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Deformations of Codimension 2 Toric Varieties

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Abstract. We prove Sturmfels' conjecture that toric varieties of codimension two have no other flat deformations than those obtained by Gröbner basis theory.

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1. Introduction

The properties of ideals with a fixed Hilbert function have been studied extensively; the most recent papers are [HP,G]. We study when an ideal has the same multigraded Hilbert function as a given toric ideal.

Let n and d be positive integers with $n > d$ and $\mathcal{A} = \{a_1, \ldots, a_n\}$ a subset of $\mathbf{N}^d \setminus \{\mathbf{0}\}$ with n different vectors. Let A be the matrix with columns a_i and suppose that $\mathrm{rank}(A) = d$. Denote by $\mathbf{N}\mathcal{A}$ the subsemigroup of $\mathbf{N}^d$ spanned by $\mathcal{A}$. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k generated by variables $x_1, \ldots, x_n$ in $\mathbf{N}^d$-degrees $a_1, \ldots, a_n$, respectively. A homogeneous ideal M is called $\mathcal{A}$-*graded* if for all $b \in \mathbf{N}^d$

$$\dim_k((S/M)_b) = \begin{cases} 1 & \text{if } b \in \mathbf{N}\mathcal{A}\,, \\ 0 & \text{otherwise.} \end{cases}$$

This means that S/M has the same multigraded Hilbert function as the *toric ring* $S/I_{\mathcal{A}}$, where $I_{\mathcal{A}}$ is the *toric ideal* equal to the kernel of the homomorphism $k[x_1, \ldots, x_n] \to k[t_1, \ldots, t_d]$ mapping x_i to $\mathbf{t}^{a_i} = t_1^{a_{i1}} \ldots t_d^{a_{id}}$ for $1 \leqslant i \leqslant n$. The paradigms of $\mathcal{A}$-graded ideals are the toric ideal and its initial ideals. An $\mathcal{A}$-graded ideal M is called *coherent* if there exist $w \in \mathbf{Q}^n$ and $(c_1, \ldots, c_n) \in (k^*)^n$ such that the ideal $\big(f(c_1x_1, \ldots, c_nx_n) \,|\, f \in M\big)$ equals the initial ideal $\mathrm{in}_w(I_{\mathcal{A}})$ of $I_{\mathcal{A}}$ with respect to the monomial order defined by the weight vector w. If M is an initial ideal of $I_{\mathcal{A}}$, then a construction from Gröbner Bases Theory gives a flat family such that the fiber over 1 is the toric ring $S/I_{\mathcal{A}}$ and the fiber over 0 is S/M. What are the other deformations of $I_{\mathcal{A}}$? The study of $\mathcal{A}$-graded ideals was initiated by Arnold [Ar], who realized that in the case $d = 1, n = 3$ the structure of such ideals is encoded

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into continued fractions. Further work in this case was done by Korkina, Post, and Roelofs [Ko, KPR].

THEOREM 1.1 ([Ar, Ko, KPR]). *If $d = 1$ and $n = 3$ then every $\mathcal{A}$-graded ideal is coherent.*

The codimension of $I_{\mathcal{A}}$ is $n - d$. In view of Lee's result that $\mathcal{A}$ has only coherent triangulations if $n - d = 2$, it is conjectured in [St1, 6.1]:

CONJECTURE 1.2 (Sturmfels 1994). *If* $\operatorname{codim}(I_{\mathcal{A}}) = 2$*, then every $\mathcal{A}$-graded ideal is coherent.*

This conjecture provides description of the structure of the $\mathcal{A}$-graded ideals and shows that the isomorphism classes of $\mathcal{A}$-graded ideals are in bijection with the vertices of the state polytope. The first example of a noncoherent $\mathcal{A}$-graded ideal was found by Eisenbud; through a systematic computer search Sturmfels [St2, Theorem 10.4] found that $(x_1^3, x_1x_2, x_2^2, x_2x_3, x_1x_4, x_1^2x_3^2, x_1x_3^4, x_2x_4^3, x_4^4)$ is a noncoherent $\mathcal{A}$-graded monomial ideal for $\mathcal{A} = \{1, 3, 4, 7\}$ and in this case $\operatorname{codim}(I_{\mathcal{A}}) = 3$. So the above conjecture cannot be extended to codimensions higher than two.

Our paper is devoted to a proof of Conjecture 1.2. The arguments in [Ar, Ko, KPR] cannot be applied for $n \geqslant 4$; some of the difficulties when $n \geqslant 4$ are outlined in [KPR, Section 8]. Our argument is broken into many steps and each step is presented in a lemma. It involves techniques from [Ar] and [PS], and relies on a detailed analysis of the syzygies of the toric ideal $I_{\mathcal{A}}$ and the syzygies of its Lawrence lifting ideal.

2. Criterion for Coherence

Fix a set $\mathcal{A}$ and denote $I = I_{\mathcal{A}}$. In this section we provide two tools for the proof of Conjecture 1.2: Lemma 2.1 gives a criterion for weak $\mathcal{A}$-gradedness and Lemma 2.2 gives a criterion for coherence. We also recall the construction of Lawrence lifting.

We say that a homogeneous ideal M is *weakly $\mathcal{A}$-graded* if for all $b \in \mathbf{N}^d$

$$\dim_k((S/M)_b) \leqslant \begin{cases} 1 & \text{if } b \in \mathbf{N}\mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that a weakly $\mathcal{A}$-graded ideal is generated by binomials (that is polynomials with at most two terms). Our first lemma shows that a weakly $\mathcal{A}$-graded ideal is generated by special binomials. A binomial $\mathbf{x}^u - \mathbf{x}^v$ in the toric ideal I is called *primitive* if there are no proper monomial factors $\mathbf{x}^{u'}$ of $\mathbf{x}^u$ and $\mathbf{x}^{v'}$ of $\mathbf{x}^v$ such that $\mathbf{x}^{u'} - \mathbf{x}^{v'} \in I$. The set of all primitive binomials is finite and is called the *Graver basis.*

LEMMA 2.1 [PS2]. *Let M be an ideal in S. The following are equivalent:*

(a) *The ideal M is weakly $\mathcal{A}$-graded.*
(b) *If $\mathbf{x}^u - \mathbf{x}^v$ is a primitive binomial in I then either M contains at least one of the monomials $\mathbf{x}^u$ and $\mathbf{x}^v$ or there is a $p \in k \setminus 0$ such that $\mathbf{x}^u - p\mathbf{x}^v \in M$.*

The Graver basis in the case $d = 1, n = 3$ considered by [Ar,Ko,KPR] is the *star*, see [Ko, Definition 2.9]; in this case Lemma 2.1 corresponds to [Ko, 2.10].

Until the end of this section we will assume that $n - d = 2$, i.e. $\operatorname{codim}(I) = 2$.

A vector $u \in \mathbf{Z}^n$ can be written uniquely as $u = u_+ - u_-$, where u_+ and u_- have nonnegative coordinates and $\operatorname{supp}(u_+) \cap \operatorname{supp}(u_-) = \emptyset$ (here $\operatorname{supp}(u) = \{i \mid \text{the } i\text{th coordinate of } u \text{ is not } 0\}$). Let $B = (b_{ij})$ be an integer $(n \times 2)$-matrix such that the following sequence is exact

$$0 \to \mathbf{Z}^2 \xrightarrow{B} \mathbf{Z}^n \xrightarrow{A} \mathbf{Z}^d .$$

Each vector α in $\mathbf{Z}^2$ corresponds to a binomial $\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-}$ in I, and every binomial in I without monomial factors can be represented uniquely in this way.

LEMMA 2.2. *Let* $\operatorname{codim}(I) = 2$ *and M be an $\mathcal{A}$-graded ideal in S. Let $\mathcal{T} \subset \mathbf{Z}^2$ be a set of vectors with the property that for some nonzero-vector $s \in \mathbf{Q}^2$ we have $\langle s, \alpha\rangle \geqslant 0$ for any $\alpha \in \mathcal{T}$. Set*

$$M' = \Big(\{\mathbf{x}^{(B\alpha)_+} | \alpha \in \mathcal{T}, \langle s, \alpha\rangle > 0\} \cup \{\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-} | \alpha \in \mathcal{T}, \langle s, \alpha\rangle = 0\}\Big).$$

If M' is weakly $\mathcal{A}$-graded and $M' \subseteq M$, then $M' = M$ and M is coherent.

Proof. Let $\alpha \in \mathcal{T}$. Let $w \in \mathbf{Q}^n$ be such that $s = B^T w$ (here B^T is the transpose of B). Then

$$\langle w, (B\alpha)_+\rangle > \langle w, (B\alpha)_-\rangle \quad \text{if and only if} \quad \langle s, \alpha\rangle > 0,$$

$$\langle w, (B\alpha)_+\rangle = \langle w, (B\alpha)_-\rangle \quad \text{if and only if} \quad \langle s, \alpha\rangle = 0.$$

Therefore,

$$in_w(\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-}) = \begin{cases} \mathbf{x}^{(B\alpha)_+} & \text{if } \langle s, \alpha\rangle > 0, \\ \mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-} & \text{if } \langle s, \alpha\rangle = 0. \end{cases}$$

By the definition of M' it follows that $M' \subseteq \operatorname{in}_w(I)$. As M' is weakly $\mathcal{A}$-graded and $\operatorname{in}_w(I)$ is $\mathcal{A}$-graded, it follows that $M' = \operatorname{in}_w(I)$. On the other hand, $M' \subseteq M$ and M is $\mathcal{A}$-graded. Hence $M' = M$ and M is coherent. □

We remark that by [St2, Proposition 1.12] if $w \in \mathbf{Q}^n$ then there exists a $w' \in \mathbf{Q}^n$ with positive coordinates such that $\operatorname{in}_{w'}(I) = \operatorname{in}_w(I)$. Thus, in the definition of coherence and in the proofs we do not need to require that the weight vector has positive coordinates.

By [PS, Remark 3.2 and Theorem 3.7], we can choose the matrix B so that the binomials corresponding to $(1, 0)$ and $(0, 1)$ are minimal generators of I. By [PS, Theorem 6.1] I is a complete intersection exactly when I is minimally generated by two elements. If I is not a complete intersection, then by [PS, Theorem 3.7] the ideal I has a unique minimal system of $\mathbf{N}^d$-homogeneous binomial generators (up to multiplying each binomial with ± 1). We call a vector $\alpha \in \mathbf{Z}^2$ *generating* if one of the following conditions is satisfied:

(1) I is a complete intersection and $\alpha \in \{\pm(1, 0), \pm(0, 1)\}$;
(2) I is not a complete intersection and the binomial corresponding to α is contained in a minimal system of generators of I.

We call α *primitive* if its binomial is primitive.

We need to recall the construction of Lawrence lifting. Let L be the matrix $\begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$, where $\mathbf{1}$ is the $(n \times n)$-identity matrix and $\mathbf{0}$ is the $(d \times n)$-zero matrix. The matrix L is called the *Lawrence lifting* of A, and the toric ideal I_L is called the Lawrence lifting of I. Then I_L is the ideal in $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ generated by $\{\mathbf{x}^u\mathbf{y}^v - \mathbf{x}^v\mathbf{y}^u \mid \mathbf{x}^u - \mathbf{x}^v \in I\}$ and $\operatorname{codim}(I_L) = 2$.

LEMMA 2.3. *The elements $y_n - 1, \ldots, y_1 - 1$ form a $k[x_1, \ldots, x_n, y_1, \ldots, y_n]/I_L$-regular sequence.*

Proof. Fix an $1 \leqslant i \leqslant n - 1$. Denote by T the ideal in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_i]$ such that

$$k[x_1, \ldots, x_n, y_1, \ldots, y_i]/T = k[x_1, \ldots, x_n, y_1, \ldots, y_n]/\big(I_L + (y_{i+1} - 1, \ldots, y_n - 1)\big).$$

The ring $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ is $\mathbf{N}^{d+n}$-graded with the degrees of the variables given by the columns of the matrix L. Deleting the last $n - i$ coordinates in $\mathbf{N}^{d+n}$ we induce an $\mathbf{N}^{d+i}$-grading in which $\deg(y_{i+1}) = \ldots = \deg(y_n) = 0$. This induces an $\mathbf{N}^{d+i}$-grading on $k[x_1, \ldots, x_n, y_1, \ldots, y_i]$ and the ideal T is $\mathbf{N}^{d+i}$-homogeneous. The elements 1 and y_i have different degrees. Therefore, if f is a polynomial and $(y_i - 1)f \in T$, then $f \in T$. □

By [St2, Theorem 7.1] I_L has a unique system of minimal homogeneous binomial generators. The Lawrence lifting is relevant to our work, because the images of the minimal binomial generators of I_L in $k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(y_1 - 1, \ldots, y_n - 1)$ form the Graver basis of I, see [St2, Theorem 7.1 and Algorithm 7.2]. We have the exact sequence

$$0 \to \mathbf{Z}^2 \xrightarrow{\begin{pmatrix} B \\ -B \end{pmatrix}} \mathbf{Z}^{2n} \xrightarrow{\begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}} \mathbf{Z}^{n+d}.$$

Therefore, the primitive vectors for I are exactly the generating vectors for I_L.

When we say that a vector α is a generating vector, we mean that α is a generating vector for the ideal I.

3. $\mathcal{A}$-Graded Ideals for Codimension 2 Toric Varieties

Fix a set $\mathcal{A}$, set $I = I_{\mathcal{A}}$, and denote by q the number of minimal generators of I. By I_L we denote the Lawrence lifting of I and by q_L the number of minimal generators of I_L. In this section we prove Conjecture 1.2. Throughout the section we assume that $n - d = codim(I) = 2$. We assume that the matrix B is chosen so that the binomials corresponding to $(1, 0)$ and $(0, 1)$ are minimal generators of I; such choice is possible by [PS, Remark 3.2 and Theorem 3.7].

Let M be a weakly $\mathcal{A}$-graded ideal and $\alpha \in \mathbf{Z}^2$. We say that α is an *M-vector* if $\mathbf{x}^{(B\alpha)_+} \in M$. We say that α is *M-gluing* if none of the monomials $\mathbf{x}^{(B\alpha)_+}$, $\mathbf{x}^{(B\alpha)_-}$ is in M; in this case there exists a $p_\alpha \in k \setminus 0$ such that $\mathbf{x}^{(B\alpha)_+} - p_\alpha \mathbf{x}^{(B\alpha)_-} \in M$. Note that the opposite vectors α and $-\alpha$ correspond to binomials which differ by sign only, therefore either at least one of the vectors α and $-\alpha$ is an M-vector, or α is M-gluing. Suppose that I is not a complete intersection: then by [PS, Theorem 3.4] for each homogeneous minimal binomial generator f of I there exist exactly two monomials in S of the same $\mathbf{N}^d$-degree as f (these monomials are the terms of f), hence if M is an $\mathcal{A}$-graded ideal and α is a generating non-M-gluing vector then exactly one of the vectors $\alpha, -\alpha$ is an M-vector. We say that two vectors *ill-match* if they are both non-M-gluing and exactly one of them is an M-vector. We say that two vectors α, β *well-match* if either α, β are M-vectors or $-\alpha, -\beta$ are M-vectors. Throughout the section we will work under the following assumption: if at least one of the vectors $(1, 0)$, $(0, 1)$ is not M-gluing, then after renumbering the quadrants and the basis vectors (if necessary) we have that $(0, 1)$ is an M-vector and $(1, 0)$ is either M-gluing or an M-vector.

We use the terminology from [PS] about the syzygies of I: the syzygies are represented by vectors, triangles, and quadrangles in $\mathbf{Z}^2$ with integer vertices and one vertex fixed at the origin $(0, 0)$. We say that a sequence $\mathbf{P} = P_1, \dots, P_r$ of quadrangles in the first or second quadrant is a *chain* if for $1 \leqslant i \leqslant r - 1$ the quadrangle P_{i+1} is a child of P_i in the master tree, see [PS, Construction 4.4]. For $1 \leqslant i \leqslant r$ denote by α_i, β_i the edges of P_i and by γ_i the longer diagonal of P_i. Then $\mathbf{P}$ is a chain exactly when for $1 \leqslant i \leqslant r - 1$ the edges of P_{i+1} are either α_i, γ_i or β_i, γ_i. When we say that the vectors α, β are edges of a quadrangle we always mean 'oriented edges' so that $\alpha + \beta$ is the longer diagonal of the quadrangle. For this reason, we say that the vector $(1, 1)$ is the longer diagonal of the unit square with edges $(1, 0)$, $(0, 1)$, and that the vector $(-1, 1)$ is the longer diagonal of the unit square with edges $(-1, 0)$, $(0, 1)$. The next lemma contains several results from [PS] which we will need.

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LEMMA 3.1. (a) *The ideal I is a complete intersection if and only if $q = 2$; I is not Cohen-Macaulay if and only if $q \geqslant 4$ if and only if I has a syzygy quadrangle.* (This follows from [PS, Theorem 6.1].)

(b) *Let $q \geqslant 4$ and δ be a generating vector of I in the first or second quadrant different from $\pm(1, 0), (0, 1)$. There exists a chain $P_1, \dots, P_r$ of syzygy quadrangles such that δ is the longer diagonal of P_r and P_1 is either the unit square with edges $(1, 0), (0, 1)$ or the unit square with edges $(-1, 0), (0, 1)$.* (This follows from [PS, proof of Corollary 4.6].)

(c) *Let P be a syzygy quadrangle. The edges and the diagonals of P are generating vectors. Each of the four triangles with edges the edges of P and one of the diagonals of P is a syzygy triangle* [PS, Corollary 3.6].

(d) *Suppose that $q = 3$. The generating vectors of I can be chosen to be $\pm(1, 0)$, $\pm(0, 1)$ and either $\pm(1, 1)$ or $\pm(-1, 1)$. In the former case the two triangles with edges $(1, 0), (0, 1), (1, 1)$ are syzygy triangles; in the latter case the two triangles with edges $(-1, 0), (0, 1), (-1, 1)$ are syzygy triangles* [PS, Remark 5.8].

LEMMA 3.2. *Let M be an $\mathcal{A}$-graded ideal. Let $\alpha, \beta, \eta = \alpha + \beta$ be generating vectors, which are edges of a syzygy triangle.*

(a) *If α and β are M-gluing vectors, then η is an M-gluing vector as well.*
(b) *If α is an M-gluing vector, but β is not, then η well-matches β.*
(c) *If α and β well-match, then η well-matches them.*

Proof. Since $q \geqslant 3$ it follows from Lemma 3.1(a) that I is not a complete intersection. By the construction of a syzygy triangle in [PS, (3.3), Theorem 3.4, Corollary 3.6] we can choose three monomials m_1, m_2, m_3 such that $m_2 - m_1$ is a monomial multiple of $\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-}$, $m_3 - m_2$ is a monomial multiple of $\mathbf{x}^{(B\beta)_+} - \mathbf{x}^{(B\beta)_-}$, $m_3 - m_1$ is a monomial multiple of $\mathbf{x}^{(B\eta)_+} - \mathbf{x}^{(B\eta)_-}$, and m_1, m_2, m_3 are all the monomials in S with $\mathbf{N}^d$-degree equal to the degree of a syzygy triangle (this triangle is one of the two syzygy triangles with edges α, β, η). Note that if $p_\alpha, p_\beta, p_\eta \in k \setminus 0$, then $m_2 - p_\alpha m_1$ is a monomial multiple of $\mathbf{x}^{(B\alpha)_+} - p_\alpha \mathbf{x}^{(B\alpha)_-}$, $m_3 - p_\beta m_2$ is a monomial multiple of $\mathbf{x}^{(B\beta)_+} - p_\beta \mathbf{x}^{(B\beta)_-}$, and $m_3 - p_\eta m_1$ is a monomial multiple of $\mathbf{x}^{(B\eta)_+} - p_\eta \mathbf{x}^{(B\eta)_-}$.

To prove (a) note that if α and β are M-gluing vectors but η is not, then $m_1, m_2, m_3 \in M$ contradicting the $\mathcal{A}$-gradedness of M. Next we prove (b). If η is M-gluing then we can apply (a) to $-\alpha, \eta, \beta = -\alpha + \eta$ and conclude that β is M-gluing, which is a contradiction. If η is non-M-gluing and we assume that β and η ill-match then m_1, m_2, m_3 are in M contradicting the $\mathcal{A}$-gradedness of M. So (b) is proved. It remains to prove (c). If α and β well-match and η is an M-gluing vector, then applying (b) to $\eta, -\alpha, \beta = \eta - \alpha$ we get a contradiction. Therefore, η is not M-gluing. As M is $\mathcal{A}$-graded, we have that at most two of the monomials m_1, m_2, m_3 are in M. Suppose that α and β are M-vectors. It follows that $m_2, m_3 \in M$. Therefore $m_1 \notin M$ and $m_3 - pm_1 \notin M$ for every $p \in k \setminus 0$. Hence

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$\mathbf{x}^{(B\eta)_-} \notin M$ and $\mathbf{x}^{(B\eta)_+} - p\mathbf{x}^{(B\eta)_-} \notin M$ for every $p \in k \setminus 0$. By $\mathcal{A}$-gradedness it follows that $\mathbf{x}^{(B\eta)_+} \in M$, so η is an M-vector. If $-\alpha, -\beta$ are M-vectors, then the previous argument shows that $-\eta$ is an M-vector. So η well-matches α, β. □

LEMMA 3.3. *Let M be an $\mathcal{A}$-graded ideal. Let $P_1, \dots, P_r$ be a chain of syzygy quadrangles. Denote by α, β the edges of P_1 and by δ the longer diagonal of P_r.*

(a) *If α and β are M-gluing vectors, then δ is an M-gluing vector as well.*
(b) *If α is an M-gluing vector, but β is not, then δ well-matches β.*
(c) *If α and β well-match, then δ well-matches them.*

Proof. For $1 \leqslant i \leqslant r$ denote by γ_i the longer diagonal of P_i and by α_i, β_i its edges. Note that for $1 \leqslant i \leqslant r-1$ the edges of P_{i+1} are either α_i, γ_i or β_i, γ_i. By Lemma 3.1(c) $\alpha_i, \beta_i, \gamma_i = \alpha_i + \beta_i$ are generating vectors, which are edges of a syzygy triangle. We argue by induction on i: at each step of the induction we apply Lemma 3.2. □

Next we prove Conjecture 1.2 in two special cases:

LEMMA 3.4. *Let M be an $\mathcal{A}$-graded ideal in S. Suppose that both* $(1, 0)$ *and* $(0, 1)$ *are M-gluing vectors. Then M is toric isomorphic to the toric ideal I.*

Proof. First, we will show that all generating vectors are M-gluing vectors. This is clear if $q = 2$. If $q = 3$ then apply Lemmas 3.1(d) and 3.2(a). Suppose that $q \geqslant 4$. Let δ be a generating vector in the first or second quadrant. Choose a chain $P_1, \dots, P_r$ of syzygy quadrangles as in Lemma 3.1(b), so the edges of P_1 are either $(1, 0), (0, 1)$ or $(-1, 0), (0, 1)$ and δ is the longer diagonal of P_r. The edges of P_1 are M-gluing, so applying Lemma 3.3(a) to the chain $P_1, \dots, P_r$ we get that δ is M-gluing.

For each generating vector δ let $p_\delta \in k \setminus 0$ be a constant such that $\mathbf{x}^{(B\delta)_+} - p_\delta \mathbf{x}^{(B\delta)_-} \in M$. Consider the ideal

$$M' = \left(\{ \mathbf{x}^{(B\delta)_+} - p_\delta \mathbf{x}^{(B\delta)_-} \mid \delta \text{ is a generating vector } \} \right) \subseteq M \ .$$

We will show that if $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$ are two monomials in S of the same $\mathbf{N}^d$-degree then there exists a nonzero constant p such that $\mathbf{x}^{\mathbf{u}} - p\mathbf{x}^{\mathbf{v}} \in M'$. We can write

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} = \sum_{i=1}^{s} \mathbf{x}^{\mathbf{w}_i} (\mathbf{x}^{(B\delta_i)_+} - \mathbf{x}^{(B\delta_i)_-}) \,,$$

where $\mathbf{x}^{(B\delta_i)_+} - p_{\delta_i} \mathbf{x}^{(B\delta_i)_-}$ is a minimal generator of I for $1 \leqslant i \leqslant s$, $\mathbf{x}^{\mathbf{u}} = \mathbf{x}^{\mathbf{w}_1} \mathbf{x}^{(B\delta_1)_+}$, $\mathbf{x}^{\mathbf{v}} = \mathbf{x}^{\mathbf{w}_s} \mathbf{x}^{(B\delta_s)_-}$, and $\mathbf{x}^{\mathbf{w}_i} \mathbf{x}^{(B\delta_i)_-} = \mathbf{x}^{\mathbf{w}_{i+1}} \mathbf{x}^{(B\delta_{i+1})_+}$ for $1 \leqslant i \leqslant s-1$. Now set

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$p = \prod_{i=1}^{s} p_{\delta_i}$. Then we have

$$\mathbf{x^u} - p\mathbf{x^v} = \sum_{i=1}^{s} \left(\prod_{j=0}^{i-1} p_{\delta_j} \right) \mathbf{x}^{\mathbf{w}_i} (\mathbf{x}^{(B\delta_i)_+} - p_{\delta_i} \mathbf{x}^{(B\delta_i)_-}),$$

(here $p_{\delta_0} = 1$). Since $\mathbf{x}^{(B\delta_i)_+} - p_{\delta_i}\mathbf{x}^{(B\delta_i)_-} \in M'$ for $1 \leqslant i \leqslant s$, it follows that $\mathbf{x^u} - p\mathbf{x^v} \in M'$. The constant p is nonzero as $p_{\delta_i} \neq 0$ for $1 \leqslant i \leqslant s$. Therefore, M' is weakly $\mathcal{A}$-graded. As M is $\mathcal{A}$-graded, we conclude that $M' = M$. Note that M contains no monomials. By [St1, Lemma 10.12] it follows that M is toric isomorphic to the toric ideal I. □

LEMMA 3.5. *If the Lawrence lifting I_L is Cohen-Macaulay and M is an $\mathcal{A}$-graded ideal in S, then M is coherent.*

Proof. By Lemma 3.4 we can assume that at least one of the vectors $(1, 0), (0, 1)$ is not M-gluing. After renumbering the quadrants and the basis vectors (if necessary) we can assume that $(0, 1)$ is an M-vector and $(1, 0)$ is either M-gluing or an M-vector.

By Lemma 3.1(a) the Cohen–Macaulayness of I_L is equivalent to $2 \leqslant q_L \leqslant 3$. Let $\mathcal{P}$ be the set consisting of the generating vectors for I_L. Recall from Section 2 that the primitive vectors for I are exactly the generating vectors for I_L. For a vector $s \in \mathbf{Q}^2$ in the first quadrant set $\mathcal{T}_s = \{ \alpha \mid \alpha \in \mathcal{P}, \langle s, \alpha \rangle \geqslant 0 \}$ and

$$M_s = \Big(\{ \mathbf{x}^{(B\alpha)_+} | \alpha \in \mathcal{T}_s, \langle s, \alpha \rangle > 0 \} \cup \{ \mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-} | \alpha \in \mathcal{T}_s, \langle s, \alpha \rangle = 0 \} \Big).$$

The ideal M_s is weakly $\mathcal{A}$-graded by Lemma 2.1, therefore Lemma 2.2 can be applied to $s, \mathcal{T}_s, M_s, M$ if $M_s \subseteq M$. We will find an $s \in \mathbf{Q}^2$ such that $M_s \subseteq M$.

Suppose that $q_L = 2$. Choose $s = (0, 1)$ if $(1, 0)$ is M-gluing and $s = (1, 1)$ otherwise. If $(1, 0)$ is M-gluing then we scale the variables so that $\mathbf{x}^{(B(1,0))_+} - \mathbf{x}^{(B(1,0))_-} \in M$. Then clearly Lemma 2.2 can be applied, so M is coherent.

Let $q_L = 3$. Applying Lemma 3.1(d) to I_L we have that the generating vectors of I_L can be chosen to be $\pm(1, 0)$, $\pm(0, 1)$ and either $\pm(1, 1)$ or $\pm(-1, 1)$. Thus, $\mathcal{P}$ is either $\{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$ or $\{\pm(1, 0), \pm(0, 1), \pm(-1, 1)\}$.

Suppose that $q = 3$. Choose

$$s = \begin{cases} (0, 1) & \text{if } (1, 0) \text{ is } M\text{-gluing}, \\ (1, 1) & \text{if } (-1, 1) \text{ is either non-generating or } M\text{-gluing}, \\ (2, 1) & \text{if } (-1, 1) \text{ is generating and } -(-1, 1) \text{ is an } M\text{-vector}, \\ (1, 2) & \text{if } (-1, 1) \text{ is a generating } M\text{-vector}. \end{cases}$$

There is at most one M-gluing generating vector; if such vector exists we denote it by ξ and scale the variables so that $\mathbf{x}^{(B\xi)_+} - \mathbf{x}^{(B\xi)_-} \in M$. Applying Lemma 3.2 we conclude that $M_s \subseteq M$ and then we apply Lemma 2.2 to $s, \mathcal{T}_s, M_s, M$. Therefore M is coherent.

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For the rest of the proof suppose that $2 = q < q_L = 3$. As in [PS, Construction 5.2], we write the binomials corresponding to $(1, 0)$ and $(0, 1)$ in the form

$$e = \mathbf{x}^{(B(1,0))_+} - \mathbf{x}^{(B(1,0))_-} = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}},$$
$$f = \mathbf{x}^{(B(0,1))_+} - \mathbf{x}^{(B(0,1))_-} = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}},$$

where in each binomial the two monomials are relatively prime, and

$$(\mathbf{u}+\mathbf{v})_+ = \mathbf{u}_+ + \mathbf{v}_+, \qquad (\mathbf{u}+\mathbf{v})_- = \mathbf{u}_- + \mathbf{v}_-,$$
$$(\mathbf{u}-\mathbf{v})_+ = \mathbf{u}_+ + \mathbf{v}_-, \qquad (\mathbf{u}-\mathbf{v})_- = \mathbf{u}_- + \mathbf{v}_+.$$

Hence the binomials corresponding to $(1, 1)$ and $(-1, 1)$ have the form

$$\mathbf{x}^{(B(1,1))_+} - \mathbf{x}^{(B(1,1))_-} = \mathbf{x}^{(\mathbf{u}+\mathbf{v})_+}\mathbf{x}^{2\mathbf{p}} - \mathbf{x}^{(\mathbf{u}+\mathbf{v})_-}\mathbf{x}^{2\mathbf{r}},$$
$$\mathbf{x}^{(B(-1,1))_+} - \mathbf{x}^{(B(-1,1))_-} = \mathbf{x}^{(\mathbf{u}-\mathbf{v})_-}\mathbf{x}^{2\mathbf{s}} - \mathbf{x}^{(\mathbf{u}-\mathbf{v})_+}\mathbf{x}^{2\mathbf{t}}.$$

Since $q = 2$ by Lemma 3.1(a) we have that I is a complete intersection. By [PS, Remark 3.2] it follows that one of the binomials e and f contains a term, which is coprime to each of the terms in the other binomial. This implies that either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1, and also that either $\mathbf{x}^{\mathbf{p}}$ or $\mathbf{x}^{\mathbf{r}}$ is 1. We consider the following two cases:

Case 1. Both $(1, 0)$ *and* $(0, 1)$ *are* M*-vectors*

Clearly, Lemma 2.2 can be applied if $\pm(-1, 1)$ are generating vectors for I_L. Suppose that $\pm(1, 1)$ are generating vectors for I_L. Since either $\mathbf{x}^{\mathbf{t}}$ or $\mathbf{x}^{\mathbf{s}}$ is 1, it follows that the monomial $\mathbf{x}^{(B(1,1))_+} = \mathbf{x}^{(\mathbf{u}+\mathbf{v})_+}\mathbf{x}^{2\mathbf{p}}$ is divided by either the monomial $\mathbf{x}^{(B(1,0))_+} = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}}$ or by the monomial $\mathbf{x}^{(B(0,1))_+} = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}}$. Thus, $\mathbf{x}^{(B(1,1))_+} \in M$, so $(1, 1)$ is an M-vector. Choose $s = (1, 1)$. We have shown that $M_s \subseteq M$. So we can apply Lemma 2.2 to $s, \mathcal{T}_s, M_s, M$. Therefore M is coherent.

Case 2. The vector $(0, 1)$ *is an* M*-vector and* $(1, 0)$ *is* M*-gluing*

By Lemma 2.1 it follows that there exists a nonzero constant p such that $\mathbf{x}^{(B(1,0))_+} - p\mathbf{x}^{(B(1,0))_-} \in M$. After scaling the variables (if necessary) we can assume that $p = 1$, so $e \in M$. We will show that $(1, 1), (-1, 1)$ are M-vectors.

We have that either $\mathbf{x}^{\mathbf{t}}$ or $\mathbf{x}^{\mathbf{s}}$ is 1. If $\mathbf{x}^{\mathbf{s}}$ is 1, then the monomial $\mathbf{x}^{(B(1,1))_+} = \mathbf{x}^{(\mathbf{u}+\mathbf{v})_+}\mathbf{x}^{2\mathbf{p}}$ is divided by the monomial $\mathbf{x}^{(B(0,1))_+} = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}}$, so $\mathbf{x}^{(B(1,1))_+} \in M$. If $\mathbf{x}^{\mathbf{t}}$ is 1, then we get the equalities

$$\mathbf{x}^{(B(1,1))_+} - (\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}})e = \mathbf{x}^{(\mathbf{u}+\mathbf{v})_+}\mathbf{x}^{2\mathbf{p}} - (\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}})e$$
$$= (\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}})(\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}) = (\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})\mathbf{x}^{(B(0,1))_+} \in M.$$

But $e \in M$, hence $\mathbf{x}^{(B(1,1))_+} \in M$. By a similar argument, using that either $\mathbf{x}^{\mathbf{r}}$ or $\mathbf{x}^{\mathbf{p}}$ is 1, we will show that $\mathbf{x}^{(B(-1,1))_+} \in M$. If $\mathbf{x}^{\mathbf{p}}$ is 1, then the monomial $\mathbf{x}^{(B(-1,1))_+} = \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{s}}$ is divided by the monomial $\mathbf{x}^{(B(0,1))_+} = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}}$, so $\mathbf{x}^{(B(-1,1))_+} \in M$. If $\mathbf{x}^{\mathbf{r}}$ is 1, then we

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have the equalities

$$\mathbf{x}^{(B(-1,1))_+} + (\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}})e = \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{2\mathbf{s}} + (\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}})e$$

$$= (\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}})(\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}}) = (\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}})\mathbf{x}^{(B(0,1))_+} \in M\,.$$

But $e \in M$, hence $\mathbf{x}^{(B(-1,1))_+} \in M$.

Choose $s = (0, 1)$. We have shown that $M_s \subseteq M$. Therefore we can apply Lemma 2.2 to $s, \mathcal{T}_s, M_s, M$. Hence M is coherent. □

Starting from here until Theorem 3.15 we assume that I_L is not Cohen-Macaulay; by Lemma 3.1(a) this is equivalent to $q_L \geqslant 4$. Also, by Lemma 3.1(a) there exists at least one syzygy quadrangle for I_L. By [PS, Corollary 4.3], a syzygy quadrangle for I is also a syzygy quadrangle for I_L. Thus the homology tree of I (which exists exactly when $q \geqslant 4$) is contained in the homology tree of I_L.

We say that Q is a *Lawrence quadrangle* if Q is in the first or second quadrant and it is a syzygy quadrangle for I_L but is not a syzygy quadrangle for I.

DEFINITION 3.6. Let Q be a syzygy quadrangle for I_L. We say that Q is a *minimal Lawrence quadrangle* if Q is in the first or second quadrant and one of the following two conditions is satisfied:

(1) $q = 2$ and Q is either the unit square with edges $(1, 0), (0, 1)$ or the unit square with edges $(-1, 0), (0, 1)$.

(2) $q \geqslant 3$, Q is not a syzygy quadrangle for I, and the two triangles with sides the edges of Q and the shorter diagonal of Q are syzygy triangles for I.

In some of the proofs we use an equivalent form (derived using Lemma 3.1) of the above definition which states that:

(1′) If $q = 2$ then the minimal Lawrence quadrangles are the unit squares with edges $(1, 0), (0, 1)$ and $(-1, 0), (0, 1)$.

(2′) If $q = 3$ and $(1, 1)$ is a generating vector, then the minimal Lawrence quadrangles are the unit square with edges $(-1, 0), (0, 1)$ and the syzygy quadrangles for I_L among the quadrangles with edges $(1, 0), (1, 1)$ and $(1, 1), (0, 1)$.

(3′) If $q = 3$ and $(-1, 1)$ is a generating vector, then the minimal Lawrence quadrangles are the unit square with edges $(1, 0), (0, 1)$ and the syzygy quadrangles for I_L among the quadrangles with edges $(-1, 0), (-1, 1)$ and $(-1, 1), (0, 1)$.

(4′) If $q \geqslant 4$ then Q is a minimal Lawrence quadrangle if and only if Q is a child (in the homology tree of I_L) of a syzygy quadrangle for I and Q is not a syzygy quadrangle for I.

LEMMA 3.7. *Let $q_L \geqslant 4$ and δ be a primitive non-generating vector for I in the first or second quadrant. There exists a chain $Q_1, \dots, Q_r$ of syzygy quadrangles for I_L starting with Q_1 a minimal Lawrence quadrangle and such that δ is the longer diagonal of Q_r.*

Proof. Recall from Section 2 that the primitive vectors for I are exactly the generating vectors for I_L. By Lemma 3.1(b) we have that there exists a chain $\mathbf{Q}' = Q'_1, \dots, Q'_s$ of syzygy quadrangles for I_L starting with Q'_1 a unit square and such that δ is the longer diagonal of Q'_s. To complete the proof it will be enough to show that $\mathbf{Q}'$ contains a minimal Lawrence quadrangle. We use the definition of a minimal Lawrence quadrangle given by (1′), (2′), (3′), (4′) in Definition 3.6. It is easy to see that $\mathbf{Q}'$ contains a minimal Lawrence quadrangle if $q \leqslant 3$. Suppose that $q \geqslant 4$. By [PS, Corollary 4.3] the homology tree of I is contained in the homology tree of I_L and they have the same root. Therefore, the chain $\mathbf{Q}'$ contains a minimal Lawrence quadrangle. □

CONSTRUCTION 3.8. Let $\alpha, \beta \in \mathbf{Z}^2$ and $\gamma = \alpha + \beta$. Set

$$\mathbf{x}^{\mathbf{p}} = \gcd(\mathbf{x}^{(B\alpha)_+}, \mathbf{x}^{(B\beta)_+}), \qquad \mathbf{x}^{\mathbf{t}} = \gcd(\mathbf{x}^{(B\alpha)_+}, \mathbf{x}^{(B\beta)_-}),$$

$$\mathbf{x}^{\mathbf{s}} = \gcd(\mathbf{x}^{(B\alpha)_-}, \mathbf{x}^{(B\beta)_+}), \qquad \mathbf{x}^{\mathbf{r}} = \gcd(\mathbf{x}^{(B\alpha)_-}, \mathbf{x}^{(B\beta)_-}).$$

As in [PS, Construction 5.2], α, β correspond to two binomials in I which we write in the form $e = \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}$ and $f = \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}}$ so that the two monomials in each binomial are relatively prime and

$$(\mathbf{u}+\mathbf{v})_+ = \mathbf{u}_+ + \mathbf{v}_+, \qquad (\mathbf{u}+\mathbf{v})_- = \mathbf{u}_- + \mathbf{v}_-,$$

$$(\mathbf{u}-\mathbf{v})_+ = \mathbf{u}_+ + \mathbf{v}_-, \qquad (\mathbf{u}-\mathbf{v})_- = \mathbf{u}_- + \mathbf{v}_+.$$

Thus, the binomials corresponding to the vectors α, β, γ are:

$$\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{r}}, \qquad \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}}\mathbf{x}^{\mathbf{p}} - \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}}\mathbf{x}^{\mathbf{r}},$$

$$(\mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}}) - (\mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}}). \tag{3.9}$$

LEMMA 3.10. *Let M be an $\mathcal{A}$-graded ideal in S and I_L the Lawrence lifting of I. Let Q be a Lawrence quadrangle, α, β the edges of Q and γ its longer diagonal. In the notation of Construction 3.8 we have that at least one of the monomials $\mathbf{x}^{\mathbf{s}}$ and $\mathbf{x}^{\mathbf{t}}$ is equal to* 1.

Proof. First we will prove the lemma in the case when Q is a minimal Lawrence quadrangle. We consider two cases:

Case 1. The ideal I is a complete intersection

By (1′) in Definition 3.6 we have that Q is a unit square and the binomials e and f correspond to its edges. By [PS, Remark 3.2] it follows that one of the binomials

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contains a term, which is coprime to each of the terms in the other binomial. This implies that either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1.

Case 2. The ideal I is not a complete intersection
By Lemma 3.1(a) we get that $q \geqslant 3$ in this case. Thus, Q satisfies condition (2) in Definition 3.6. As in [PS, Construction 5.2] we have that the longer and shorter diagonals of Q are represented respectively by the binomials

$$g = \mathbf{x}^{(\mathbf{u}+\mathbf{v})_+}\mathbf{x}^{2\mathbf{p}} - \mathbf{x}^{(\mathbf{u}+\mathbf{v})_-}\mathbf{x}^{2\mathbf{r}}, \quad h = \mathbf{x}^{(\mathbf{u}-\mathbf{v})_+}\mathbf{x}^{2\mathbf{t}} - \mathbf{x}^{(\mathbf{u}-\mathbf{v})_-}\mathbf{x}^{2\mathbf{s}}.$$

Denote by $\mathbf{F}$ the minimal free resolution of $k[x_1, \ldots, x_n]/I$ over the ring $S = k[x_1, \ldots, x_n]$. Let $\mathbf{G}$ be the minimal free resolution of $k[x_1, \ldots, x_n, y_1, \ldots, y_n]/I_L$ over $k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ which is constructed as in [PS, Theorem 5.5]. Set

$$\bar{\mathbf{G}} = \mathbf{G} \otimes k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(y_1 - 1, \ldots, y_n - 1).$$

By [PS, Constructions 5.1, 5.2 and Theorems 5.4, 5.5], we have the following complex

$$0 \to S \xrightarrow{\begin{pmatrix} -\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{t}} \\ \mathbf{x}^{\mathbf{r}} \\ -\mathbf{x}^{\mathbf{p}} \end{pmatrix}} S^4 \xrightarrow{\begin{pmatrix} \mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{r}} & -\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{r}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}} \\ -\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{s}} & 0 & 0 \\ 0 & 0 & \mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{r}} \end{pmatrix}} S^4 \xrightarrow{(e\ f\ g\ h)} S, \qquad (3.11)$$

which is a subcomplex of $\bar{\mathbf{G}}$ and the basis elements of the free modules in (3.11) are basis elements in $\bar{\mathbf{G}}$ as well.

We assume that the minimal free resolution $\mathbf{F}$ is constructed as in [PS, Theorem 5.5]. By [PS, Corollary 4.3], if $q \geqslant 4$ then the homology tree of I is contained in the homology tree of I_L which induces an inclusion of $\mathbf{F}$ in $\bar{\mathbf{G}}$. If $q = 3$ we apply [PS, Remark 5.8] to get an inclusion of $\mathbf{F}$ in $\bar{\mathbf{G}}$. Lemma 2.3 implies that $\bar{\mathbf{G}}$ is a (possibly non-minimal) free resolution of $k[x_1, \ldots, x_n]/I$ over $k[x_1, \ldots, x_n]$. By [Ei, Theorem 20.2], $\mathbf{F}$ is a direct summand in $\bar{\mathbf{G}}$. Now consider (3.11). The basis element in (3.11) in homological degree 3 corresponds to the quadrangle Q via [PS, Constructions 5.1, 5.2 and Theorem 5.4]. On the other hand, the basis elements in $\mathbf{F}$ in homological degree 3 correspond to the syzygy quadrangles for I via [PS, Constructions 5.1, 5.2 and Theorems 5.4, 5.5]. Since Q is not a syzygy quadrangle for I and since $\bar{\mathbf{G}}$ has length 3 it follows that the matrix of the third differential in (3.11) contains an invertible element, that is, one of the monomials $\mathbf{x}^{\mathbf{s}}$, $\mathbf{x}^{\mathbf{t}}$, $\mathbf{x}^{\mathbf{r}}$, $\mathbf{x}^{\mathbf{p}}$ is 1. The monomials $\mathbf{x}^{\mathbf{r}}$ and $\mathbf{x}^{\mathbf{p}}$ are entries in the matrix

$$\begin{pmatrix} -\mathbf{x}^{\mathbf{v}_-}\mathbf{x}^{\mathbf{t}} & -\mathbf{x}^{\mathbf{v}_+}\mathbf{x}^{\mathbf{s}} \\ \mathbf{x}^{\mathbf{u}_-}\mathbf{x}^{\mathbf{s}} & \mathbf{x}^{\mathbf{u}_+}\mathbf{x}^{\mathbf{t}} \\ 0 & 0 \\ \mathbf{x}^{\mathbf{p}} & \mathbf{x}^{\mathbf{r}} \end{pmatrix},$$

which appears as a submatrix of the second differential in (3.11); this matrix gives the action of the differential on the two triangles with edges e, f, h. By the choice of Q these two triangles are syzygy triangles for I. Hence the above matrix is contained in the differential of the minimal free resolution $\mathbf{F}$, therefore it cannot have invertible entries. So $\mathbf{x}^{\mathbf{r}}$ and $\mathbf{x}^{\mathbf{p}}$ are not invertible. It follows that at least one of the monomials $\mathbf{x}^{\mathbf{s}}$ and $\mathbf{x}^{\mathbf{t}}$ is equal to 1.

Thus, the lemma is proved in the case when Q is a minimal Lawrence quadrangle. Order the Lawrence quadrangles so that if P is a child of P' in the homology tree of I_L (see [PS, Construction 4.5]) then $P' \prec P$. We will finish the proof by induction on this order. Let Q be an arbitrary Lawrence quadrangle. Applying Lemma 3.7 to the longer diagonal of Q we get that there exists a chain $Q_1, \dots Q_r$ of Lawrence quadrangles starting with a minimal Lawrence quadrangle Q_1 and $Q = Q_r$. If $r = 1$ then Q is a minimal Lawrence quadrangle and we are done. Suppose that $r > 1$ and denote $Q' = Q_{r-1}$. Applying Construction 3.8 to Q' we obtain monomials $\mathbf{x}^{\mathbf{s}'}, \mathbf{x}^{\mathbf{t}'}, \mathbf{x}^{\mathbf{p}'}, \mathbf{x}^{\mathbf{r}'}, \mathbf{x}^{\mathbf{u}'_+}, \mathbf{x}^{\mathbf{u}'_-}, \mathbf{x}^{\mathbf{v}'_+}, \mathbf{x}^{\mathbf{v}'_-}$. Let α', β' and γ' be the two edges and the longer diagonal of Q'. Then the two edges of Q are either α', γ' or β', γ'. We consider these two cases separately:

Subcase 1. Let α', γ' be the edges of Q.

Applying Construction 3.8 to Q we get that

$$\mathbf{x}^{\mathbf{s}} = \gcd(\mathbf{x}^{(B\alpha')_-}, \mathbf{x}^{(B\gamma')_+}) = \gcd(\mathbf{x}^{\mathbf{v}'_+}, \mathbf{x}^{\mathbf{s}'}),$$

$$\mathbf{x}^{\mathbf{t}} = \gcd(\mathbf{x}^{(B\alpha')_+}, \mathbf{x}^{(B\gamma')_-}) = \gcd(\mathbf{x}^{\mathbf{v}'_-}, \mathbf{x}^{\mathbf{t}'}).$$

By the induction hypothesis the lemma holds for Q', that is, either $\mathbf{x}^{\mathbf{s}'}$ or $\mathbf{x}^{\mathbf{t}'}$ is 1. Hence, either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1.

Subcase 2. Let β', γ' be the edges of Q.

Applying Construction 3.8 to Q we get that

$$\mathbf{x}^{\mathbf{s}} = \gcd(\mathbf{x}^{(B\beta')_-}, \mathbf{x}^{(B\gamma')_+}) = \gcd(\mathbf{x}^{\mathbf{u}'_+}, \mathbf{x}^{\mathbf{t}'}),$$

$$\mathbf{x}^{\mathbf{t}} = \gcd(\mathbf{x}^{(B\beta')_+}, \mathbf{x}^{(B\gamma')_-}) = \gcd(\mathbf{x}^{\mathbf{u}'_-}, \mathbf{x}^{\mathbf{s}'}).$$

By the induction hypothesis the lemma holds for Q', that is, either $\mathbf{x}^{\mathbf{s}'}$ or $\mathbf{x}^{\mathbf{t}'}$ is 1. Hence either $\mathbf{x}^{\mathbf{s}}$ or $\mathbf{x}^{\mathbf{t}}$ is 1. □

We obtain an analogue to Lemma 3.2 for Lawrence quadrangles:

LEMMA 3.12. *Let M be an $\mathcal{A}$-graded ideal in S. Let Q be a Lawrence quadrangle, α, β the edges of Q, and γ its longer diagonal.*

(a) *If α, β well-match then γ well-matches them.*
(b) *If α is M-gluing and β is not M-gluing, then γ well-matches β.*

Proof. The vectors α, β, γ correspond to binomials in I which we write as in (3.9). By Lemma 3.10, we have that either $\mathbf{x^t}$ or $\mathbf{x^s}$ is 1 in (3.9). It follows that at least one of the monomials $\mathbf{x}^{(B\alpha)_+}, \mathbf{x}^{(B\beta)_+}$ divides $\mathbf{x}^{(B\gamma)_+}$ and also that at least one of the monomials $\mathbf{x}^{(B\alpha)_-}, \mathbf{x}^{(B\beta)_-}$ divides $\mathbf{x}^{(B\gamma)_-}$. Therefore, (a) holds. It remains to prove part (b). Since α is M-gluing by hypothesis, after scaling the variables (if necessary) we can assume that

$$e = \mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-} = \mathbf{x}^{\mathbf{u}_+}\mathbf{x^t}\mathbf{x^p} - \mathbf{x}^{\mathbf{u}_-}\mathbf{x^s}\mathbf{x^r} \in M\,.$$

First we consider the case when β is an M-vector, that is $\mathbf{x}^{(B\beta)_+} = \mathbf{x}^{\mathbf{v}_+}\mathbf{x^s}\mathbf{x^p} \in M$. If $\mathbf{x^t} = 1$, then we have

$$\mathbf{x}^{(B\gamma)_+} - (\mathbf{x}^{\mathbf{v}_+}\mathbf{x^p})e = (\mathbf{x}^{\mathbf{u}_+}\mathbf{x^p})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x^p}) - (\mathbf{x}^{\mathbf{v}_+}\mathbf{x^p})e$$

$$= (\mathbf{x}^{\mathbf{v}_+}\mathbf{x^p})(\mathbf{x}^{\mathbf{u}_-}\mathbf{x^s}\mathbf{x^r}) = \mathbf{x}^{(B\beta)_+}(\mathbf{x}^{\mathbf{u}_-}\mathbf{x^r}) \in M\,,$$

and as $e \in M$ we get that $\mathbf{x}^{(B\gamma)_+} \in M$. If $\mathbf{x^s} = 1$, then we have $\mathbf{x}^{\mathbf{v}_+}\mathbf{x^p} = \mathbf{x}^{(B\beta)_+} \in M$, so $\mathbf{x}^{(B\gamma)_+} = (\mathbf{x}^{\mathbf{u}_+}\mathbf{x^p})(\mathbf{x}^{\mathbf{v}_+}\mathbf{x^p}) \in M$. Thus, γ well-matches β.

Now consider the case when $-\beta$ is an M-vector, that is $\mathbf{x}^{(B\beta)_-} = \mathbf{x}^{\mathbf{v}_-}\mathbf{x^t}\mathbf{x^r} \in M$. If $\mathbf{x^s} = 1$, then we have

$$\mathbf{x}^{(B\gamma)_-} + (\mathbf{x}^{\mathbf{v}_-}\mathbf{x^r})e = (\mathbf{x}^{\mathbf{u}_-}\mathbf{x^r})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x^r}) + (\mathbf{x}^{\mathbf{v}_-}\mathbf{x^r})e$$

$$= (\mathbf{x}^{\mathbf{v}_-}\mathbf{x^r})(\mathbf{x}^{\mathbf{u}_+}\mathbf{x^t}\mathbf{x^p}) = \mathbf{x}^{(B\beta)_-}(\mathbf{x}^{\mathbf{u}_+}\mathbf{x^p}) \in M\,,$$

and as $e \in M$ we get that $\mathbf{x}^{(B\gamma)_-} \in M$. If $\mathbf{x^t} = 1$, then $\mathbf{x}^{\mathbf{v}_-}\mathbf{x^r} = \mathbf{x}^{(B\beta)_-} \in M$, so $\mathbf{x}^{(B\gamma)_-} = (\mathbf{x}^{\mathbf{u}_-}\mathbf{x^r})(\mathbf{x}^{\mathbf{v}_-}\mathbf{x^r}) \in M$. Thus, γ well-matches β. □

Lemmas 3.3 and 3.12 imply the following result:

LEMMA 3.13. *Let M be an $\mathcal{A}$-graded ideal. Let $r \geqslant 1$ and $P_1, \ldots, P_r$ be a chain of syzygy quadrangles for I_L in the first or second quadrant. Denote by α, β the edges of P_1 and by δ the longer diagonal of P_r.*

(a) *If α and β well-match, then δ well-matches them.*
(b) *If α is an M-gluing vector, but β is not, then δ well-matches β.*

Proof. Let s be the smallest number such that P_s is a Lawrence quadrangle, or set $s = r + 1$ if the chain contains no Lawrence quadrangle. If $s \geqslant 2$ then apply Lemma 3.3 to the chain $P_1, \ldots, P_{s-1}$. The proof is completed by induction: if Lemma 3.13 holds for $P_1, \ldots, P_j$ for some $r > j \geqslant s - 1$ then by Lemma 3.12 it follows that Lemma 3.13 holds for $P_1, \ldots, P_{j+1}$ as well. □

LEMMA 3.14. *Let M be an $\mathcal{A}$-graded ideal in S. Suppose that at least one of the vectors* $(1, 0), (0, 1)$ *is not M-gluing (so by assumption* $(0, 1)$ *is an M-vector and* $(1, 0)$ *is either an M-vector or M-gluing) and that I_L is not Cohen–Macaulay. Consider the set $\mathcal{P}$ of all primitive vectors for I in the first and second quadrants.*

(a) *Let $\alpha, \beta \in \mathcal{P}$ be in the second quadrant and the angle between α and* $(-1, 0)$ *be smaller than the angle between β and* $(-1, 0)$. *Suppose that α is either an M-vector or M-gluing. Then β is an M-vector.*
(b) *There exists at most one M-gluing vector in the intersection of $\mathcal{P}$ and the second quadrant.*

Proof. Denote by $\mathcal{G}$ the set of all generating vectors for I in the first or second quadrant. First, we prove that (a) holds if $\alpha, \beta \in \mathcal{G}$ and $q \leqslant 3$. If $q = 2$ then we are done by the assumption that $(0, 1)$ is an M-vector. If $q = 3$, then apply Lemmas 3.1(d) and 3.2.

Recall from Section 2 that the primitive vectors for I are the generating vectors for I_L.

Suppose that $(-1, 0)$ is M-gluing. Applying Lemma 3.1(b) to I_L, and then applying Lemma 3.13, we conclude that all primitive vectors in the first or second quadrant which are different from $\pm(1, 0)$ are M-vectors; in particular, Lemma 3.14 holds. Now suppose that $(-1, 0)$ is not M-gluing and therefore by assumption $(1, 0)$ is an M-vector.

By [PS, Corollary 4.7] there exists a total order $\prec$ on the vectors in $\mathcal{P}$ such that if P is a syzygy quadrangle for I or a minimal Lawrence quadrangle then its longer diagonal is bigger in the order $\prec$ than its edges. The first vectors in the order are $(0, 1), (1, 0), (-1, 0)$ and we can assume that $(0, 1) \prec (1, 0) \prec (-1, 0)$. This induces a partial order on the set of pairs of elements in $\mathcal{P}$ in the following way: Let $\sigma, \sigma', \tau, \tau' \in \mathcal{P}$. Suppose that $\sigma \preceq \sigma'$ and $\tau \preceq \tau'$. We say that $\{\sigma, \sigma'\} \preceq \{\tau, \tau'\}$ if $\sigma \preceq \tau$ and $\sigma' \preceq \tau'$.

Let $\alpha, \beta \in \mathcal{P}$ be in the second quadrant and the angle between α and $(-1, 0)$ be smaller than the angle between β and $(-1, 0)$. By Lemma 3.1(b) applied to I_L, there exists a chain $\mathbf{P} = P_1, \dots, P_r$ such that $P_1, \dots, P_r$ are syzygy quadrangles for I_L, P_1 is the unit square with edges $(-1, 0), (0, 1)$, and α is the longer diagonal of P_r. Similarly, by Lemma 3.1(b) there exists a chain $\mathbf{T} = T_1, \dots, T_s$ such that $T_1, \dots, T_s$ are syzygy quadrangles for I_L, T_1 is the unit square with edges $(-1, 0), (0, 1)$ and β is the longer diagonal of T_s. Then $T_1 = P_1$. Denote by R the last common quadrangle in $\mathbf{P}$ and $\mathbf{T}$. Let i, j be such that $1 \leqslant i \leqslant s, 1 \leqslant j \leqslant r$ and $R = P_j = T_i$. Let μ, ν be the edges of R and $\xi = \mu + \nu$ be its longer diagonal. Suppose that the angle between μ and $(-1, 0)$ is smaller than the angle between ν and $(-1, 0)$. By [PS, Construction 4.4], if $i < s$ then the edges of T_{i+1} are ν, ξ and also if $j < r$ then the edges of P_{j+1} are ξ, μ.

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We will prove (a) by induction on the considered order. Part (a) holds if $\beta = (0, 1)$ since $(0, 1)$ is an M-vector by assumption. Assume that $\beta \neq (0, 1)$. Since α is either M-gluing or an M-vector and $(1, 0)$ is an M-vector, it follows that $\alpha \neq (-1, 0)$.

Since $\mu \prec \alpha$ we have that $\{\mu, \beta\} \prec \{\alpha, \beta\}$. If μ is either M-gluing or an M-vector, then by the induction hypothesis we have that β is an M-vector. Suppose that $-\mu$ is an M-vector. If $-\xi$ is an M-vector or ξ is M-gluing, then applying Lemma 3.13 to the chain $P_{j+1}, \ldots, P_r$ we conclude that $-\alpha$ is an M-vector which is a contradiction. Therefore, ξ is an M-vector. Since $\nu \prec \xi \preceq \alpha, \beta$ we have that $\{\nu, \xi\} \prec \{\alpha, \beta\}$; hence by the induction hypothesis it follows that ν is an M-vector. Therefore, applying Lemma 3.13 to the chain $T_{i+1}, \ldots, T_s$ we conclude that β is an M-vector. Thus, (a) is proved.

Now we prove (b). Suppose that there exist more than one M-gluing vectors in the intersection of $\mathcal{P}$ and the second quadrant. Let α, β be the two smallest (in the order $\prec$) M-gluing vectors which are in $\mathcal{P}$ and in the second quadrant. Suppose that the angle between α and $(-1, 0)$ is smaller than the angle between β and $(-1, 0)$. Note that $\alpha \neq (-1, 0)$ and $\beta \neq (0, 1)$. Suppose that $\alpha = \xi$ (in the above notation). Since α, β are chosen to be the two smallest M-gluing vectors and $\nu \prec \beta$, $\nu \prec \alpha$ it follows that ν is not M-gluing. Applying Lemma 3.13 to the chain $T_i, \ldots, T_s$ we conclude that β is not M-gluing which is a contradiction. Hence $\alpha \neq \xi$. Similar argument shows that $\beta \neq \xi$. Then ξ, μ, ν are smaller than each of α, β; hence ξ, μ, ν are not M-gluing. Therefore, at least one of the pairs $\{\xi, \mu\}$, $\{\xi, \nu\}$, and $\{\mu, \nu\}$ consists of well-matching vectors. If ξ, μ well-match, then applying Lemma 3.13 to the chain $P_{j+1}, \ldots, P_r$ we get that α is not M-gluing which is a contradiction. If ξ, ν or μ, ν well-match, then applying Lemma 3.13 to the chain $T_{i+1}, \ldots, T_s$ or $T_i, \ldots, T_s$ respectively, we get that β is not M-gluing which is a contradiction as well. Therefore, there cannot exist two M-gluing vectors in the intersection of $\mathcal{P}$ with the second quadrant. □

We are ready to prove our main result.

THEOREM 3.15. *If* $codim(I_{\mathcal{A}}) = 2$ *and* M *is an* $\mathcal{A}$*-graded ideal in* S*, then* M *is coherent.*

Proof. If $(1, 0)$ and $(0, 1)$ are M-gluing then apply Lemma 3.4. If I_L is Cohen–Macaulay, then apply Lemma 3.5. Suppose that I_L is not Cohen–Macaulay and that at least one of the vectors $(1, 0), (0, 1)$ is not M-gluing. Recall that in this case after renumbering the quadrants and the basis vectors (if necessary) we assume that $(0, 1)$ is an M-vector and $(1, 0)$ is either M-gluing or an M-vector. As in Lemma 3.14 consider the set $\mathcal{P}$ consisting of the primitive vectors for I in the first and second quadrants. The primitive vectors for I are the generating vectors for I_L. Applying Lemma 3.1(b) to I_L, and Lemma 3.13 (if $q = 3$ then applying also Lemmas 3.1(d) and 3.2) we conclude that every vector $\eta \neq (1, 0)$, which is in the intersection of $\mathcal{P}$ and the first quadrant is an M-vector. Combining this fact with

Lemma 3.14 we see that there exists a vector $s \in \mathbf{Q}^2$ such that the following three conditions are satisfied:

(i) if $\alpha \in \mathcal{P}$ and $\langle \alpha, s \rangle > 0$ then α is an M-vector;
(ii) if $\alpha \in \mathcal{P}$ and $\langle \alpha, s \rangle < 0$ then $-\alpha$ is an M-vector;
(iii) if $\alpha \in \mathcal{P}$ and $\langle \alpha, s \rangle = 0$ then α is M-gluing.

If there exists a vector $\zeta \in \mathcal{P}$ such that $\langle s, \zeta \rangle = 0$ then for some nonzero constant $p \in k \setminus 0$ we have $\mathbf{x}^{(B\zeta)_+} - p\mathbf{x}^{(B\zeta)_-} \in M$. After scaling the variables (if necessary) we can assume that $p = 1$. Hence, condition (iii) above becomes:

(iii′) if $\alpha \in \mathcal{P}$ and $\langle \alpha, s \rangle = 0$ then $\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-} \in M$.
Set

$$\mathcal{T} = \{\, \alpha \mid \langle s, \alpha \rangle \geqslant 0 \text{ and either } \alpha \text{ or } -\alpha \text{ is in } \mathcal{P} \,\},$$

$$M' = \Big(\{\mathbf{x}^{(B\alpha)_+} | \alpha \in \mathcal{T}, \langle s, \alpha \rangle > 0\} \cup \{\mathbf{x}^{(B\alpha)_+} - \mathbf{x}^{(B\alpha)_-} | \alpha \in \mathcal{T}, \langle s, \alpha \rangle = 0\,\} \Big).$$

The ideal M' is weakly $\mathcal{A}$-graded by Lemma 2.1. By (i),(ii),(iii′) we have that $M' \subseteq M$. Applying Lemma 2.2 to M, M', s, and $\mathcal{T}$ we get that M is coherent. □

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