A THEOREM ON A FINITE DIFFERENCE OPERATOR AND ITS CONNECTION WITH THE POISSON DISTRIBUTION

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The well-known Taylor expansion of a function around a point a can be formally written as

$$(1.1)\qquad f(a+x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\frac{d}{da}\right)^n f(a) = e^{x \cdot d/da} \cdot f(a).$$

The last expression is just a symbolic form and is valid, as we know, under certain restrictive conditons. We shall study the situation when the differential operator d/da is replaced by the finite difference operator Δ_h/h, where the operator Δ_h is defined by

$$\Delta_h f(a) = f(a+h) - f(a).$$

In general,

$$\begin{aligned}(\Delta_h)^n f(a) = \Delta_h^n f(a) &= \Delta_h[\Delta_h^{n-1} f(a)] \\ &= f(a+nh) - \binom{n}{1} f(a+(n-1)h) + \ldots + (-1)^n f(a).\end{aligned}$$

Then we have the following theorem.

THEOREM 1. *If the function f is continuous and bounded for $0 \leqq x < \infty$, $0 \leqq a < \infty$, then $e^{x \cdot \Delta_h/h} f(a)$ tends uniformly to $f(a+x)$ throughout any finite intervals of values of a and x, as h tends to zero. That is to say,*

$$(1.2)\qquad \lim_{h \to 0} e^{x \cdot \Delta_h/h} f(a) = f(a+x).$$

The above theorem has been proved by Hille [3] as a special case of a theorem on semigroups. We give an independent proof following Bernstein's method [1] of proving Weierstrass' theorem on the approximation of continuous functions of polynomials. We also point out an interpretation of the theorem in terms of the Poisson distribution.

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Let us introduce the translation operator T_h defined by

$$T_h f(t) = f(t+h); \quad (T_h)^n = T_h^n f(t) = f(t+nh).$$

We observe that

(2.1) $$\Delta_h = T_h - 1.$$

From (1.2) and (2.1) we get

$$e^{x \cdot \Delta_h/h} f(a) = e^{x/h(T_h-1)} f(a) = e^{-x/h} e^{x/h \cdot T_h} f(a) = e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k T_h^k f(a)$$

$$= e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k f(a+kh).$$

Thus the contention of Theorem 1 becomes equivalent to the following theorem:

THEOREM 2. *The infinite series*

$$e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k f(a+kh)$$

tends uniformly to $f(a+x)$ throughout any finite intervals of values of a and x, as h tends to zero.

PROOF. We have the identiy

(2.2) $$e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k = e^{-x/h} e^{x/h} = 1.$$

Taking two successive derivatives we get

(2.3) $$e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k (x-kh) = 0,$$

(2.4) $$e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k (x-kh)^2 = xh.$$

From (2.2) we also get the identity

$$f(a+x) = e^{-x/h} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x}{h}\right)^k f(a+x).$$

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We have

$$\left|f(a+x)-e^{-x/h}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{h}\right)^k f(a+kh)\right|$$
$$= e^{-x/h}\left|\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{h}\right)^k [f(a+x)-f(a+kh)]\right|$$
$$\leqq e^{-x/h}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{h}\right)^k |f(a+x)-f(a+kh)|.$$

Let us divide the sum into two parts, namely $\sum_{k'}$ corresponding to the k's satisfying $|x-kh| \leqq \delta$ and $\sum_{k''}$ to the k's satisfying $|x-kh| > \delta$, where δ is determined by the condition that $|f(a+x)-f(a+kh)| < \frac{1}{2}\varepsilon$ for $|x-kh| \leqq \delta$. Then we have

$$e^{-x/h}\sum_{k'}\frac{1}{k!}\left(\frac{x}{h}\right)^k |f(a+x)-f(a+kh)| < \frac{\varepsilon}{2}\, e^{-x/h}\sum_{k'}\frac{1}{k!}\left(\frac{x}{h}\right)^k$$
$$\leqq \frac{\varepsilon}{2}\, e^{-x/h}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{h}\right)^k = \varepsilon/2.$$

From (2.4) we get

$$e^{-x/h}\sum_{k''}\frac{1}{k!}\left(\frac{x}{h}\right)^k \delta^2 < xh$$

and

(2.5) $$e^{-x/h}\sum_{k''}\frac{1}{k!}\left(\frac{x}{h}\right)^k < \frac{xh}{\delta^2}.$$

Also as f is bounded, $|f| < M$, we get by (2.5)

$$e^{-x/h}\sum_{k''}\frac{1}{k!}\left(\frac{x}{h}\right)^k |f(a+x)-f(a+kh)| < 2M e^{-x/h}\sum_{k''}\frac{1}{k!}\left(\frac{x}{h}\right)^k < 2M\frac{xh}{\delta^2} < \frac{\varepsilon}{2}$$

if $h < \varepsilon\delta^2/4Mx$. Thus it follows that as h tends to zero

$$\left|f(a+x)-e^{-x/h}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x}{h}\right)^k f(a+kh)\right|$$

tends uniformly to zero in the intervals considered. This completes the proof of Theorem 2 and hence of Theorem 1.

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We discuss the probabilistic interpretation of the theorems which we write as

(3.1) $$\lim_{h\to 0}\sum_{k=0}^{\infty}\frac{(x/h)^k}{k!}e^{-x/h}f(a+kh) = f(a+x).$$

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Define

$$p_k = \frac{(x/h)^k}{k!} e^{-x/h} \tag{3.2}$$

as the probability for the variable t to assume the special value $t_k = kh$. Then we can write (3.1) as

$$\lim_{h\to 0} \sum_{k=0}^{\infty} p_k f(a+t_k) = f(a+x) \tag{3.3}$$

and we can write (2.3) and (2.4) as

$$\sum_{k=0}^{\infty} p_k t_k = x \tag{3.4}$$

and

$$\sum_{k=0}^{\infty} p_k (t_k - x)^2 = xh \tag{3.5}$$

respectively.

Now (3.2) defines a Poisson distribution, for by putting $x/h = \lambda$ we get

$$p_k = \frac{\lambda^k}{k!} e^{-\lambda}.$$

Now (3.4) shows that the mathematical expectation of the discrete values t_k is x. Thus x is the mean value of the variable t.

As for (3.3) this equation tells us that as h tends to zero, that is to say, as the "points of interpolation" $t_k = kh$ are chosen increasingly close to each other, the mathematical expectation of the discrete function values $f(a+t_k)$ tends to the definite value $f(a+x)$. The equation (3.5) gives the variance of the variable t. Thus $\sigma^2 = xh$, and this shows how the standard deviation decreases as we choose closer points of interpolation on the t-axis.

Now Tchebycheff's inequality [2] states that

$$P(|t-x| > A\sigma) < 1/A^2$$

where $P(|t-x| > A\sigma)$ denotes the probability that the variable t should differ from its mean x by a quantity of modulus $> A\sigma$. For our special distribution we find

$$\sum_{k''} \frac{(x/h)^k}{k!} e^{-x/h} = \sum_{k''} p_k = P(|t-x| > \delta) < \frac{\sigma^2}{\delta^2} = \frac{xh}{\delta^2}$$

for $\sigma^2 = xh$, in accordance with (3.5). Thus (2.5) can be regarded as a special case of Tchebycheff's general inequality.

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