\,ds = \sum_{i=1}^{n}\sum_p \log p \sum_{k=1}^{\infty} \alpha_i(p)^k p^{-k} e^{-p^k/x}.$$

Since the terms with $k \geqslant 2$ converge absolutely, we only need to estimate

$$\sum_p \lambda(p)\frac{\log p}{p}e^{-p/x}.$$

Let x be a constant with $(\log N)^{16/(1-\alpha)} \leqslant x \leqslant N^{1/4}$. Then the error term in (3.2) is $O_{n,x,\alpha}(1)$. On the other hand,

$$\sum_{p \leqslant x} \frac{\log p}{p}(1-e^{-p/x}) < 1, \quad \sum_{p > x} \frac{\log p}{p}e^{-p/x} \ll 1.$$

Hence we obtain an approximation of $\frac{L'}{L}(1,\rho)$ as a sum over a short interval, which can be summarized as follows.

PROPOSITION 3.3. *Suppose that $L(s,\rho)$ is entire and free from zeros in the rectangle $[\alpha,1]\times[-(\log N)^2,(\log N)^2]$. If N is sufficiently large, then for any constant x with $(\log N)^{16/(1-\alpha)}\leqslant x\leqslant N^{1/4}$,*

$$\frac{L'}{L}(1,\rho)=-\sum_{p\leqslant x}\frac{\lambda(p)\log p}{p}+O_{n,x,\alpha}(1).$$

Because we lack the GRH, we cannot use the above result directly. We extend the result of Kowalski and Michel to isobaric automorphic representations of $\mathrm{GL}(n)$.

Let $n=n_1+\cdots+n_r$, and let $S(q)$ be a set of isobaric representations $\pi=\pi_1\boxplus\pi_2\boxplus\cdots\boxplus\pi_r$, where each π_j is a cuspidal automorphic representation of $\mathrm{GL}(n_j)/\mathbb{Q}$ and satisfies the Ramanujan–Petersson conjecture at the finite places. We assume that given two representations $\pi,\pi'\in S(q)$, for each j, π_j is not equivalent to any π'_k if $n_j=n_k$. Moreover, $S(q)$ satisfies the following conditions.

(1) There exists $e>0$ such that for $\pi=\pi_1\boxplus\pi_2\boxplus\cdots\boxplus\pi_r\in S(q)$, $\mathrm{Cond}(\pi_1)\cdots\mathrm{Cond}(\pi_r)\leqslant q^e$.

(2) There exists $d>0$ such that $|S(q)|\leqslant q^d$.

(3) The Γ-factors of π_j are of the form $\prod_{k=1}^{n_j}\Gamma(s/2+\alpha_k)$ where $\alpha_k\in\mathbb{R}$.

For $\alpha\geqslant 3/4$ and $T\geqslant 2$, let

$$N(\pi;\alpha,T)=|\{\rho:L(\rho,\pi)=0,\ \mathrm{Re}(\rho)\geqslant\alpha,\ |\mathrm{Im}(\rho)|\leqslant T\}|.$$

Then, clearly, $N(\pi;\alpha,T)=N(\pi_1;\alpha,T)+\cdots+N(\pi_r;\alpha,T)$.

THEOREM 3.4. *For some $B\geqslant 0$,*

$$\sum_{\pi\in S(q)}N(\pi;\alpha,T)\ll T^B q^{c_0(1-\alpha)/(2\alpha-1)}.$$

One can choose any $c_0>c_0'$, where $c_0'=5n'e/2+d$ with $n'=\max\{n_i\}_{1\leqslant i\leqslant r}$.

Proof. Let $S(q)_j$ be the set of cuspidal automorphic representations consisting of the jth component of π. By assumption, $|S(q)_j|=|S(q)|$ for all $j=1,2,\ldots,r$. Then, clearly, $\mathrm{Cond}(\pi_j)\leqslant q^e$ and $|S(q)_j|\leqslant q^d$. So

$$\sum_{\pi\in S(q)}N(\pi;\alpha,T)=\sum_{\pi\in S(q)}\sum_{j=1}^{r}N(\pi_j;\alpha,T)=\sum_{j=1}^{r}\sum_{\pi_j\in S(q)_j}N(\pi_j;\alpha,T).$$

Now we apply the result of Kowalski and Michel [KM02] to the inner sum. They assumed that the Γ-factors of π_j are the same; however, that assumption is used only to obtain the convexity bound in [KM02, Lemma 10], and our Γ-factors provide the same convexity bound. Hence our result follows. □

In the following, we apply the above result to a family of Artin L-functions. In this case, the Γ-factors are a product of $\Gamma(s/2)$ and $\Gamma((s+1)/2)$.

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4. Extreme values of $\frac{L'}{L}(1, \rho)$

In this section, we describe how to obtain extreme positive and negative values of $\frac{L'}{L}(1, \rho)$ in a general setting. Let $f(x, t)$, K_t, $\widehat{K_t}$ and G be as in § 2. Suppose $K_t \in \mathfrak{K}(n, G, r_1, r_2)$. Let ρ be the $(n-1)$-dimensional complex representation of the Galois group $\mathrm{Gal}(\widehat{K_t}/\mathbb{Q})$ given by $\zeta_{K_t}(s) = \zeta(s)L(s, \rho, t)$. Then the conductor of ρ is $|d_{K_t}|$. Now we assume that ρ is modular, i.e. an automorphic representation of GL_{n-1}; this is called the strong Artin conjecture. The discriminant of $f(x, t)$ is a polynomial in t. We expect the regular Galois extension property to imply that the field discriminant $|d_{K_t}|$ will increase with respect to t.

Assumption 4.1. $\log |d_{K_t}| \gg_f \log |t|$.

4.1 Extreme positive values of $\frac{L'}{L}(1, \rho)$

Let G be a finite group having the Gal_T property, and let $f(x, t) \in \mathbb{Z}[t][x]$ be an irreducible polynomial of degree n whose splitting field over $\mathbb{Q}(t)$ is a regular extension with Galois group G. Let K_t be the number field obtained by adjoining a root of $f(x, t)$ to the rational number field $\mathbb{Q}$ for a specialization $t \in \mathbb{Z}$, and let $\widehat{K_t}$ be its Galois closure. Let $L(s, \rho, t) = \sum_{l=1}^{\infty} \lambda(l, t) l^{-s}$ be the Artin L-function $\zeta_{K_t}(s)/\zeta(s)$.

Note that the conductor of $L(s, \rho, t)$ is $|d_{K_t}|$, and for an unramified prime p, $\lambda(p, t) = N(p, t) - 1$ where $N(p, t)$ is the number of solutions of $f(x, t) \equiv 0 \pmod p$. Hence $-1 \leqslant \lambda(p, t) \leqslant n - 1$.

The Galois group $\mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G$ acts on the set $X = \{x_1, x_2, \ldots, x_n\}$ of roots of $f(x, t)$ transitively. Let G_0 be the set of all $g \in G$ with no fixed points. Then G_0 is not empty and $|G_0|/|G| \geqslant 1/n$ (see [Ser03, p. 430]). Choose any $g_0 \in G_0$ and let $[g_0]$ be the conjugacy class of g_0 in G. If the Frobenius element of p belongs to $[g_0]$, then $f(x, t) \equiv 0 \pmod p$ has no root and hence $\lambda(p, t) = -1$.

Since $f(x, t)$ gives rise to a regular extension, by Theorem 2.1 there is a constant c_f (depending on f) such that for any prime $p \geqslant c_f$ there is an integer i_p such that for any $t \equiv i_p \pmod p$ with $\mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G$, the Frobenius element of p belongs to $[g_0]$. For $X > 0$, let $y = (\log X)/(\log \log X)$ and $M = \prod_{c_f \leqslant p \leqslant y} p$. Note that $M \ll e^y = e^{(\log X)/(\log \log X)} \ll_\epsilon X^\epsilon$ for any $\epsilon > 0$.

Let i_M be an integer such that $i_M \equiv i_p \pmod p$ for all $c_f \leqslant p \leqslant y$. Thus, if $t \equiv i_M \pmod M$, then for all $c_f \leqslant p \leqslant y$, Frob_p belongs to $[g_0]$ and $\lambda(p, t) = -1$.

Assume that the discriminant of $f(x, t)$ is a polynomial in t of degree D. Then there is a constant C such that $|d_{K_t}| \leqslant Ct^D$. We define a set $L(X)$ of positive numbers given by

$$L(X) = \left\{ \frac{X}{2} < t < X \,\middle|\, t \equiv i_M \pmod M,\ \mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G \right\}.$$

Under the strong Artin conjecture, every t in $L(X)$ corresponds to an automorphic L-function of $\mathrm{GL}(n-1)$ over $\mathbb{Q}$. However, it is possible that different $t \in L(X)$ correspond to the same automorphic L-function, i.e. $\zeta_{K_{t_1}}(s) = \zeta_{K_{t_2}}(s)$, in which case we say that K_{t_1} and K_{t_2} are arithmetically equivalent. We make the following assumption.

Assumption 4.2. There exists a finite set $T \subset \mathbb{Z}$, depending only on f, such that the $L(s, \rho, t)$ are distinct for all $t \in L(X) \backslash T$.

In order to verify the assumption, we use the following theorem.

Theorem 4.3 [Kli98]. *Let $K/\mathbb{Q}$ be a number field of degree $n \leqslant 11$. Let $\widehat{K}$ be the Galois closure and assume that there exists a non-conjugate field K' which is arithmetically equivalent to K.*

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Then, up to conjugacy, only the following cases are possible for $G = \mathrm{Gal}(\hat{K}/\mathbb{Q})$*:*

- $n = 7$ *and* $G = \mathrm{GL}_3(2)$*;*
- $n = 8$ *and* $G = \mathbb{Z}/8\mathbb{Z} \rtimes (\mathbb{Z}/8\mathbb{Z})^\times$ *or* $G = \mathrm{GL}_2(3)$*;*
- $n = 11$ *and* $G = \mathrm{PSL}_2(11)$.

In §§ 6–9, we consider explicit examples of families of number fields. We show that the K_t are not isomorphic. In those cases, the degree of K_t is less than 7 and hence, by the above theorem, the number fields are not arithmetically equivalent. We may have to place more conditions on $L(X)$ to obtain the property that the K_t are not isomorphic. In any case, we shall show that $X^{1-\epsilon} \ll |L(X)| \ll X$ for any fixed $\epsilon > 0$.

Let $c_0 = 5(n-1)D/2 + 1$. We may replace the $(n-1)$ in c_0 by a smaller constant if ρ is not irreducible. Choose α with $c_0(1-\alpha)/(2\alpha - 1) < 98/100$. By applying Theorem 3.4 to $L(X)$ with $e = D$, $d = 1$ and $T = (\log CX^D)^2$, we deduce that every automorphic L-function, excluding exceptional $O(X^{98/100})$ L-functions, has a zero-free region $[\alpha, 1] \times [-(\log |d_{K_t}|)^2, \log |d_{K_t}|)^2]$. Let us denote by $\widehat{L}(X)$ the set of automorphic L-functions with this zero-free region.

Applying Proposition 3.3 to the L-function $L(s, \rho, t)$ in $\widehat{L}(X)$ with $x = (\log CX^D)^{16/(1-\alpha)}$, we obtain

$$\begin{aligned}\frac{L'}{L}(1, \rho, t) &= -\sum_{p \leqslant x} \frac{\lambda(p, t) \log p}{p} + O_{n,x,\alpha}(1) \\ &= \sum_{c_f \leqslant p \leqslant y} \frac{\log p}{p} - \sum_{y < p \leqslant x} \frac{\lambda(p, t) \log p}{p} + O_{n,x,\alpha}(1) \\ &= \log\log X - \sum_{y < p \leqslant x} \frac{\lambda(p, t) \log p}{p} + O(\log\log\log X), \end{aligned} \tag{4.4}$$

where we have used the fact that $\sum_{p \leqslant y} (\log p)/p = \log y + O(1)$ and that $y = (\log X)/(\log\log X)$.

Now we sum the logarithmic derivatives $\frac{L'}{L}(1, \rho, t)$ over $\widehat{L}(X)$; that is, we consider

$$\sum_{L(s,\rho,t) \in \widehat{L}(X)} \frac{L'}{L}(1, \rho, t).$$

We need to deal with the sum

$$\sum_{L(s,\rho,t) \in \widehat{L}(X)} \sum_{y < p \leqslant x} \frac{\lambda(p, t) \log p}{p} = \sum_{y < p \leqslant x} \frac{\log p}{p} \sum_{L(s,\rho,t) \in \widehat{L}(X)} \lambda(p, t).$$

In the next section, we prove the following proposition.

PROPOSITION 4.5. *For all* $y < p \leqslant x$,

$$\sum_{L(s,\rho,t) \in \widehat{L}(X)} \lambda(p, t) \ll \frac{|\widehat{L}(X)|}{\sqrt{p}} + \frac{|\widehat{L}(X)|}{(\log X)^{1/2}},$$

where the implied constant is independent of p.

Proposition 4.5 implies that

$$\sum_{L(s,\rho,t)\in\widehat{L}(X)}\sum_{y<p\leqslant x}\frac{\lambda(p,t)\log p}{p} \ll |\widehat{L}(X)|\sum_{y<p\leqslant x}\frac{\log p}{p^{3/2}}+\frac{|\widehat{L}(X)|}{(\log X)^{1/2}}\sum_{y<p\leqslant x}\frac{\log p}{p}$$
$$\ll \frac{|\widehat{L}(X)|}{y^{1/2}}+\frac{|\widehat{L}(X)|\log\log X}{(\log X)^{1/2}}.$$

Hence we have

$$\sum_{L(s,\rho,t)\in\widehat{L}(X)}\frac{L'}{L}(1,\rho,t)=|\widehat{L}(X)|\log\log X+O(|\widehat{L}(X)|\log\log\log X).$$

Now note that $|d_{K_t}|\leqslant Ct^D$ and $t<X$. So if there are only finitely many L-functions with $\frac{L'}{L}(1,\rho,t)\geqslant\log\log|d_{K_t}|+O(\log\log\log|d_{K_t}|)$, they cannot reach the average value $\log\log X$ as X increases. Therefore we have proved the following result under Assumptions 4.1 and 4.2.

Theorem 4.6. *There are infinitely many $L(s,\rho,t)$ in $\mathfrak{K}(n,G,r_1,r_2)$ such that*

$$\frac{L'}{L}(1,\rho,t)\geqslant\log\log|d_{K_t}|+O(\log\log\log|d_{K_t}|).$$

4.2 Extreme negative values of $\frac{L'}{L}(1,\rho)$

To generate a $\frac{L'}{L}(1,\rho,t)$ that is negative but whose absolute value is large, we need to manipulate $\lambda(p,t)$ so that $\lambda(p,t)=n-1$ for all primes p between c_f and $y=(\log X)/(\log\log X)$ in (4.4).

Since $f(x,t)$ gives rise to a regular Galois extension, Theorem 2.1 tells us that for any prime $p\geqslant c_f$ there is an integer s_p such that for any $t\equiv s_p \pmod p$ with $\mathrm{Gal}(\widehat{K_t}/\mathbb{Q})\simeq G$, the Frobenius element of p is the identity in G. For $X>0$, let $M=\prod_{c_f\leqslant p\leqslant y}p$. Let s_M be an integer such that $s_M\equiv s_p \pmod p$ for all $c_f\leqslant p\leqslant y$. So, if $t\equiv s_M \pmod M$, for all $c_f\leqslant p\leqslant y$ we have that p splits completely in $\widehat{K_t}$ and $\lambda(p,t)=n-1$.

We define $L(X)$ and $\widehat{L}(X)$ as in § 4.1. Then, as in (4.4),

$$\frac{L'}{L}(1,\rho,t)=-(n-1)\log\log X-\sum_{y<p\leqslant x}\frac{\lambda(p,t)\log p}{p}+O(\log\log\log X).$$

By Proposition 4.5,

$$\sum_{L(s,\rho,t)\in\widehat{L}(X)}\frac{L'}{L}(1,\rho,t)=-(n-1)|\widehat{L}(X)|\log\log X+O(|\widehat{L}(X)|\log\log\log X).$$

Thus we have proved the following result under Assumptions 4.1 and 4.2.

Theorem 4.7. *There are infinitely many L-functions $L(s,\rho,t)$ in $\mathfrak{K}(n,G,r_1,r_2)$ with*

$$\frac{L'}{L}(1,\rho,t)\leqslant-(n-1)\log\log|d_{K_t}|+O(\log\log\log|d_{K_t}|).$$

5. Proof of Proposition 4.5

For a fixed prime p, consider the equation $f(x,t)\equiv 0 \pmod p$. Now we view $f(x,t)$ as an algebraic curve over $\mathbb{Z}/p\mathbb{Z}$. Let A_i be the number of $t \pmod p$ such that $\lambda(p,t)=i$, i.e. such

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that $f(x,t) \equiv 0 \pmod p$ has $i+1$ roots. Then we have

$$\sum_{i=-1}^{n-1} A_i = p + O(1),$$

where $O(1)$ is bounded by D, the degree of the discriminant of $f(x,t)$.

Recall Weil's celebrated theorem on rational points of a curve over a finite field (see, e.g., [Sch04, p. 75]).

THEOREM 5.1. *Let $f(x,y) \in \mathbb{F}_p[x,y]$ be absolutely irreducible and of total degree $d > 0$. Let N be the number of zeros of f in $\mathbb{F}_p \times \mathbb{F}_p$. Then*

$$|N - p| \leqslant (d-1)(d-2)\sqrt{p} + c(d)$$

for some constant $c(d)$.

Weil's theorem implies that

$$\sum_{i=-1}^{n-1} (i+1) A_i = p + O(\sqrt{p}).$$

Hence we obtain

$$\sum_{i=-1}^{n-1} i A_i = O(\sqrt{p}). \tag{5.2}$$

Now we define $Q_i = \{X/2 < t < X \mid t \in L(X) \text{ and } t \equiv i \pmod p\}$ and write

$$L(X) = Q_0 \cup Q_1 \cup \cdots \cup Q_{p-1}.$$

Let R be a finite subset of $\{0,1,2,\ldots,p-1\}$ such that $k \in R$ if and only if p is ramified for $t \in Q_k$. We prove the following in the examples of §§ 6–9:

$$|Q_i| = c_p \frac{|L(X)|}{p} + O\bigg(\frac{|L(X)|}{p(\log X)^{1/2}}\bigg) \quad \text{for } i \notin R, \tag{5.3}$$

where c_p is a constant that is close to 1 and independent of i. (We can show that $1/2 < c_p < 2$.)

Since $\sum_{L(s,\rho,t) \in \widehat{L}(X)} \lambda(p,t) = \sum_{L(s,\rho,t) \in L(X)} \lambda(p,t) + O(X^{98/100})$, in order to prove Proposition 4.5 it is enough to show that

$$\sum_{L(s,\rho,t) \in L(X)} \lambda(p,t) \ll \frac{|L(X)|}{\sqrt{p}} + \frac{|L(X)|}{(\log X)^{1/2}}.$$

If $k \in R$,

$$\bigg|\sum_{L(s,\rho,t) \in Q_k} \lambda(p,t)\bigg| \leqslant (n-1)\frac{|L(X)|}{p} + O(1).$$

If $k \notin R$, then p is unramified for all $t \in Q_k$, and $\lambda(p,t) = j(k)$ for a unique $j(k)$. In that case,

$$\sum_{L(s,\rho,t) \in Q_k} \lambda(p,t) = j(k) c_p \frac{|L(X)|}{p} + O\bigg(\frac{|L(X)|}{p(\log X)^{1/2}}\bigg).$$

Hence

$$\sum_{L(s,\rho,t) \in L(X)} \lambda(p,t) = \sum_{k \in R} \sum_{L(s,\rho,t) \in Q_k} \lambda(p,t) + \sum_{k \notin R} \sum_{L(s,\rho,t) \in Q_k} \lambda(p,t).$$

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Here

$$\sum_{k\in R}\sum_{L(s,\rho,t)\in Q_k}\lambda(p,t)\ll\frac{|L(X)|}{p},$$

where the implied constant is independent of p. On the other hand,

$$\begin{aligned}\sum_{k\notin R}\sum_{L(s,\rho,t)\in Q_k}\lambda(p,t)=\sum_{k\notin R}j(k)|Q_k| &= c_p\frac{|L(X)|}{p}\sum_{k\notin R}j(k)+O\bigg(\frac{|L(X)|}{(\log X)^{1/2}}\bigg)\\ &= c_p\frac{|L(X)|}{p}\sum_{j=-1}^{n-1}jA_j+O\bigg(\frac{|L(X)|}{(\log X)^{1/2}}\bigg).\end{aligned}$$

By (5.2),

$$\sum_{L(s,\rho,t)\in L(X)}\lambda(p,t)\ll\frac{|L(X)|}{\sqrt{p}}+\frac{|L(X)|}{(\log X)^{1/2}}.$$

For Proposition 4.5, we do not need to calculate A_i. Nevertheless, we prove the following theorem in the continuation paper [CKc].

Theorem 5.4. *Let G be the Galois group of the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$. Fix a prime p. Let C_i be the union of conjugacy classes C in G such that the trace of ρ at C is equal to i. Then*

$$A_i=\frac{|C_i|}{|G|}p+O(\sqrt{p}).$$

Remark 5.5. This is a refinement of Weil's theorem. It can also be thought of as a reciprocity law. It may be helpful to compare this result with the Chebotarev density theorem: fix t; then the number of $p<x$ such that $\lambda(p,t)=i$ is asymptotic to $\frac{|C_i|}{|G|}\frac{x}{\log x}$ as $x\to\infty$.

In §§ 6–8, we look at explicit examples from [Cho, CKa, CKb] for which the strong Artin conjecture and Assumptions 4.1 and 4.2 hold. For the A_4 and C_5 cases, we were not able to verify Assumption 4.2.

6. Cyclic and dihedral extensions

Cyclic and dihedral extensions satisfy the Gal_T property. Hence, given a cyclic or dihedral group G, there exists a polynomial $f(x,t)\in\mathbb{Z}[t][x]$ whose splitting field over $\mathbb{Q}(t)$ is a regular Galois extension and whose Galois group is G. We give some details for quadratic and cyclic cubic extensions.

6.1 Quadratic extensions

Consider $K_t=\mathbb{Q}[\sqrt{t}]$ for t square-free and $t\equiv 1 \pmod 4$. Consider, for $M=4\prod_{3\leqslant p\leqslant y}p$,

$$L(X)_1=\left\{\frac{X}{2}<t<X\;\middle|\;t\text{ square-free and }t\equiv s_M\ (\mathrm{mod}\ M)\right\},$$

$$L(X)_2=\left\{\frac{X}{2}<t<X\;\middle|\;t\text{ square-free and }t\equiv i_M\ (\mathrm{mod}\ M)\right\}.$$

Then Assumptions 4.1 and 4.2 are clear. We verify (5.3) in the case of $L(X)_2$:

$$Q_i = \left\{ \frac{X}{2} < t < X \;\middle|\; t \text{ square-free}, \ t \equiv i_M \ (\mathrm{mod}\ M), \ t \equiv i \ (\mathrm{mod}\ p) \right\}.$$

Since $p > y$, $(p, M) = 1$; and if $i \neq 0$, then by [Dai06, p. 248] we have

$$|Q_i| = \frac{3}{\pi^2} \prod_{q|M} (1 - q^{-2})^{-1} \frac{X}{M} (1 - p^{-2})^{-1} \frac{1}{p} + O(X^{1/2}) = c_p \frac{|L(X)_2|}{p} + O(X^{1/2}),$$

where $c_p = (1 - p^{-2})^{-1}$ and $1 < c_p < 2$. Since $p \ll (\log X)^{16/(1-\alpha)}$, it follows that $X^{1/2} \ll |L(X)_2|/p(\log X)^{1/2}$. Here we have considered real quadratic fields. However, the same argument is applicable to imaginary quadratic fields. So Theorems 4.6 and 4.7 can now be stated as follows.

THEOREM 6.1. (1) *There are infinitely many real and infinitely many imaginary quadratic fields* $\mathbb{Q}(\sqrt{t})$ *with*

$$\frac{L'}{L}(1, \chi_t) \leqslant -\log\log|t| + O(\log\log\log|t|).$$

(2) *There are infinitely many real and infinitely many imaginary quadratic fields* $\mathbb{Q}(\sqrt{t})$ *with*

$$\frac{L'}{L}(1, \chi_t) \geqslant \log\log|t| + O(\log\log\log|t|).$$

6.2 Cyclic cubic extensions

Consider

$$f(x, t) = x^3 - tx^2 - (t + 3)x - 1$$

for $t \in \mathbb{Z}^+$. Its discriminant is $g(t)^2$ with $g(t) = t^2 + 3t + 9$. Then $K_t/\mathbb{Q}$ is a C_3 Galois extension, and $L(s, \rho, t) = L(s, \chi_t)L(s, \overline{\chi_t})$ where χ_t and $\overline{\chi_t}$ are two non-principal characters of C_3. The conductor f_{χ_t} of χ_t is $g(t)$ when $g(t)$ is square-free. Hence we can see that $\chi_t \not\cong \chi_{t'}$ for $t \neq t'$. Note also that $\frac{L'}{L}(1, \rho, t) = 2\,\mathrm{Re}(\frac{L'}{L}(1, \chi_t))$. Consider, for $M = 6\prod_{5 \leqslant p \leqslant y} p$,

$$L(X)_1 = \left\{ \frac{X}{2} < t < X \;\middle|\; g(t) \text{ square-free}, \ t \equiv s_M \ (\mathrm{mod}\ M) \right\}$$

$$L(X)_2 = \left\{ \frac{X}{2} < t < X \;\middle|\; g(t) \text{ square-free}, \ t \equiv i_M \ (\mathrm{mod}\ M) \right\}.$$

Then Assumptions 4.1 and 4.2 are clear. We verify (5.3) in the case of $L(X)_2$:

$$Q_i = \left\{ \frac{X}{2} < t < X \;\middle|\; g(t) \text{ square-free}, \ t \equiv i_M \ (\mathrm{mod}\ M), \ t \equiv i \ (\mathrm{mod}\ p) \right\}.$$

Define R' to be the set of solutions $t \ (\mathrm{mod}\ p)$ for $g(t) \equiv 0 \ (\mathrm{mod}\ p)$. Then R' has at most two elements. So it is enough to consider $i \notin R'$. Since $p > y$, $(p, M) = 1$; and for $i \notin R'$, by [Duk04] we have

$$\begin{aligned}|Q_i| &= \prod_{q \nmid M} \left(1 - \left(1 + \left(\frac{-3}{q}\right)\right) q^{-2}\right) \frac{X}{2M} \left(1 - \left(1 + \left(\frac{-3}{p}\right)\right) p^{-2}\right)^{-1} \frac{1}{p} + O(X^{2/3}\log X) \\ &= c_p \frac{|L(X)_2|}{p} + O(X^{2/3}\log X),\end{aligned}$$

where $c_p = (1 - (1 + (\frac{-3}{p}))p^{-2})^{-1}$ and $1/2 < c_p < 2$. Since $p \ll (\log X)^{16/(1-\alpha)}$, we have $X^{2/3}\log X \ll |L(X)_2|/p(\log X)^{1/2}$. So Theorems 4.6 and 4.7 can now be stated as follows.

Theorem 6.2. (1) *There are infinitely many* $L(s,\rho,t)$ *with*

$$\frac{L'}{L}(1,\rho,t) \leqslant -2\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|),$$
$$\mathrm{Re}\left(\frac{L'}{L}(1,\chi_t)\right) \leqslant -\log\log f_{\chi_t} + O(\log\log\log f_{\chi_t}).$$

(2) *There are infinitely many* $L(s,\rho,t)$ *with*

$$\frac{L'}{L}(1,\rho,t) \geqslant \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|),$$
$$\mathrm{Re}\left(\frac{L'}{L}(1,\chi_t)\right) \geqslant \frac{1}{2}\log\log f_{\chi_t} + O(\log\log\log f_{\chi_t}).$$

6.3 Dihedral and cyclic extensions

For higher-degree extensions, we recall the following explicit examples from [CKa, Dai06]:

$$\begin{aligned}
&\mathfrak{K}(6,C_6,6,0): f(x,t) = x^6 - 2tx^5 - 5(t+3)x^4 - 20x^3 + 5tx^2 + 2(t+3)x + 1,\\
&\mathfrak{K}(4,C_4,4,0): f(x,t) = x^4 - 2tx^3 - 6x^2 + 2tx + 1,\\
&\mathfrak{K}(5,D_5,5,0): f(x,t) = x^5 - tx^4 + (2t-1)x^3 - (t-2)x^2 - 2x + 1, \quad t > 7,\\
&\mathfrak{K}(5,D_5,1,2): f(x,t) = x^5 + tx^4 - (2t+1)x^3 + (t+2)x^2 - 2x + 1, \quad t \geqslant 0,\\
&\mathfrak{K}(4,D_4,4,0): f(x,t) = x^4 - tx^3 - x^2 + tx + 1,\\
&\mathfrak{K}(4,D_4,2,1): f(x,t) = x^4 - tx^3 + 3x^2 - tx + 1,\\
&\mathfrak{K}(4,D_4,0,2): f(x,t) = x^4 - x^3 + (t+2)x^2 - x + 1,\\
&\mathfrak{K}(3,D_3,3,0): f(x,t) = (x-t)(x-4t)(x-9t) - t,\\
&\mathfrak{K}(3,D_3,1,1): f(x,t) = x^3 + tx - 1.
\end{aligned}$$

For the cyclic and dihedral Galois extensions, the representation ρ is no longer irreducible. We need to show that the irreducible components of the representations ρ are not equivalent for different t by computing their Artin conductors.

In the case of the simplest sextic fields, let K_t be a sextic field obtained by adjoining a root of $f(x,t)$ to $\mathbb{Q}$. Here we do not need to specialize t as in [CKa], since we do not need to find units. Note that the discriminant of $f(x,t)$ is $6^6(t^2+3t+9)^5$. We assume that t^2+3t+9 is square-free. Then the cubic field L_t of K_t is the simplest cubic field with the field discriminant $(t^2+3t+9)^2$ and the quadratic field M_t is $\mathbb{Q}(\sqrt{t^2+3t+9})$.

Let σ be the generator of $\mathrm{Gal}(K_t/\mathbb{Q}) \simeq C_6$, then $L_t = K_t^{\langle\sigma^3\rangle}$ and $M_t = K_t^{\langle\sigma^2\rangle}$. Let χ be the generator of the group of characters for $\mathrm{Gal}(K_t/\mathbb{Q})$ with $\chi(\sigma) = e^{2\pi i/6}$. Then $\mathrm{Ind}_{\langle\sigma^2\rangle}^{\langle\sigma\rangle} 1_{\langle\sigma^2\rangle} = 1_{\langle\sigma\rangle} + \chi^3$, $\mathrm{Ind}_{\langle\sigma^3\rangle}^{\langle\sigma\rangle} 1_{\langle\sigma^3\rangle} = 1_{\langle\sigma\rangle} + \chi^2 + \chi^4$ and $\mathrm{Ind}_{\langle\sigma^3\rangle}^{\langle\sigma\rangle}\varphi = 1_{\langle\sigma\rangle} + \chi + \chi^5$, where φ is the non-trivial representation for $\langle\sigma^3\rangle$. Hence the Artin conductor of χ^3 equals the field discriminant of M_t, and the Artin conductors of χ^2 and χ^4 are both t^2+3t+9. The Artin conductors of χ and χ^5 are equal to $(t^2+3t+9)\sqrt{N(\mathfrak{b})}$ where $\mathfrak{b}$ is the Artin conductor of φ. Since the product of Artin conductors of $\chi, \chi^2, \ldots, \chi^5$ is at most $6^6(t^2+3t+9)^5$, $N(\mathfrak{b})$ is a bounded constant. Hence for t^2+3t+9 square-free, as t increases, the Artin conductors also increase. Thus we have verified that the irreducible components are not equivalent.

In the case of the simplest quartic fields, we assume that t^2+4 is square-free. When t^2+4 is square-free, the field discriminant d_{K_t} equals $2^4(t^2+4)^3$, and K_t has the unique quadratic subfield $M_t = \mathbb{Q}(\sqrt{t^2+4})$. Let $H \simeq C_2$ be the unique subgroup of order 2 in C_4. Then $\mathrm{Ind}_H^{C_4} 1_H = 1 + \chi^2$ where χ is the generator of the group of characters for C_4 with $\chi(\sigma) = e^{2\pi i/4}$.

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Hence the Artin conductor $f(\chi^2)$ of χ^2 equals t^2+4 when t^2+4 is square-free. Since $d_{K_t} = f(\chi)f(\chi^2)f(\chi^3)$ and $\chi^3 = \overline{\chi}$, we have $f(\chi) = f(\chi^3) = 2^2(t^2+4)$, and thus we have verified that the irreducible components are not equivalent.

In the case of $\mathfrak{K}(5, D_5, 1, 2)$, $\rho = \sigma_1 \oplus \sigma_2$ where σ_1 and σ_2 are the two 2-dimensional representations of D_5. Assume that $4t^3+28t^2+24t+47$ is square-free. In [CKa], we showed that their Artin conductors are both $4t^3+28t^2+24t+47$. The case of $\mathfrak{K}(5, D_5, 5, 0)$ is dealt with in the same way.

When $G = D_4$, the representation ρ is decomposed into a sum of the non-trivial 1-dimensional representation χ and the 2-dimensional representation ψ of D_4. For the case of $\mathfrak{K}(4, D_4, 4, 0)$, we assume that $(t^2-4)(4t^2+9)$ is square-free. In [CKa], it is shown that the Artin conductor of χ is $|t^2-4|$ and the Artin conductor of ψ is $|(t^2-4)(4t^2+9)|$. For the case of $\mathfrak{K}(4, D_4, 2, 1)$, the Artin conductor of χ is $|t^2-4|$ and the Artin conductor of ψ is $|(t^2-4)(4t^2-25)|$ if $(t^2-4)(25-4t^2)$ is square-free. For the case of $\mathfrak{K}(4, D_4, 0, 2)$, the Artin conductor of χ is $|1-4t|$ and the Artin conductor of ψ is $|(1-4t)(t+2)(t+6)|$ if $(1-4t)(t+2)(t+6)$ is square-free.

The strong Artin conjecture is valid in all of the above cases. We recall the definition of the set $L(X)$ in each case (writing only the extreme positive-value version).

$$\mathfrak{K}(6, C_6, 6, 0) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; t^2+3t+9 \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(4, C_4, 4, 0) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; t^2+4 \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(5, D_5, 5, 0) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; 4t^3-28t^2+24t-47 \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(5, D_5, 1, 2) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; 4t^3+28t^2+24t+47 \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(4, D_4, 4, 0) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; (t^2-4)(4t^2+9) \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(4, D_4, 2, 1) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; (t^2-4)(25-4t^2) \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(4, D_4, 0, 2) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; (1-4t)(t+2)(t+6) \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(3, D_3, 3, 0) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; t \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$
$$\mathfrak{K}(3, D_3, 1, 1) : L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; 4t^3+27 \text{ square-free and } t \equiv i_M \pmod{M} \right\}.$$

In these cases we proved Assumption 4.1, namely that $\log|d_{K_t}| \gg \log|t|$. Let us now make some remarks about Assumption 4.2. For cyclic and dihedral extensions, we computed the Artin conductors of the irreducible components of the representation ρ; hence we verified Assumption 4.2 as a byproduct in these cases.

In the case of $\mathfrak{K}(3, D_3, 3, 0)$, Daileda [Dai06] used the set

$$L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; t \text{ and } (36t^2+1)(400t^2-27) \text{ square-free and } t \equiv i_M \pmod{M} \right\}$$

and showed that Assumption 4.2 holds. In this case, only the lower bound on $|L(X)|$ is obtained, so it is not clear how to prove (5.3). We therefore use the set

$$L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; t \text{ square-free and } t \equiv i_M \pmod{M} \right\}.$$

Then (5.3) can be proved as in the quadratic case. We verify Assumption 4.2 as follows. Note that $f(x,t)$ is an Eisenstein polynomial. So $p \mid t$ if and only if p is totally ramified in K_t: if $p \mid t$, then by [Coh93, p. 315] p is totally ramified; conversely, if p is totally ramified and $p \nmid t$, then $f(x,t) \equiv (x+a)^3 \pmod{p}$. If we compare the coefficients of $f(x,t) \pmod{p}$, we obtain a contradiction. Therefore, the K_t are distinct for all $t \in L(X)$.

Now we show that (5.3) holds for the remaining cases. For C_6 and C_4, it can be verified as in the case of cyclic cubic fields. Consider the case of $\mathfrak{K}(5, D_5, 1, 2)$:

$$Q_i = \left\{ \frac{X}{2} < t < X \;\middle|\; 4t^3 + 28t^2 + 24t + 47 \text{ square-free}, \ t \equiv i_M \pmod{M}, \ t \equiv i \pmod{p} \right\}.$$

Let R' be the set of solutions $t \pmod{p}$ of $4t^3 + 28t^2 + 24t + 47 \equiv 0 \pmod{p}$. Then R' has at most three elements. Hence it is enough to consider $i \notin R'$. For $i \notin R'$, by [CKb, Lemma 8.3] we have

$$\begin{aligned} |Q_i| &= \prod_{q \nmid M} \left(1 - \frac{\rho(q^2)}{q^2}\right)\left(1 - \frac{\rho(p^2)}{p^2}\right)^{-1} \frac{1}{p}\frac{X}{2M} + O\left(\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1/2}\right) \\ &= c_p \frac{|L(X)|}{p} + O\left(\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1/2}\right), \end{aligned}$$

where $\rho(p^2)$ is the number of solutions of $4t^3 + 28t^2 + 24t + 47 \equiv 0 \pmod{p}^2$ and $c_p = (1 - \rho(p^2)/p^2)^{-1}$. Here $1 < c_p < 2$. Since $(1-\epsilon)\log X < \log(X/pM) < \log X$ for any $\epsilon > 0$, we have

$$\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1/2} \ll \frac{|L(X)|}{p(\log X)^{1/2}}.$$

We can show (5.3) for the cases of $\mathfrak{K}(3, D_3, 1, 1)$ and $\mathfrak{K}(5, D_5, 5, 0)$ by the same argument.

Next, consider the case of $\mathfrak{K}(4, D_4, 0, 2)$:

$$Q_i = \left\{ \frac{X}{2} < t < X \;\middle|\; (1-4t)(t+2)(t+4) \text{ square-free}, \ t \equiv i_M \pmod{M}, \ t \equiv i \pmod{p} \right\}.$$

Let R' be the set of solutions $t \pmod{p}$ of $(1-4t)(t+2)(t+4) \equiv 0 \pmod{p}$. Then R' has at most three elements. Hence it is enough to consider $i \notin R'$. For $i \notin R'$, [Nai76, Theorem B] gives

$$\begin{aligned} |Q_i| &= \prod_{q \nmid M} \left(1 - \frac{\rho(q^2)}{q^2}\right)\left(1 - \frac{\rho(p^2)}{p^2}\right)^{-1} \frac{1}{p}\frac{X}{2M} + O\left(\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1}\right) \\ &= c_p \frac{|L(X)|}{p} + O\left(\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1}\right), \end{aligned}$$

where $\rho(p^2)$ is the number of solutions of $(1-4t)(t+2)(t+4) \equiv 0 \pmod{p^2}$ and $c_p = (1 - \rho(p^2)/p^2)^{-1}$. Here $1 < c_p < 2$. Since $(1-\epsilon)\log X < \log(X/pM) < \log X$ for any $\epsilon > 0$, we have

$$\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1} \ll \frac{|L(X)|}{p(\log X)^{1/2}}.$$

We can show (5.3) for the case of $\mathfrak{K}(4, D_4, 2, 1)$ by the same argument.

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For the case of $\mathfrak{K}(4, D_4, 4, 0)$, consider

$$Q_i = \left\{ \frac{X}{2} < t < X \;\middle|\; (t^2-4)(4t^2+9) \text{ square-free},\ t \equiv i_M \pmod{M},\ t \equiv i \pmod{p} \right\}.$$

We define R' similarly to the previous cases. For $i \notin R'$, [Nai76, Theorem C] implies that

$$\begin{aligned} |Q_i| &= \prod_{q \nmid M} \left(1 - \frac{\rho(q^2)}{q^2}\right)\left(1 - \frac{\rho(p^2)}{p^2}\right)^{-1} \frac{1}{p}\frac{X}{2M} + O\left(\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1}\right) \\ &= c_p \frac{|L(X)|}{p} + O\left(\frac{X}{pM}\left(\log \frac{X}{pM}\right)^{-1}\right), \end{aligned}$$

where $\rho(p^2)$ is the number of solutions of $(t^2-4)(4t^2+9) \equiv 0 \pmod{p}^2$ and $c_p = (1 - \rho(p^2)/p^2)^{-1}$. Hence we have verified (5.3) for this case.

Therefore Theorems 4.6 and 4.7 are valid for the above examples.

Remark 6.3. We do not include the simplest quintic fields considered in [CKa], because we cannot verify Assumption 4.2 in that case. For the extreme class number problem, we assumed that $P_t = t^4 + 5t^3 + 15t^2 + 25t + 25$ is cube-free and we proved that the number of possible repetitions is $O(X^\epsilon)$. However, in order to prove Assumption 4.2, we have to assume that P_t is square-free, and we need the following difficult folklore conjecture: $\#\{1 < t < X \mid P_t \text{ is square-free}\} = cX + O(X/(\log X)^d)$ for some constants c and d.

7. S_4 Galois extensions

Consider the following polynomials from [Cho, CKb]:

$$\begin{aligned} &\mathfrak{K}(4, S_4, 4, 0) : f(x,t) = (x-t)(x-4t)(x-9t)(x-16t) - t, \\ &\mathfrak{K}(4, S_4, 2, 1) : f(x,t) = x^2(x-10t)(x-18t) + t, \\ &\mathfrak{K}(4, S_4, 0, 2) : f(x,t) = x^4 + tx^2 + tx + t. \end{aligned}$$

Assumption 4.1 for these three polynomials was verified in [Cho, CKb]. All these polynomials generate regular Galois extensions and the strong Artin conjecture is true.

We define $L(X)$ as

$$L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; t \text{ square-free},\ t \equiv i_M \pmod{M} \right\}.$$

Note that in these cases, $f(x,t)$ is an Eisenstein polynomial, so as in the $\mathfrak{K}(3, D_3, 3, 0)$ case, we can show that $p \mid t$ if and only if p is totally ramified in K_t. Hence the K_t are not isomorphic for all $t \in L(X)$, and Assumption 4.2 holds. In these cases, (5.3) is verified as in the quadratic case. Therefore Theorems 4.6 and 4.7 are valid unconditionally.

8. A_5 Galois extension

Consider the polynomial $f(x,t) = x^5 + 5(5t^2-1)x - 4(5t^2-1)$ where $5t^2-1$ is square-free. Here $\mathrm{disc}(f(x,t)) = 2^8 5^6 t^2 (5t^2-1)^4$.

We claim that the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is an A_5 regular extension. We need to show that the Galois group of $f(x,t)$ over $\overline{\mathbb{Q}}(t)$ is A_5. First, $f(x,t)$ is irreducible over $\overline{\mathbb{Q}}(t)$ since it is an Eisenstein polynomial with respect to $t\sqrt{5}+1$ as a polynomial over $\overline{\mathbb{Q}}(t)$. Since the discriminant is a square in $\overline{\mathbb{Q}}(t)$, the Galois group is a subgroup of A_5. It is enough to show that the following sextic resolvent of $f(x,t)$ has no root in $\overline{\mathbb{Q}}(t)$:

$$\theta(y) = (y^3 + b_2 y^2 + b_4 y + b_6)^2 - 2^{10} \operatorname{disc}(f(x,t)) y \tag{8.1}$$

where $b_2 = -100(5t^2-1)$, $b_4 = 6000(5t^2-1)^2$ and $b_6 = 4000(5t^2-1)^3$. If $\theta(y)$ has a root α in $\overline{\mathbb{Q}}(t)$, we have $(\alpha^3 + b_2\alpha^2 + b_4\alpha + b_6)^2 = 2^{10}\operatorname{disc}(f(x,t))\alpha$. Hence α should be a square in $\overline{\mathbb{Q}}(t)$. Since α is a divisor of b_6^2, the possible degrees of α are 0, 2, 4 and 6. When the degree is 0, then such an α cannot be a root of $\theta(y)$. If the degree is 4 or 6, then the degree of the left-hand side and the degree of the right-hand side in (8.1) do not match. Hence the only possible forms of α are $a(t\sqrt{5}+1)^2$ and $a(t\sqrt{5}-1)^2$ for some algebraic number $a \in \overline{\mathbb{Q}}$. With the help of a computer algebra system such as PARI, we can check that each of these cannot be a root of $\theta(y)$. Hence, the splitting field of $f(x,t)$ over $\mathbb{Q}(t)$ is an A_5 regular extension.

If $K_t = \mathbb{Q}[\alpha_t]$ for $t \in \mathbb{Z}$, then K_t has signature $(1,2)$. Let $\widehat{K_t}$ be the Galois closure, and let $G = \operatorname{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq A_5$. Then G has a subgroup H isomorphic to A_4 such that $\widehat{K_t}^H = K_t$. Let $\operatorname{Ind}_H^G 1_H = 1 + \rho$ be the induced representation of G by the trivial representation of H, where ρ is the 4-dimensional representation of A_5, so that $L(s,\rho,t) = \zeta_{K_t}(s)/\zeta(s)$. Now, by [Kim04, p. 498], ρ is equivalent to a twist of $\sigma \otimes \sigma^\tau$ by a character, where σ and σ^τ are the icosahedral 2-dimensional representations of $\tilde{A}_5 \simeq \mathrm{SL}_2(\mathbb{F}_5)$. Since K_t is not totally real, σ and σ^τ are odd. Hence, by [KW09, Corollary 10.2], σ and σ^τ are modular, i.e. they are attached to cuspidal representations π and π^τ of $\mathrm{GL}_2/\mathbb{Q}$. By [Ram00], the functorial product $\pi \boxtimes \pi^\tau$ is a cuspidal representation of $\mathrm{GL}_4/\mathbb{Q}$. Hence $L(s,\rho,t)$ is a cuspidal automorphic L-function of $\mathrm{GL}_4/\mathbb{Q}$.

Let

$$L(X) = \left\{ \frac{X}{2} < t < X \;\middle|\; 5t^2-1 \text{ square-free, } t \text{ even, } t \equiv i_M \pmod{M} \right\}.$$

We now prove Assumptions 4.1 and 4.2. Since $5t^2-1$ is square-free, for every prime divisor p of $5t^2-1$, $f(x,t)$ is an Eisenstein polynomial with respect to p and p does not divide the index of α_t. This implies that d_{K_t} is divisible by $(5t^2-1)^4$, and Assumption 4.1 is satisfied. To verify Assumption 4.2, by the remark after Assumption 4.2, it suffices to show that the K_t are not isomorphic; for this, we prove that the totally ramified primes p in K_t are exactly prime divisors of $5t^2-1$. Since $f(x,t)$ is an Eisenstein polynomial with respect to each prime divisor of $5t^2-1$, p is totally ramified in K_t. Conversely, assume that a prime p is totally ramified and is not a prime divisor of $5t^2-1$. If $p=2$, then $f(x,t) \equiv x(x^4+1) \bmod 2$ since t is even. Hence p is not totally ramified. Now assume that $p \neq 2$ and p is not a prime divisor of $5t^2-1$. Then we should have $f(x,t) \equiv (x+a)^5 \bmod p$ with $a \not\equiv 0 \bmod p$. This forces $p=5$ and $a=4$. However, by the Newton polygon method, we see that $5\mathbb{Z}_{K_t} = \mathfrak{p}_1\mathfrak{p}_2^4$ with two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

We can verify (5.3) as in the cubic field case, and thus Theorems 4.6 and 4.7 are valid.

9. A_4 Galois extensions

Consider the following polynomials from [CKb, Smi00]:

$$\mathfrak{K}(4, A_4, 0, 2) : f(x,t) = x^4 - 8tx^3 + 18t^2x^2 + 1$$
$$\mathfrak{K}(4, A_4, 4, 0) : f(x,t) = x^4 + 18tx^3 + (81t^2+2)x^2 + 2t(54t^2+1)x + 1.$$

These polynomials generate regular Galois extensions, and the strong Artin conjecture is true. In the first case, Assumption 4.1 was verified in [CKb]. The second case is similar. Note that $\operatorname{disc}(f(x,t)) = 16^2t^2(27t^2-4)^2(27t^2+4)^2$; so if t is square-free, then the Newton polygon argument shows that $t \mid d_{K_t}$. (If $p \mid t$, then $p\mathbb{Z}_{K_t} = \mathfrak{p}^2$ for a prime ideal $\mathfrak{p}$.) Hence $\log d_{K_t} \gg \log |t|$. We define $L(X)$ as follows:

$$\mathfrak{K}(4, A_4, 0, 2) : L(X) = \left\{ \frac{X}{2} < t < X \,\middle|\, 27t^4 + 1 \text{ cube-free and } t \equiv i_M \pmod{M} \right\}$$

$$\mathfrak{K}(4, A_4, 4, 0) : L(X) = \left\{ \frac{X}{2} < t < X \,\middle|\, t \text{ square-free and } t \equiv i_M \pmod{M} \right\}.$$

In the case of $\mathfrak{K}(4, A_4, 0, 2)$, (5.3) is verified using [CKb, Lemma 8.3]. The case of $\mathfrak{K}(4, A_4, 4, 0)$ is similar to the quadratic case.

Assumption 4.2 remains to be proved. Hence Theorems 4.6 and 4.7 are valid modulo Assumption 4.2.

10. Conditional result under zero-density hypothesis

Up to now, we have obtained the average value of logarithmic derivatives of Artin L-functions in a family. In this section, we assume a zero-density hypothesis and evaluate the logarithmic derivative of a single Artin L-function.

We use the same notation as in § 4: let $f(x, t) \in \mathbb{Z}[t][x]$ be an irreducible polynomial of degree n whose splitting field over $\mathbb{Q}(t)$ is a regular extension with Galois group G. Let K_t and $\widehat{K_t}$ be as in § 4. Let $L(s, \rho, t) = \sum_{l=1}^{\infty} \lambda(l, t) l^{-s}$ be the Artin L-function $\zeta_{K_t}(s)/\zeta(s)$. For simplicity of notation, let $L(s, \rho) = L(s, \rho, t)$, $\lambda(p) = \lambda(p, t)$ and $N = |d_{K_t}|$.

If we assume the Artin conjecture and GRH for $L(s, \rho)$, then by [Duk03] we get

$$\frac{L'}{L}(1, \rho) = -\sum_{p \leqslant (\log N)^{2+\epsilon}} \frac{\lambda(p) \log p}{p} + O_{n,x}(1).$$

We show under a certain zero-density hypothesis (Conjecture 10.4) that if $w = (\log N)(\log \log N)^2$ and $x = (\log N)^{2+\epsilon}$, then

$$\sum_{w < p < x} \frac{\lambda(p) \log p}{p} = O(1). \tag{10.1}$$

Proof of (10.1). By partial summation,

$$\sum_{w<p<x} \frac{\lambda(p) \log p}{p} = -\frac{1}{w} \sum_{p<w} \lambda(p) \log p + \frac{1}{x} \sum_{p<x} \lambda(p) \log p + \int_w^x \frac{\sum_{p<u} \lambda(p) \log p}{u^2} \, du. \tag{10.2}$$

Then $(1/w) \sum_{p<w} \lambda(p) \log p = O(1)$, and $(1/x) \sum_{p<x} \lambda(p) \log p = O(1)$ since $\sum_{p<x} \log p = O(x)$.

Let $\varrho = \beta + i\gamma$ run over the zeros of $L(s, \rho)$ in the critical strip of height up to T, with $1 \leqslant T \leqslant u$. Then, by [IK04, p. 112],

$$\psi(\rho, u) = \sum_{n \leqslant u} \lambda(n) \Lambda(n) = -\sum_{|\gamma| \leqslant T} \frac{u^{\varrho} - 1}{\varrho} + O\left(\frac{u \log u}{T} \log(u^{n-1} N) \right). \tag{10.3}$$

Here $\psi(\rho, u) = \sum_{p \leqslant u} \lambda(p) \log p + \sum_{p^k \leqslant u,\, k \geqslant 2} \lambda(p^k) \log p$. Since $\lambda(l) \leqslant d_{n-1}(l)$, where $\zeta(s)^{n-1} = \sum_{l=1}^{\infty} d_{n-1}(l) l^{-s}$ and $d_{n-1}(l) \leqslant d(l)^{n-1}$, we have $\lambda(p^k) \leqslant (k+1)^{n-1}$. Hence

$$\sum_{p^k \leqslant u, k \geqslant 2} \lambda(p^k) \log p \ll \sum_{p \leqslant \sqrt{u}} \log p \sum_{k < (\log u)/(\log p)} (k+1)^{n-1} \ll \sqrt{u} (\log u)^n.$$

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So this error term contributes $O(1)$ to the integral in (10.2). Therefore, we can use $\psi(\rho, u)$ in the integral in (10.2). We apply (10.3) with $T = (\log N)(\log\log N)^2$. The error term $O(((u \log u)/T)\log(u^{n-1}N))$ gives rise to

$$\int_w^x \left((n-1)\frac{u(\log u)^2}{(\log N)(\log\log N)^2} + \frac{u\log u}{(\log\log N)^2}\right)\frac{du}{u^2},$$

which is $O(1)$.

The sum $\sum_{|\gamma|\leqslant T} 1/\varrho$ is bounded by $(\log N)\sum_{k=1}^{T} 1/k \ll \log N \log T$ and gives rise to

$$(\log N)(\log T)\int_w^x \frac{du}{u^2} \ll \frac{\log N \log T}{w} = O(1).$$

Now we assume the following zero-density hypothesis for $L(s, \rho)$ (cf. [Roj03, p. 6]).

CONJECTURE 10.4. For $u \geqslant (\log(n-1)\log N)^\kappa$ and $\kappa \geqslant 1$,

$$\sum_{|\gamma|\leqslant T} \frac{u^\varrho}{\varrho} \leqslant u^{1-c/(\log(n-1)\log N)^\kappa}\, T^{d/(\log(n-1)\log N)^\kappa}$$

for some positive constants c and d which are independent of $L(s, \rho)$.

Remark 10.5. Conjecture 10.4 follows from the GRH if u is large. However, if u is small, of size $(\log N)^a$ with $a \leqslant 2$, which is under consideration, it does not follow from the GRH.

If $T = (\log N)(\log\log N)^2$, then

$$T^{d/(\log(n-1)\log N)^\kappa} = O(1).$$

Let $b = c/(\log((n-1)\log N)^\kappa$; then, under Conjecture 10.4,

$$\int_w^x \left(\sum_{|\gamma|\leqslant T}\frac{u^\varrho}{\varrho}\right)\frac{du}{u^2} \ll \int_w^x u^{-1-b}\,du \ll w^{-b} = O(1).$$

Hence, under the zero-density hypothesis, we have proved (10.1). □

Since

$$\sum_{(\log N)/(\log\log N)\leqslant p\leqslant w} \frac{\log p}{p} \ll \log\log\log N,$$

we have

$$\frac{L'}{L}(1, \rho) = -\sum_{c_f\leqslant p<(\log N)/(\log\log N)} \frac{\lambda(p)\log p}{p} + O(\log\log\log N). \tag{10.6}$$

Because $-1 \leqslant \lambda(p) \leqslant n-1$, we have the following result.

THEOREM 10.7. *Under the Artin conjecture, the GRH and Conjecture 10.4 for $L(s, \rho)$, the upper and lower bounds for $\frac{L'}{L}(1, \rho, t)$ are*

$$\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|), \quad -(n-1)\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|),$$

respectively.

For $X > 0$, let $y = (\log X)/100$ and define M, i_M and s_M as in § 4. So for all $c_f \leqslant p \leqslant y$, if $t \equiv s_M \pmod M$, then p splits completely in $\widehat{K_t}$ and $\lambda(p, t) = n-1$; if $t \equiv i_M \pmod M$, then $\lambda(p,t) = -1$. Assume that the discriminant of $f(x,t)$ is a polynomial in t of degree D. Then there is a constant C such that $|d_{K_t}| \leqslant Ct^D$. So $\log|d_{K_t}| \ll \log t$. We define a set $L(X)_i$ by

$$L(X)_i = \left\{ \frac{X}{2} < t < X \;\middle|\; t \equiv i_M \pmod{M},\ \mathrm{Gal}(\widehat{K_t}/\mathbb{Q}) \simeq G \right\}.$$

We define $L(X)_s$ similarly. Note that for $X/2 < t < X$, $(\log|d_{K_t}|)/(\log\log|d_{K_t}|) \leqslant y = (\log X)/100$ for sufficiently large X. Hence we can control $\lambda(p)$ for $c_f \leqslant p < (\log N)/(\log\log N)$. In particular, for $t \in L(X)_s$, $\lambda(p) = n-1$; for $t \in L(X)_i$, $\lambda(p) = -1$. Therefore we have proved the following theorem.

THEOREM 10.8. *Under the Artin conjecture, the GRH and Conjecture 10.4 for $L(s,\rho,t)$, for all $t \in L(X)_i$ we have $\frac{L'}{L}(1,\rho,t) = \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|)$, and for all $t \in L(X)_s$ we have $\frac{L'}{L}(1,\rho,t) = -(n-1)\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|)$.*

Let us assume the strong Artin conjecture for $L(s,\rho,t)$ instead of the Artin conjecture and GRH. In addition, suppose that Assumptions 4.1 and 4.2 are true. Then, by applying Theorem 3.4 to $L(X)_s$ and $L(X)_i$, we obtain sets $\widehat{L}(X)_s$ and $\widehat{L}(X)_i$ where every automorphic L-function has a zero-free region $[\alpha,1] \times [-(\log|d_{K_t}|)^2, \log|d_{K_t}|)^2]$. Then we have the following result.

THEOREM 10.9. *Under the strong Artin conjecture, Assumptions 4.1 and 4.2 and Conjecture 10.4, for $t \in \widehat{L}(X)_s$ we have $\frac{L'}{L}(1,\rho,t) = -(n-1)\log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|)$, and for $t \in \widehat{L}(X)_i$ we have $\frac{L'}{L}(1,\rho,t) = \log\log|d_{K_t}| + O(\log\log\log|d_{K_t}|)$.*

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REFERENCES

Cho P. J. Cho, *The strong Artin conjecture and number fields with large class numbers*, Quart. J. Math., to appear.

CKa P. J. Cho and H. H. Kim, *Dihedral and cyclic extensions with large class numbers*, J. Théor. Nombres Bordeaux, to appear.

CKb P. J. Cho and H. H. Kim, *Application of the strong Artin conjecture to class number problem*, Canad. J. Math., to appear.

CKc P. J. Cho and H. H. Kim, *Weil's theorem on rational points over finite fields and Artin L-functions*, Conference Proceedings of Tata Institute, to appear.

Coh93 H. Cohen, *A course in computational algebraic number theory* (Springer, Berlin, 1993).

Dai06 R. C. Daileda, *Non-abelian number fields with very large class numbers*, Acta Arith. **125** (2006), 215–255.

Duk03 W. Duke, *Extreme values of Artin L-functions and class numbers*, Compositio Math. **136** (2003), 103–115.

Duk04 W. Duke, *Number fields with large class groups*, in *Number theory*, CRM Proceedings and Lecture Notes, vol. 36 (American Mathematical Society, Providence, RI, 2004), 117–126.

Iha06 Y. Ihara, *On the Euler–Kronecker constants of global fields and primes with small norms*, in *Algebraic geometry and number theory*, Progress in Mathematics, vol. 253 (Birkhäuser, Boston, MA, 2006), 407–451.

IMS09 Y. Ihara, V. K. Murty and M. Shimura, *On the logarithmic derivatives of Dirichlet L-functions at* $s = 1$, Acta Arith. **137** (2009), 253–276.

IK04 H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53 (American Mathematical Society, Providence, RI, 2004).

KW09 C. Khare and J. P. Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), 485–504.

Kim04 H. H. Kim, *An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type*, Invent. Math. **156** (2004), 495–502.

Kli98 N. Klingen, *Arithmetical similarities*, Oxford Mathematical Monographs (Oxford University Press, New York, 1998).

KM02 E. Kowalski and P. Michel, *Zeros of families of automorphic L-functions close to* 1, Pacific J. Math. **207** (2002), 411–431.

MM M. Mourtada and V. K. Murty, *An omega theorem for* $\frac{L'}{L}(1, \chi_D)$, Int. J. Number Theory, to appear.

Nai76 M. Nair, *Power free values of polynomials*, Mathematika **23** (1976), 159–183.

Ram00 D. Ramakrishnan, *Modularity of the Rankin-Selberg L-series, and multiplicity one for* SL(2), Ann. of Math. **152** (2000), 45–111.

Roj03 M. Rojas, *Dedekind zeta functions and the complexity of Hilbert's Nullstellensatz*, Preprint (2003), arXiv:math/0301111.

Sch04 W. Schmidt, *Equations over finite fields: an elementary approach* (Kendrick Press, Heber City, UT, 2004).

Ser03 J. P. Serre, *On a theorem of Jordan*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), 429–440.

Ser08 J. P. Serre, *Topics in Galois theory*, Research Notes in Mathematics (A K Peters, Wellesley, MA, 2008).

Smi00 G. W. Smith, *Polynomials over* $\mathbb{Q}(t)$ *and their Galois groups*, Math. Comp. **230** (2000), 775–796.

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